THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 60

1953

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND SEATTLE, WASH.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 1

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JANUARY 1953

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

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THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of College Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

VIEWS AND APPROXIMATIONS ON DIFFERENTIAL EQUATIONS*

R. P. AGNEW, Cornell University

1. Introduction. The English mathematician Andrew Russell Forsyth, 1858–1942, wrote sixteen books including his six-volume *Theory of differential equations* published from 1890 to 1906. In an obituary of Forsyth, E. H. Neville† (p. 237) said that the six-volume treatise is the most imposing single-handed work of any English mathematician. Forsyth believed that differential equations should be solved, and he set himself the task of collecting all effective methods for obtaining solutions of differential equations. Neville† (p. 242) said that Forsyth is more to blame than anyone else if to this day an applied mathematician always asks whether and how a differential equation can be solved instead of sometimes asking what the equation tells us about a function which satisfies it.

We should not ignore the necessity of knowing and teaching standard procedures for obtaining useful formulas for solutions of differential equations of important types. But we should hold the view that such procedures represent only one step, sometimes very useful and sometimes useless, in the process of learning about a function by deriving a differential equation which it satisfies and then squeezing information from the differential equation. Our attention in this paper is to be focused on methods of obtaining information from differential equations without solving the equations in any traditional sense. Sections 2, 3, and 4 indicate, by consideration of examples, methods independent of numerical calculations.

2. A first-order equation. To illustrate by an example the advantage of looking at a differential equation without solving it, we consider the amusing equation

$$y'(x) = A \sin x \sin y, \tag{2.1}$$

where A is a positive constant. Simple theorems imply that when a and b are given constants there is one and only one function $y(x)$, defined over $-\infty < x < \infty$, for which $y(a) = b$ and (2.1) hold. Since $\sin x$ and $\sin y$ are alternately positive and negative over intervals of length π, we convert the whole x, y plane into

* This is the first part of a lecture given June 26, 1951, by invitation of the program committee, at the joint meeting of the Mathematical Association of America and the American Society for Engineering Education held at Michigan State College. The remainder of the lecture dealt with numerical approximations.

† Neville, E. H. 1942, Andrew Russell Forsyth. (Obituary). Journal of the London Mathematical Society, vol. 17, pp. 237–256.

a checkerboard of squares having sides of length π as in Figure 1. The graph of $y(x)$ has positive slope wherever it lies in a black (or shaded) square, negative slope wherever it lies in a white square, and zero slope wherever it intersects a boundary between black and white squares. It is immediately obvious from this fact, as it is from the equation (2.1) itself, that $y=n\pi$ is a solution of (2.1) for each integer n. Since no two graphs of solutions $y(x)$ can intersect unless they coincide, we obtain the very significant fact that if the graph of a solution $y(x)$ contains one point between two consecutive lines in the set $y=n\pi$,

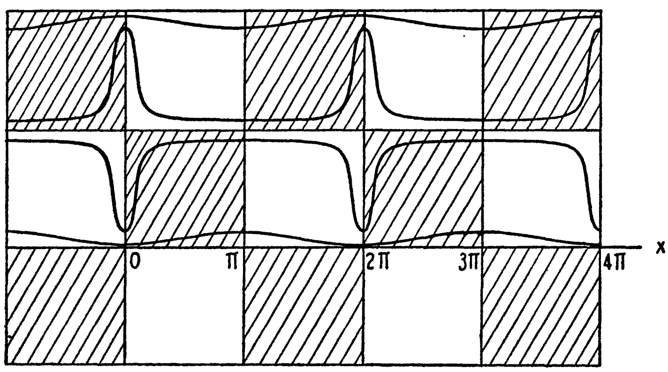

FIG. 1

then the whole graph lies between these lines. The symmetries in $\sin x$ and $\sin y$ imply that the lines $x=m\pi$, $m=1, 2, \cdots$, are lines of symmetry of graphs of solutions, and that solutions have period 2π. Figure 1 shows the nature of graphs of some of the solutions of (2.1) when A is a rather large positive constant.

The standard way to learn about the solutions of (2.1) is to separate the variables by dividing by $\sin y$, integrate, and simplify. Depending on the formula one uses for the integral of the cosecant, we arrive at one or the other of the equivalent formulas

$$y = \pm 2 \tan^{-1} e^{c_1 - A\cos x} \tag{2.2}$$

$$y = \pm \cot^{-1} \sinh (c_2 + A \cos x). \tag{2.3}$$

These formal calculations should be accompanied by consideration of the possibility that $\sin y$ may be 0, and of the manner in which the inverse trigonometric functions and the signs are to be used to isolate different solutions $y(x)$. In this case, information obtained by looking at the differential equation and the formulas obtained by solving the equation are both helpful. Each aids the other.

A more complicated equation such as

$$y' = \frac{A \sin x \sin y}{1 + x^2 + y^2} \tag{2.4}$$

resists solution, but consideration of the checkerboard and slopes yields fundamental ideas about its solutions. Perhaps the most important single fact about solutions of (2.4)is the fact that for each solution $\lim_{x\to\infty} y(x)$ must exist. If $y(x)$ satisfies (2.4), then $|y'(x)| < |A| x^{-2}$; hence, when $0 < u < v$,

$$\begin{aligned} |y(v) - y(u)| &= \left| \int_u^v y'(x)dx \right| \leqq \int_u^v |y'(x)| dx \\ &< |A| \int_u^v x^{-2}dx = A\left(\frac{1}{u} - \frac{1}{v}\right). \end{aligned} \tag{2.5}$$

Hence, when u and v become infinite, $y(v) - y(u) \to 0$ and therefore $\lim_{x\to\infty} y(x)$ exists.

3. Some views of the equation of the harmonic oscillator. Let us suppose that a function $y(x)$ in which we are interested satisfies the conditions $y(0) = 0$, $y'(0) = 1$ and the differential equation

$$y''(x) + y(x) = 0. \tag{3.1}$$

This equation is linear, and we should know, or be able to discover very quickly, that each solution of (3.1) has the form $y = c_1 \cos x + c_2 \sin x$ and that the supplementary conditions determine c_1 and c_2 to make

$$y = \sin x. \tag{3.2}$$

By accident of our training, we know so many tables and properties of sin x that we correctly consider the problem to be completely solved.

Suppose we were totally ignorant of the trigonometric solution of (3.1). We could seek power series solutions of (3.1) and find by standard procedures that

$$y(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{3.3}$$

Of course we can use (3.3) to compute y for any given value of x and hence to make tables and graphs. But the behavior of $y(x)$ for large values of x, and in particular the existence of a strange number $\pi = 3.141592653589793 \cdots$ such that $y(x + 2\pi) = y(x)$, is certainly not evident from (3.3). The fundamental fact is that one has learned very little when one has first reached (3.3). One who feels that differential equations should be solved and discarded may be partly disillusioned by discovering that the best way to learn about the function defined by (3.3) is to show that it satisfies (3.1) and then proceed as in the next paragraph.

Suppose another person, ignorant of the trigonometric solution, should write (3.1) in the form $y''(x)=-y(x)$ and try to sketch an approximation to the graph of $y(x)$. As x starts at 0 and increases, the graph (Figure 2) starts at the origin and runs upward with slope beginning at 1 and (because $y''(x)=-y(x)$ is the rate of change of slope) decreasing at an increasing rate. Hence the slope must reach 0 for some value x_1 of x. As x increases beyond x_1, the slope continues to decrease and the graph must intersect the x axis at some point for which, say $x=x_2$. Because $y''(x)$ is a function of y only, it can be concluded that the graph over the interval $0\leqq x\leqq x_2$ is symmetric about the line $x=x_1$ and hence that $x_2=2x_1$ and $y'(x_2)=-1$. Because of this and the fact that $y''(x)$ is an odd function of y, it can be concluded that $y(x_2+x)=-y(x)$ when $0\leqq x\leqq x_2$. Thus when $x_4=2x_2=4x_1$ we have $y(x_4)=0$ and $y'(x_4)=1$. This and the fact that $y''(x)$ is a

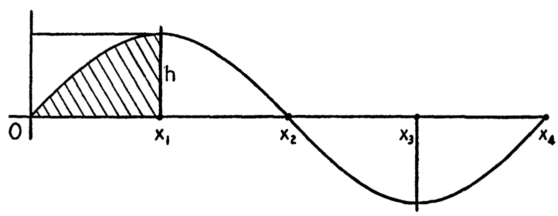

FIG. 2

function of y only imply that the graph to the right of x_4 must be a copy of the graph to the right of 0. Thus we have an idea of the nature of the graph from 0 to x_4, and we know the fundamental fact that $y(x)$ is periodic, that is, $y(x+x_4)=y(x)$. If we multiply (3.1) by $2[y'(x)]$, we obtain

$$\frac{d}{dx}\{[y'(x)]^2+[y(x)]^2\}=0. \tag{3.4}$$

The quantity in braces is therefore a constant which must be 1 because $y(0)=0$, $y'(0)=1$. Therefore

$$[y'(x)]^2+[y(x)]^2=1. \tag{3.5}$$

Since $y'(x_1)=0$ and $y(x_1)>0$, it follows that $y(x_1)=1$. Again using (3.1), we find that

$$\int_0^{x1} y(x)dx=-\int_0^{x1} y''(x)dx=-y'(x_1)+y'(0)=1. \tag{3.6}$$

If, for a rough first approximation, we assume that the area of the shaded region in Figure 1 is two-thirds of the area of the enclosing rectangle of length x_1 and height $h=1$, we obtain $1=(2/3)\cdot 1\cdot x_1$ or $x_1=3/2$ and $x_2=3$. Thus it is not unreasonable to guess that x_2, normally called π, is about 3. Numerical methods

can be used to approximate x_1 and the function $y(x)$ over the interval $0 \leqq x \leqq x_1$. We now find ourselves developing the theory of the function $y(x)$, which is sin x, and $y'(x)$, which is cos x, with the aid of the differential equation (3.1), but we do not have time to continue the development here.

Let us now apply different points of view to a more complicated equation of the type $y''(x)=f(y)$ where $f(y)$ is an odd continuous increasing function of y, say

$$y''(x) = -y^3\sqrt{1+y^2+\cos^2 y} \tag{3.7}$$

where we assume as before the simple conditions $y(0)=0$, $y'(0)=1$. This equation is nonlinear. A power series expansion of $y(x)$ could not be obtained easily and could not provide much illumination even if one had it. An attempt to solve (3.7) by the standard device of multiplying by $2y'(x)$ and integrating leads, as it usually does, to the uninformative inverse of a function defined by an integral of a function of an integral not expressible in terms of elementary functions. It thus appears that we are not being successful in attempts to get information about $y(x)$ by solving (3.7). But if we will simply turn to the equation (3.7) itself and apply the same considerations involving slopes and rates of change of slope that we applied to (3.1), we see that $y(x)$ is periodic and that the graph over a period looks much like the graph in Figure 2, being somewhat straighter near its intersections with the x axis. Methods of approximation can be used to approximate x_1, $x_2=2x_1$, $x_4=4x_1$, and the graph over a period. For the still more complicated equation

$$y''(x) = -y^3\sqrt{1+x^2+y^2+\cos^2 y} \tag{3.8}$$

considerations of slopes and changes of slope will show that the symmetries and periodicities have disappeared and will give preliminary ideas about the manner in which $y(x)$ oscillates as x increases.

4. An eigenvalue problem. For each real constant λ, let $y_\lambda(x)$ denote the real function $y(x)$ for which $y(-1)=0$, $y'(-1)=1$, and

$$y'' = -(\lambda - x^2)y. \tag{4.1}$$

Each solution of (4.1) which vanishes when $x=-1$ has the form $Cy_\lambda(x)$ where C is a constant, and we focus our attention on the functions $y_\lambda(x)$. If $y_\lambda(1)=0$, then λ is an eigenvalue of a special eigenvalue problem, and $Cy_\lambda(x)$ is an eigenfunction belonging to λ for each $C\neq 0$. For some purposes, it is better to write (4.1) in the form $-y''+x^2y=\lambda y$, but this need not concern us here. However much theory one may know concerning this and related problems, it is of interest to see that qualitative information is readily obtained, from (4.1) itself, about all functions $y_\lambda(x)$ as well as about eigenvalues and eigenfunctions. If $\lambda \leqq 0$, then $y_\lambda'(x)$ increases from 1 and $y_\lambda(x)$ increases from 0 as x increases from -1. Thus if $\lambda \leqq 0$, then $y_\lambda(1)$ cannot be 0 and λ cannot be an eigenvalue. If $0<\lambda<1$, then $y_\lambda'(x)$ is increasing over $-1<x<-\lambda^{1/2}$, decreasing over $-\lambda^{1/2}$

$<x<\lambda^{1/2}$, and increasing over $\lambda^{1/2}<x<1$, provided (as turns out to be the case) $y_\lambda(x)>0$ over $0<x<1$. The effect on $y_\lambda(x)$ of an increase in λ is not immediately discernible from (4.1), because an increase in λ forces $y_\lambda''(x)$ and hence $y_\lambda(x)$ to increase more rapidly at the start, and the resulting increase in y in (4.1) tends, when $|x|<\lambda^{1/2}$, to make y'' smaller and to make y' decrease more rapidly. To help us with our difficulties, we use a standard device. Supposing $\mu\neq\lambda$, we write

$$y_\lambda'' = -(\lambda - x^2)y_\lambda, \qquad y_\mu'' = -(\mu - x^2)y_\mu. \tag{4.2}$$

Multiplying these equations by y_μ and $-y_\lambda$ respectively and adding gives

$$y_\mu y_\lambda'' - y_\lambda y_\mu'' = (\mu - \lambda)y_\mu y_\lambda. \tag{4.3}$$

The left member of (4.3) is the derivative with respect to x of $y_\mu(x)y_\lambda'(x) - y_\lambda(x)y_\mu'(x)$. Hence, since $y_\mu(-1)=y_\lambda(-1)=0$,

$$y_\mu(x)y_\lambda'(x) - y_\lambda(x)y_\mu'(x) = (\mu - \lambda)\int_{-1}^{x} y_\mu(t)y_\lambda(t)dt. \tag{4.4}$$

When $y_\mu(x)\neq 0$, we can divide by $[y_\mu(x)]^2$ to obtain

$$\frac{d}{dx}\frac{y_\lambda(x)}{y_\mu(x)} = \frac{\mu-\lambda}{[y_\mu(x)]^2}\int_{-1}^{x} y_\mu(t)y_\lambda(t)dt. \tag{4.5}$$

The two formulas (4.4) and (4.5) are excellent sources of information. If $\lambda<\mu$ and $y_\lambda(x)>0$ over an interval $-1<x<x_0$, then (4.5) and the fact that $\lim_{x\to-1}[y_\lambda(x)/y_\mu(x)]=1$ imply that $y_\lambda(x)/y_\mu(x)$ increases from 1 as x increases over $-1<x<x_0$ and hence $y_\lambda(x)>y_\mu(x)$ over $-1<x<x_0$. All eigenvalues are greater than 1 since, as crude numerical calculations or the formula

$$y_1(x) = e^{-1/2}e^{-x^2/2}\int_{-1}^{x} e^{u^2}du \tag{4.6}$$

will show, $y_1(x)>0$ when $x>-1$; and, moreover, $y_\lambda(x)>y_1(x)$ when $0<\lambda<1$ and $x>-1$. If $\lambda\leqq 1$, then, in the interval $-1\leqq x\leqq 1$ where our interest lies, $y_\lambda'(x)$ is decreasing when $y_\lambda(x)$ is positive and increasing when $y_\lambda(x)$ is negative.

Using information derived above, and with repeated appeals to the fact involving rate of change of slope expressed in (4.1), it is quite easy to sketch schematically correct graphs of functions $y_\lambda(x)$ for many values of λ greater than 1. It is very instructive to see how the number of oscillations of $y_\lambda(x)$ about the x axis increases as λ grows, and to see how it happens that, for roughly equally spaced values of λ^2, we have eigenvalues for which $y_\lambda(1)=0$. We cannot give an account of all useful facts made intuitively evident by these graphs. For example, it becomes very clear that values of λ for which $y_\lambda'(1)=0$ (or, more generally, those for which $y_\lambda'(1)=ky_\lambda(1)$ where k is a given constant) must interlace with the values of λ for which $y_\lambda(1)=0$.

A DIFFERENT APPROACH TO THE NON-EUCLIDEAN GEOMETRIES

C. M. FULTON, University of California, Davis

The starting point for our approach to the trigonometry of the non-Euclidean geometries is a simple question. What is the nature of a 3-space in which the ordinary spherical trigonometry holds? By the latter we mean the trigonometry of a trihedral angle imbedded in our space. To formulate our problem more precisely we put it on an axiomatic basis. We take Hilbert's axioms of incidence, order, and congruence [**1**, pp. 25–31] or any equivalent set of axioms. Since we want to leave out the axioms of parallelism later on, we need an axiom of continuity independent of any assumption on parallel lines. We have a choice of Dedekind's axiom [**3**, p. 37] or the axioms of Eudoxus and Cantor [**1**, pp. 41–45]. Upon adding the Euclidean axiom of parallels [**1**, p. 55] the system of axioms for the Euclidean geometry is complete. The spherical trigonometry can now be derived in the usual fashion. If we omit the axiom of parallelism, the other axioms together with the spherical trigonometry still constitute a consistent geometric system. In this paper we investigate the geometries to which this system gives rise.

We state first some geometric facts which are true in the system we are considering and which will be referred to later. The dihedral angle between two planes is measured by the angle between two lines, one in each plane, perpendicular to the line of intersection at a common point. This measure is independent of the point chosen on the line of intersection [**5**, p. 42]. Furthermore there are three theorems of solid geometry which can be shown without reference to parallels [**5**, p. 43]: (I) If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to every line through this point in the plane of the two lines. (II) If a line is perpendicular to a given plane, every plane containing this line is perpendicular to the given plane. (III) Any line which lies in one face of a right dihedral angle and is perpendicular to the edge is perpendicular to the other face.

We now construct a familiar configuration. Let ABC be a right triangle having the right angle at C. Set $BC=a$, $CA=b$, $AB=c$. Consider a segment VA of arbitrary length perpendicular to AB and AC. The line VA is then perpendicular to the plane ABC (I). Also plane ABC is perpendicular to plane VCA (II). Since BC is perpendicular to CA it follows that BC is perpendicular to plane VCA (III). Especially BC is perpendicular to VC. Moreover plane VBC is perpendicular to plane VCA (II). Thus $V-ABC$ is a trihedral angle whose dihedral angle VC is right. The dihedral angle VA is measured by the angle A of triangle ABC. All four triangles in our configuration are right triangles. We finally set $VB=p$, $VC=q$ and for the face angles of the trihedral angle $BVC=a'$, $CVA=b'$, $AVB=c'$. The trihedral angle $V-ABC$ will be used throughout the following discussion.

The ordinary congruence theorems are valid in our geometric system [**1**, pp. 33–38]. Hence two right triangles are congruent if any two sides of one are

respectively equal to the corresponding sides of the other. In other words, if two sides are given the other parts of the right triangle are completely determined. Now the axioms of continuity allow us to measure distances and angles. More exactly we can establish a one-to-one correspondence between certain sets of real numbers and distances or angles respectively [**1**, pp. 42–47]. If then the measures of two sides of a right triangle are given, these sides are determined. So are the remaining parts of the triangle and hence their measures. Especially the measure of an angle or a given function of it will be a certain function of the measures of the given sides.

We apply the last conclusion to the right triangle ABC and set $\sin A = S(a, c)$ where S stands for a certain unknown function. Similarly $\sin a' = S(a, p)$ and $\sin c' = S(c, p)$. From spherical trigonometry we know that $\sin A = \sin a'/\sin c'$. Hence we have $S(a, c) = S(a, p)/S(c, p)$. We can show that in this relation p remains arbitrary. It must be greater than c, however [**1**, p. 38]. After the triangle ABC has been drawn the direction of VA is fixed. The length of VA is found from triangle VAB once p is chosen. Thus our trihedral angle can be constructed selecting p arbitrarily. Because of this we can put $S(a, p) = s(a)$ for some value of p with s representing an unknown function. Hence

$$\sin A = \frac{s(a)}{s(c)} \cdot \tag{1}$$

With arguments analogous to the preceding ones we set $\tan A = T(a, b)$. Using $\tan A = \tan a'/\sin b'$ and (1) we obtain $T(a, b) = T(a, q)s(q)/s(b)$. Again the trihedral angle can be completed selecting q at will. We therefore set $T(a, q)s(q) = t(a)$. Then

$$\tan A = \frac{t(a)}{s(b)} \cdot \tag{2}$$

Making use of $\cos A = \tan b'/\tan c'$ and (2) we find

$$\cos A = \frac{t(b)}{t(c)} \cdot \tag{3}$$

We now equate the right side of (3) to the expression for $\cos A$ derived by dividing (1) and (2). Hence $s(a)s(b)/t(a)t(b) = s(c)/t(c)$. Upon introducing a new unknown function f by means of the defining identity

$$f(a) = \frac{s(a)}{t(a)} \tag{4}$$

this becomes

$$f(a)f(b) = f(c). \tag{5}$$

We note as a simple consequence of (1), (3), (4), (5)

$$\frac{\cos A}{\sin B} = f(a). \tag{6}$$

In order to find one more relation between our unknown functions we substitute (1) and (3) in the identity $\sin^2 A + \cos^2 A = 1$. With the aid of (4) and (5) we can write $s^2(a) + f^2(a)s^2(b) = s^2(c)$. We equate the left side to the expression we obtain from it interchanging a and b. Thus

$$\frac{f^2(a) - 1}{s^2(a)} = \frac{f^2(b) - 1}{s^2(b)} = e. \tag{7}$$

Here we have set the common ratio equal to e. The quantities a and b are the measures of the legs of a right triangle. After a value has been chosen for a the value of b can still be selected arbitrarily. Consequently e is a universal constant for the relation (7) between the unknown functions f and s. We also see that in (1), (2), (3), (4) only ratios of s and t are involved. This makes it permissible to consider s multiplied by a factor such that $|e| = 1$. Since only the square of s occurs in (7) we can also assume s positive. According as the numerators in (7) are positive or negative e will be 1 or -1. There is a third possibility too, namely $e = 0$. In this case s could not be normalized by the above procedure. However, no such necessity will arise.

As the last step in our deduction we establish a functional equation for f. For this purpose we draw in triangle ABC the altitude $CD = z$ which divides the hypotenuse into two segments $AD = x$ and $BD = y$. We set angle $ACD = \theta$ and angle $BCD = \phi$. These angles being complementary we have $\tan \theta \tan \phi = 1$. On using (2) for expressing this fact we see that

$$t(x)t(y) = s^2(z). \tag{8}$$

Applying (5), (7), (8) in succession we conclude that

$$\begin{aligned} f(x + y) &= f(a)f(b) = f^2(z)f(x)f(y) = [1 + es^2(z)]f(x)f(y) \\ &= [1 + et(x)t(y)]f(x)f(y), \end{aligned}$$

or in accordance with (4)

$$f(x + y) = f(x)f(y) + es(x)s(y). \tag{9}$$

For a special case we make ABC an isosceles right triangle and x will be equal to y in (9). It then follows from (7) that

$$f(2x) + 1 = 2f^2(x). \tag{10}$$

We have arrived at the functional equations (9) and (10) by means of the right triangle ABC. But the equations are meaningful only if such a triangle exists for arbitrarily chosen x and y. For a proof let our notations apply to oblique triangles ABC. Because of (6) $\cos \theta / \sin A = f(x)$. Now in our geometric system the angles θ and A are necessarily acute angles [**1**, p. 48]. Hence if we

hold x constant θ is a continuous decreasing function of A. On the other hand it is geometrically obvious that A is a continuous increasing function of z. Thus θ is a continuous decreasing function of z for a fixed x. More generally, $C=\theta+\phi$ will be a continuous decreasing function of z for given values of x and y. Assume for definiteness that x and y are given such that x is greater than y. In this case θ is greater than ϕ. If we make θ equal to one half of a right angle, the right triangle ACD is determined by x and θ [1, p. 38]. In this way we find a value of z corresponding to an angle C less than a right angle. Again, make ϕ equal to one half of a right angle. This time we find a value of z corresponding to an angle C greater than a right angle. Since C is a continuous and decreasing function of z, there will be exactly one z between the two values considered for which C is a right angle. If x and y are equal the two values of z simply coincide. This completes the proof.

We return now to the right triangle ABC. Using (3) we find two expressions for $\cos B$ in triangles BCD and ABC. With a manipulation similar to the preceding we then see that

$$f(c-y)=f(c)f(y)-es(c)s(y). \tag{11}$$

As before c and y may take on arbitrary values with the only restriction that c is greater than y. Addition of (9) and (11) with an obvious change of letters leads to

$$f(x+y)+f(x-y)=2f(x)f(y). \tag{12}$$

Two conditions are usually imposed on the f-function in the functional equations (10) and (12). We show that our unknown function satisfies these conditions. First, f has to be continuous. We know already that A and θ are continuous functions of z. Thus in the relation $\cos A/\sin\theta=f(z)$ due to (6) $f(z)$ is a continuous function of z. Second, $f(x)$ in (10) must be positive. This becomes obvious from $\cos\theta/\sin A=f(x)$, where θ is one half of a right angle for the special case contemplated in (10). Under the stated conditions the solutions of our functional equations are [2, p. 52]

$$\cosh\frac{x}{k},\quad \cos\frac{x}{k},\quad 1$$

with an arbitrary constant k. This constant can be made equal to unity for a suitable choice of the unit of length. The values of e in (7) corresponding to the above solutions are 1, -1, 0. For the first two solutions the functions s and t are found from (7) and (4). They are $\sinh a$ and $\tanh a$, $\sin a$ and $\tan a$, respectively. As a glance at (1), (2), (3), (5), (6) will show, we are dealing with formulas of hyperbolic and elliptic trigonometry. These formulas will more than suffice to build up these trigonometries. The elliptic geometry, however, does not satisfy all the axioms we had to start with. Yet it appears as a solution of our original problem. The explanation is this. Our axioms are true for the elliptic

case, if we restrict ourselves to a limited region of our space. In other words, all the segments involved should not exceed a certain length [**4**, pp. 6, 28, 29]. Clearly all the triangles considered can be made small enough and all conclusions will apply to elliptic geometry. The third solution of our functional equations does not allow us to find s and t in the available equations. But we can infer from (6) that A and B are complementary angles in this case. This is known to be equivalent to the Euclidean axiom of parallels [**3**, p. 46]. Thus the ordinary Euclidean geometry constitutes the third solution of our original problem. It was clear from the beginning of course that it had to be among the solutions.

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THE TREATMENT OF BOUNDARY PROBLEMS BY MATRIX METHODS

MARK LOTKIN, Ballistic Research Laboratories

1. Introduction. Various methods leading to the approximate solution of systems of ordinary linear differential equations with boundary conditions are available; some of the most useful ones are Taylor series expansions, collocation, Galerkin's method, least squares solution, *etc.*

The purpose of this article is two-fold. First it is intended to acquaint the reader with the least squares approach, and, secondly, it is shown how the use of matrix procedures may be employed advantageously to give a more compact description of this least squares method. This method is essentially an application of the Rayleigh-Ritz principle in that the problem to be solved is replaced by a related variational problem for which approximate solutions may be found with relative ease [**1**].

As a by-product it is interesting to note that computations carried out for a number of cases seem to indicate a superiority of the least squares method over the ones mentioned above.

It is a pleasure to acknowledge that certain suggestions made by B. Garfinkel helped materially in the preparation of this paper.

2. The least squares method. Let us consider the linear differential equation of order n

$$L(D)y \equiv \sum_{k=0}^{n} p_k(x)D^{n-k}y = r(x), \tag{1}$$

where $D^k \equiv d^k/dx^k$, $k=0, 1, 2, \cdots, n$, $D^0y=y$, and the p_k, r are continuous functions almost everywhere in some interval I: $x_0 \leqq x < x_1$. We are interested in the solution $y(x)$ of (1) which satisfies boundary conditions that assure the unique existence of this solution. Suppose these conditions are of the form

$$\Lambda(D)y(a) = \rho; \tag{2}$$

here

$$a \equiv \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{bmatrix}$$

is a column of prescribed positions $a_k \epsilon I$, $k=1, 2, \cdots, p$, and

$$y(a) \equiv \begin{bmatrix} y(a_1) \\ y(a_2) \\ \cdot \\ \cdot \\ \cdot \\ y(a_p) \end{bmatrix}$$

Further, $\Lambda(D)$ is a matrix of n rows and p columns, and its elements are polynomials in D of degree not exceeding $n-1$. Finally, ρ is a column of n constants, not all zero.

To obtain an approximate solution of the p-point boundary problem of the type described above we define a sequence $y_s(x)$, $s=1, 2, \cdots$ of approximations to the solutions in the following manner:

$$y_s(x) = \sum_{k=0}^{s} Y_k(x)c_k; \tag{3}$$

here the Y_k are arbitrary functions of class C^{n-1}, with Y_0 satisfying the boundary conditions (2), the Y_k for $k \geqq 1$ satisfying the associated homogeneous boundary conditions, and $c_0=1$, while the c_k, $k \geqq 1$, are unknown constants to be properly adjusted so as to make y_s approximate solutions of (1). By defining

$$Y \equiv \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_s \end{bmatrix} = (Y_i)$$

we may write (3) in the form $y_s = Y_0 + Y^T c$, and express the homogeneous bound-

ary conditions as $\Lambda(D)Y^T(a)=0$, with Y^T denoting the transpose of Y: $Y^T \equiv (Y_1Y_2 \cdots Y_s)=(Y_j)$.

While now the y_s clearly satisfy the conditions (2):

$$\Lambda(D)y_s(a) = \Lambda(D)Y_0(a) + \Lambda(D)Y^T(a)c = \rho,$$

they will not, in general, fulfill the differential equation (1), so that the function

$$\epsilon(x) = L(D)y_s - r \tag{4}$$

will not vanish everywhere in I.

A variety of methods may now be employed in an effort to make $\epsilon(x)$ small in I [2]. What is proposed here is to achieve this by the method of least squares, minimizing, namely, the function

$$U(c) = \int_{x_0}^{x_1} \epsilon^T(t)\epsilon(t)dt. \tag{5}$$

By (4) and (3)

$$\begin{aligned}\epsilon &= (LY^T)c - (r - LY_0)\\ &= Ac - b,\end{aligned}$$

if we put $A(x)=LY^T(x)$, $b(x)=r(x)-LY_0(x)$. It follows that

$$U(c) = c^T\left(\int A^TAdt\right)c - 2c^T\left(\int A^Tbdt\right) + \int b^Tbdt. \tag{6}$$

For a minimum of $U(c)$ it is necessary that

$$\nabla U \equiv \begin{bmatrix} \partial U/\partial c_1 \\ \cdot \\ \cdot \\ \cdot \\ \partial U/\partial c_s \end{bmatrix} = 0.$$

Now for a quadratic form c^TBc

$$\nabla(c^TBc) = (B + B^T)c.$$

Since in our case $B=\int A^TAdt$ is symmetrical,

$$\nabla c^T\left(\int A^TAdt\right)c = 2\left(\int A^TAdt\right)c.$$

For a linear form c^TE

$$\nabla(c^TE) = E.$$

Consequently we obtain the normal equations

$$\left(\int A^TAdt\right)c = \int A^Tbdt,$$

i.e.

$$\left(\int_{x_0}^{x_1} LY(t)\cdot LY^T(t)dt\right)c = \int_{x_0}^{x_1} LY(t)\left[r(t) - LY_0(t)\right]dt. \tag{7}$$

The determination of the vector c is thus reduced to the solution of the system (7) of linear equations with the symmetrical matrix $M = \int LY\cdot LY^T dt$.

Written out in more detail equations (7) look like this:

$$\int_{x_0}^{x_1}\begin{bmatrix} (LY_1)^2 & LY_1\cdot LY_2 & LY_1\cdot LY_3 & \cdot \\ \cdot & (LY_2)^2 & LY_2\cdot LY_3 & \cdot \\ \cdot & \cdot & (LY_3)^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} dt\cdot c = \int_{x_0}^{x_1}\begin{bmatrix} LY_1(r - LY_0) \\ LY_2(r - LY_0) \\ LY_3(r - LY_0) \\ \cdot \end{bmatrix} dt.$$

The case $s+1$ thus differs from the case s only in that the terms due to LY_{s+1} are inserted, without, however, changing the previous terms.

3. Example. Suppose we want to solve the differential equation

$$[D^2 - (1 - 4/x)]y = 2\sinh x \tag{8}$$

in the interval $0 \leqq x \leqq 1$, subjecting the solution $y(x)$ to the boundary conditions

$$\begin{pmatrix} 1 & -D+2 \\ eD & -D+1 \end{pmatrix} y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e \\ 0 \end{pmatrix}. \tag{9}$$

Taking $Y_0 = a_0x + b_0x^2$, we find, by (9) that

$$a_0 = -\tfrac{1}{2}e, \qquad b_0 = -\tfrac{1}{2}.$$

Next we assume the Y_i, $i = 1, 2, \cdots, s$ to be of the form

$$Y_i = a_ix^i + b_ix^{i+1} + x^{i+2}.$$

The homogeneous boundary conditions then show that $a_i = 1$, $b_1 = e-2$, $b_i = -2$ for $i \geqq 2$, so that

$$\begin{aligned} Y_1 &= x[1 + (e-2)x + x^2], \\ Y_i &= x^i(1-x)^2, & i \geqq 2. \end{aligned}$$

It follows that

$$\begin{aligned} LY = {} & i(i-1)x^{i-2} + [i(i+1)b_i + 4]x^{i-1} \\ & + [(i+1)(i+2) - (1 - 4b_i)]x^i + (4 - b_i)x^{i+1} - x^{i+2}, \end{aligned}$$

and LY^T similarly, with the column index j replacing the row index i. Consesequently,

$$LY\cdot LY^T = \sum_{k=0}^{8} d_k x^{i+j+k-4},$$

with d_k denoting certain constants that depend on i, j, b_i, b_j; thus:

$$d_0 = i\cdot j(i-1)(j-1),$$
$$d_1 = j(j-1)[i(i+1)b_i+4]+i(i-1)[j(j+1)b_j+4], \cdots,$$
$$d_7 = b_i+b_j-8,$$
$$d_8 = 1.$$

The elements m_{ij} of M are now easily obtained in the form

$$m_{ij} = \sum_{k=0}^{8} d_k/(i+j+k-3);$$

the result is, for $s=3$,

$$M = \begin{bmatrix} 113.169252 & 2.82366572 & 1.22204903 \\ \cdot & .696825397 & .350000000 \\ \cdot & \cdot & .298845599 \end{bmatrix}.$$

Having calculated the matrix M we next proceed to compute the right-hand side $\int_{x_0}^{x_1} LY(r-Y_0)dt$ of (7); for our example we find

$$\begin{bmatrix} 39.5069050 \\ .950329464 \\ .414999435 \end{bmatrix}.$$

The components of c found as the solution of (7) are shown in the first column of Table 1, accurate to six decimals. The second column of this table contains the values of the c_i as found by Galerkin's Method:

Table 1. Comparison of Solutions. Vector c

c	Least Squares		Galerkin's Method		Exact Solution
	$s=3$	$s=4$	$s=3$	$s=4$	
c_1	.350607	.350606	.350598	.350601	.350606
c_2	−.083317	−.084951	−.082594	−.083449	−.085168
c_3	.052537	.060539	.052006	.055174	.062392
c_4		−.007879		−.002992	−.012271

$$\left(\int_{x_0}^{x_1} Y\cdot LY^T dt\right) c = \int_{x_0}^{x_1} Y(r-LY_0)dt. \tag{10}$$

The last column exhibits the exact values of the c_i; the latter are easily calculated by expanding the exact solution of (8) and (9):

$$y(x) = (xe^x/6) - (x^2e^{-x}/2), \tag{11}$$

and comparing the coefficients of the successive powers of x with the coefficients resulting from the approximate solution y_s, as defined by (3).

A comparison of these solutions indicates the superiority of the least squares method, the convergence to the exact values proceeding faster than in Galerkin's Method.

This superiority becomes even more apparent when the solutions are compared throughout the interval of integration, as shown in Table 2.

Table 2. Comparison of Solutions. Function y

x	0	.2	.4	.6	.8	1.0
Least Squares						
$s=3$	0	.024348	.045817	.083409	.152962	.269109
$s=4$	0	.024339	.045828	.083427	.152954	.269107
Galerkin						
$s=3$	0	.024361	.045835	.083422	.152953	.269083
$s=4$	0	.024353	.045832	.083423	.152953	.269092
Exact Solution	0	.024339	.045829	.083426	.152954	.269107

The square of the residuals, in the least squares procedure, drops from $727\cdot 10^{-6}$, for $s=3$, to $2\cdot 10^{-6}$, for $s=4$; in the Galerkin Method the corresponding numbers are $1113\cdot 10^{-6}$ and $440\cdot 10^{-6}$.

4. Systems of equations. The method described above may be applied, with suitable modifications, to systems of differential equations

$$f(D)y = r, \tag{12}$$

whose solutions are subject to boundary conditions

$$\phi(D)y(a) = \rho. \tag{13}$$

In (12), $f(D)$ is an $m\cdot m$ matrix, $m\geqq 1$, with elements that are polynomials in D of degree $n\geqq 1$, y is a vector of functions $y_1, y_2, \cdots, y_m$. In (13), a again denotes a column of prescribed points $a_1, a_2, \cdots, a_p$, $a_k \epsilon I$. $\phi(D)$ is an $n\cdot ms$ matrix whose elements are polynomials in D of degree not exceeding $n-1$, and $y(a)$ is to be formed as follows:

$$y(a) = \begin{Bmatrix} y(a_1) \\ \cdot \\ \cdot \\ \cdot \\ y(a_p) \end{Bmatrix}, \quad y(a_k) = \begin{Bmatrix} y_1(a_k) \\ \cdot \\ \cdot \\ \cdot \\ y_m(a_k) \end{Bmatrix}, \qquad k = 1, 2, \cdots, p.$$

As before, the sth approximation to the solution may be defined in the form

$$y_s(x) = Y_0(x) + Y^T(x)c. \tag{14}$$

Here

$$y_s = \begin{Bmatrix} y_{1s} \\ \cdot \\ \cdot \\ \cdot \\ y_{ms} \end{Bmatrix}, \qquad Y_0 = \begin{Bmatrix} Y_{10} \\ \cdot \\ \cdot \\ \cdot \\ Y_{m0} \end{Bmatrix},$$

and the independent Y_{i0} must be chosen such that

$$\phi(D)Y_0(a) = \rho; \tag{15}$$

further, the $m \cdot ms$ matrix Y^T must be constructed in the following manner: Let

$$Y_i = \begin{Bmatrix} Y_{i1} \\ \cdot \\ \cdot \\ \cdot \\ Y_{is} \end{Bmatrix}, \qquad i = 1, 2, \cdots, m,$$

have independent components $Y_{ij}(x)$, and let

$$Y = \begin{Bmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Y_m \end{Bmatrix} \tag{16}$$

denote the diagonal matrix of ms rows and m columns. Then

$$Y^T = \begin{Bmatrix} \overline{Y}_1 & & & \\ & \overline{Y}_2 & & \\ & & \ddots & \\ & & & \overline{Y}_m \end{Bmatrix},$$

with $\overline{Y}_1 = (Y_{i1}, Y_{i2}, \cdots, Y_{is})$, $i = 1, 2, \cdots, m$.

The Y_{ij} must be selected in such a manner that Y^T satisfies the n homogeneous boundary conditions

$$\phi(D)Y^T(a) = \phi(D) \begin{Bmatrix} \overline{Y}_1(a_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{Y}_m(a_1) \\ \cdots & \cdots & \cdots \\ \overline{Y}_1(a_p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{Y}_m(a_p) \end{Bmatrix} = 0. \tag{17}$$

This prescription will again assure the satisfaction of the conditions (13) by the approximating functions y_s.

The vector c, in (14), is composed of m vectors $c^{(i)}$ each having s components $c_j^{(i)}$:

$$c = \begin{vmatrix} c^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ c^{(m)} \end{vmatrix}, \qquad c^{(i)} = \begin{vmatrix} c_1^{(i)} \\ \cdot \\ \cdot \\ \cdot \\ c_s^{(i)} \end{vmatrix}, \qquad i = 1, 2, \cdots, m.$$

Minimizing the residuals $\epsilon = \phi(D)y_s - r$ by the method of least squares now leads to the normal equations

$$\left(\int_{x_0}^{x_1} fY \cdot fY^T dt\right) c = \int_{x_0}^{x_1} fY(r - fY_0)dt. \tag{18}$$

Care must be exercised to carry out the required matrix operation in the proper manner. Thus

$$fY = \begin{vmatrix} f_{11} & \cdots & f_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_{m1} & \cdots & f_{mm} \end{vmatrix} \begin{vmatrix} Y_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & Y_m \end{vmatrix} = \begin{vmatrix} f_{11}Y_1 & \cdots & f_{1m}Y_m \\ \cdot & & \\ \cdot & & \\ f_{m1}Y_1 & \cdots & f_{mm}Y_m \end{vmatrix},$$

and similarly for $f\overline{Y}$. Since fY is an $ms \cdot m$ matrix, $f\overline{Y}$ an $m \cdot ms$ matrix, $M = \int_{x_0}^{x_1} fYfY^T dt$ is a symmetric matrix of ms rows.

5. Example of a system. Let us apply this method to the system

$$\begin{pmatrix} D^2 & x^2D + 3x \\ 2 & x^3D^2 + x^3 - 2x \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{19}$$

with the boundary conditions

$$\begin{pmatrix} D & -1 & 0 & 0 \\ 1 & 0 & -2D/\pi & 0 \end{pmatrix} \begin{vmatrix} y_1(0) \\ y_2(0) \\ y_1(\pi/2) \\ y_2(\pi/2) \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{20}$$

Here $m = n = p = 2$. Suppose we take

$$Y_{10} = x - (x^3/2) + bx^5, \qquad Y_{20} = 1 - (x^2/6) + (x^4/120),$$

with b to be determined such that (20) is fulfilled. We find

$$b = 2(3\pi^2 - 4\pi - 8)/5\pi^4.$$

Next we choose Y_1, Y_2 to be of the form

$$Y_{1j} = x^{j+1} + b_j x^{j+2}$$
$$Y_{2j} = x^j, \qquad \text{for } j = 1, 2, 3, 4,$$

taking $s = 4$. The b_j must be computed from conditions (17); these lead to

$$b_j = -(2/\pi)(j + 1)/(j + 2).$$

It is seen that the normal equations (18), for the case (19), (20), have the solution

$$c^{(1)} = \begin{vmatrix} -0.00107 \\ 0.00670 \\ -0.01361 \\ 0.01500 \end{vmatrix}, \qquad c^{(2)} = \begin{vmatrix} 0.00593 \\ -0.01967 \\ 0.01859 \\ -0.01078 \end{vmatrix}.$$

The exact values are $c_4^{(1)} = 0.00453$, all other $c_j^{(i)} = 0$, $i = 1, 2$; $j = 1, \cdots, 4$, as may be verified by expanding the exact solution

$$y(x) = \begin{pmatrix} x \cos x \\ (1/x) \sin x \end{pmatrix}$$

in Taylor series about $x = 0$. Table 3 contains again a comparison of the approximating function with the exact solution, in the basic interval $(0, \pi/2)$.

Table 3. Comparison of Solutions

	0	$\pi/16$	$\pi/8$	$3\pi/16$	$\pi/4$	$3\pi/8$	$\pi/2$
Exact Solution	0	.1926	.3628	.4898	0.5554	0.4508	0
	1	.9936	.9745	.9432	0.9003	0.7842	0.6366
Approximate Solution	0	.1926	.3628	.4901	0.5572	0.4724	0.1205
	1	.9941	.9747	.9423	0.8978	0.7740	0.6067

References

1. For a description of this principle see, for example, R. Courant, Variational Methods for the Solution of Problems of Equilibrium and Vibrations, Bulletin, A. M. S., vol. 49, 1943.

2. Such methods are the Collocation Method, Galerkin's Method, etc., described in Frazer, Duncan, Collar, Elementary Matrices.

A NEW METHOD OF EVALUATING $\zeta(2n)$

G. T. WILLIAMS, Ballistic Research Laboratories

All the extant determinations of $\sum_{\nu=1}^{\infty} (1/\nu^{2n})$ turn on arguments quite remote from the series itself, which generally makes its adventitious appearance in the next to the last line of the proof. For instance, Riemann's functional equation is obtained by transforming the series into a definite integral, which is in turn transformed into a contour integral, which is transformed back into a series (of residue terms) which is luckily of the same form as the original one; or,

alternatively, the function $1/(e^z-1)-1/z$ is expanded about its symmetrically-paired poles, the summand is written as an infinite series and the order of summation is inverted, yielding a power-series, whose coefficients, which turn out to be $\zeta(2n)$, are equated with those of the generating function, these being assumed to be known (or, at least, knowable) constants.

It is consequently hoped that the following approach, which is wholly elementary and deals only with manipulations of the defining series, may be of intrinsic as well as didactic interest. Except for digressions in (8) and (12), where the passage to the limit is justified, the argument moves ahead under its own momentum, once the initial steps are taken. The convolutions obtained (Theorems I and II) seem never to have been explicitly formulated before, and Theorem III is apparently entirely new.

THEOREM I:

$$\zeta(2)\zeta(2n-2)+\zeta(4)\zeta(2n-4)+\cdots+\zeta(2n-2)\zeta(2)=(n+\tfrac{1}{2})\zeta(2n)$$
$$(n=2,3,4,\cdots).$$

The left-hand side, written out at length, is the limit, as $N\to\infty$, of

$$\sum_{\nu=1}^{N}\sum_{\mu=1}^{N}\left\{\frac{1}{\mu^2}\cdot\frac{1}{\nu^{2n-2}}+\frac{1}{\mu^4}\cdot\frac{1}{\nu^{2n-4}}+\cdots+\frac{1}{\mu^{2n-2}}\cdot\frac{1}{\nu^2}\right\} \tag{1}$$

(N a positive integer). Summing the geometric series within braces, and taking note of the exceptional case $\mu=\nu$, this becomes

$$\sum_{\nu}\left\{\sum_{\mu}{}'\frac{\nu^{2-2n}-\mu^{2-2n}}{\mu^2-\nu^2}+(n-1)\frac{1}{\nu^{2n}}\right\}. \tag{2}$$

Throughout the present discussion, all sums run from 1 to N, unless otherwise indicated, and an accent on an inner $\sum$ indicates that the index (in this case, μ) does not take on the value of the index of the outer sum (ν, here). Ignoring the term on the far right, (2) is equal to

$$\sum_{\nu}\sum_{\mu}{}'\frac{\nu^{2-2n}}{\mu^2-\nu^2}+\sum_{\nu}\sum_{\mu}{}'\frac{\mu^{2-2n}}{\nu^2-\mu^2}. \tag{3}$$

Inverting the order of summation in the second double sum, and noting that the condition $\mu\neq\nu$ is the same as the condition $\nu\neq\mu$, (3) may be written as

$$\sum_{\nu}\frac{1}{\nu^{2n-2}}\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2}+\sum_{\mu}\frac{1}{\mu^{2n-2}}\sum_{\nu}{}'\frac{1}{\nu^2-\mu^2}=2\sum_{\nu}\frac{1}{\nu^{2n-2}}\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2}, \tag{4}$$

the latter form arising out of an interchange of the dummy indices in the second sum. Combining (1)–(4), we find that

$$\sum_\nu \sum_\mu \left\{\frac{1}{\mu^2}\cdot\frac{1}{\nu^{2n-2}} + \cdots + \frac{1}{\mu^{2n-2}}\cdot\frac{1}{\nu^2}\right\} = (n-1)\sum_\nu \frac{1}{\nu^{2n}} + 2\sum_\nu \frac{1}{\nu^{2n-2}}\sum_\mu{}' \frac{1}{\mu^2-\nu^2}. \tag{5}$$

Now,

$$\begin{aligned} 2\nu\sum_\mu{}'\frac{1}{\mu^2-\nu^2} &= \sum_\mu{}'\frac{1}{\mu-\nu} - \sum_\mu{}'\frac{1}{\mu+\nu} \\ &= \sum_{\mu=1}^{\nu-1}\frac{1}{\mu-\nu} + \sum_{\mu=\nu+1}^{N}\frac{1}{\mu-\nu} - \sum_{\mu=1}^{N}\frac{1}{\mu+\nu} + \frac{1}{2\nu} \\ &= -\sum_{\mu=1}^{\nu-1}\frac{1}{\mu} + \sum_{\mu=1}^{N-\nu}\frac{1}{\mu} - \sum_{\mu=\nu+1}^{N+\nu}\frac{1}{\mu} + \frac{1}{2\nu} \\ &= -\sum_{\mu=1}^{N+\nu}\frac{1}{\mu} + \frac{1}{\nu} + \sum_{\mu=1}^{N-\nu}\frac{1}{\mu} + \frac{1}{2\nu} \\ &= \frac{3}{2\nu} - \left\{\frac{1}{N-\nu+1} + \frac{1}{N-\nu+2} + \cdots + \frac{1}{N+\nu}\right\}. \end{aligned}$$

When we substitute this into (5), we get

$$\begin{aligned} &\sum_\mu \frac{1}{\mu^2}\sum_\nu \frac{1}{\nu^{2n-2}} + \cdots + \sum_\mu \frac{1}{\mu^{2n-1}}\sum_\nu \frac{1}{\nu^2} = \\ &(n+\tfrac{1}{2})\sum_\nu \frac{1}{\nu^{2n}} - \sum_\nu \frac{1}{\nu^{2n-2}}\left\{\frac{1}{N-\nu+1} + \frac{1}{N-\nu+2} + \cdots + \frac{1}{N+\nu}\right\}. \end{aligned} \tag{7}$$

Finally,

$$0 < \frac{1}{N-\nu+1} + \frac{1}{N-\nu+2} + \cdots + \frac{1}{N+\nu} < \frac{2\nu}{N-\nu+1},$$

and so

$$\begin{aligned} 0 &< \sum_\nu \frac{1}{\nu^{2n-1}}\left\{\frac{1}{N-\nu+1} + \cdots + \frac{1}{N+\nu}\right\} \\ &< 2\sum_\nu \frac{1}{\nu^{2n-2}}\cdot\frac{1}{N-\nu+1} \leqq 2\sum \frac{1}{\nu(N-\nu+1)} \qquad \left(n \geqq \frac{3}{2}\right) \\ &= \frac{2}{N+1}\sum\left\{\frac{1}{\nu} + \frac{1}{N-\nu+1}\right\} = \frac{4}{N+1}\sum \frac{1}{\nu} \\ &< \frac{4}{N+1}(1+\log N) \to 0 \qquad (N\to\infty). \end{aligned} \tag{8}$$

Statements (7) and (8), taken together, complete the proof.

By successive applications of the convolution just established, one may now express $\zeta(2n)$ as a rational multiple of $\{\zeta(2)\}^n$. The theorem tells us nothing, however, about $\zeta(2)$, itself. We can circumvent this difficulty by considering an allied function,

$$\xi(s) \equiv \sum_{\nu=0}^{\infty} (-)^{\nu} \frac{1}{(2\nu+1)^s} \qquad (s > 0).$$

It, too, has a convolution, given by

THEOREM II:

$$\xi(1)\xi(2n-1) + \xi(3)\xi(2n-3) + \cdots + \xi(2n-1)\xi(1)$$
$$= \left(n - \frac{1}{2}\right) \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^{2n}} \qquad (n = 1, 2, 3, \cdots).$$

Now, it is a well-known fact that

$$\xi(1) \equiv 1 - \frac{1}{3} + \frac{1}{5} - + \cdots = \arctan 1 = \frac{\pi}{4} \cdot$$

Hence, taking $n=1$ in the above theorem, we find

$$\sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^2} = 2\{\xi(1)\}^2 = \frac{\pi^2}{8} \cdot$$

Coupling this with the transparent identity,

$$\sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^s} = (1 - 2^{-s})\zeta(s) \qquad (s > 1),$$

the familiar value, $\zeta(2) = \pi^2/6$, emerges at once.

The proof of Theorem II proceeds along the same lines as that of Theorem I, though the details are a bit more intricate. In the first place, we have, as in (1)–(5) *supra*,

$$\sum_{\nu}\sum_{\mu} (-)^{\nu+\mu} \left\{ \frac{1}{2\mu+1} \cdot \frac{1}{(2\nu+1)^{2n-1}} + \cdots + \frac{1}{(2\mu+1)^{2n-1}} \cdot \frac{1}{2\nu+1} \right\}$$
$$(9) \qquad = n \sum_{\nu} \frac{1}{(2\nu+1)^{2n}} + 2 \sum_{\nu} \frac{1}{(2\nu+1)^{2n-1}} \sum_{\mu}{}' (-)^{\nu+\mu} \frac{2\mu+1}{(2\mu+1)^2 - (2\nu+1)^2} .$$

Here, all sums run from 0 to N, and the accent has the same significance as before. Again,

$$4\sum_{\mu}{}' (-)^{\nu+\mu} \frac{2\mu+1}{(2\mu+1)^2-(2\nu+1)^2}$$

$$(10) \quad = \sum_{\mu}{}' (-)^{\nu+\mu} \frac{2\mu+1}{(\mu+\nu+1)(\mu-\nu)} = \sum_{\mu}{}' (-)^{\nu+\mu} \frac{1}{\mu-\nu} + \sum_{\mu}{}' (-)^{\nu+\mu} \frac{1}{\mu+\nu+1}$$

$$= -\frac{1}{2\nu+1} + (-)^{N-\nu} \left\{\frac{1}{N-\nu+1} - \frac{1}{N-\nu+2} + - \cdots + \frac{1}{N+\nu+1}\right\},$$

the last step involving reasoning analogous to (6). Equations (9) and (10) now yield

$$\sum_{\mu} (-)^{\mu} \frac{1}{2\mu+1} \sum_{\nu} (-)^{\nu} \frac{1}{(2\nu+1)^{2n-1}} + \cdots + \sum_{\mu} (-)^{\mu} \frac{1}{(2\mu+1)^{2n-1}} \sum_{\nu} (-)^{\nu} \frac{1}{2\nu+1}$$

$$(11) \quad = \left(n-\frac{1}{2}\right) \sum_{\nu} \frac{1}{(2\nu+1)^{2n}}$$

$$+ \frac{1}{2} \sum_{\nu} (-)^{N-\nu} \frac{1}{(2\nu+1)^{2n-1}} \left\{\frac{1}{N-\nu+1} - \frac{1}{N-\nu+2} + - \cdots + \frac{1}{N+\nu+1}\right\}.$$

But,

$$0 < \frac{1}{N-\nu+1} - \frac{1}{N-\nu+2} + - \cdots + \frac{1}{N+\nu+1} \leqq \frac{1}{N-\nu+1},$$

and so the absolute value of the last sum in (11) is

$$\leqq \sum_{\nu} \frac{1}{(2\nu+1)^{2n-1}} \cdot \frac{1}{N-\nu+1} \leqq \sum_{\nu} \frac{1}{(2\nu+1)(N-\nu+1)} \qquad (n \geqq 1)$$

$$(12) \quad = \frac{2}{2N+3} \sum_{\nu} \left\{\frac{1}{2\nu+1} + \frac{1}{2(N-\nu+1)}\right\}$$

$$= \frac{2}{2N+3} \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2N+1} + \frac{1}{2N+2}\right\} \to 0 \ (N \to \infty),$$

as with (8), above. From (11) and (12), the proof of Theorem II follows at once.

As we have already seen, the first convolution serves to give $\zeta(2n)$, recursively, as a rational multiple of $\{\zeta(2)\}^n$, which is $\{\pi^2/6\}^n$. Now, it may be shown (by, *e.g.*, one of the methods alluded to in our opening paragraph) that

$$(13a) \qquad \zeta(2n) = (-)^{n-1} \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!},$$

where these so-called Bernoulli numbers are the coefficients in the power-series expansion of

(13b) $$\frac{z}{e^z - 1} \equiv \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}.$$

($N.B.$, $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, $B_4=-\frac{1}{30}$, $B_5=0$, $B_6=\frac{1}{42}$, and so on.) If there were any simple, closed expression for these numbers, (13) would represent an indisputable improvement over Theorem I; but the fact is that they themselves are most readily computed by means of recurrence relationships also. For example, (13b) may be stated symbolically as

(13c) $$\frac{z}{e^z - 1} = e^{Bz}$$

whence,

$$ze^{-z} = e^{Bz} - e^{(B-1)z},$$

and, equating coefficients of like powers of z,

(14) $$(B - 1)^n = B_n + (-)^n n.$$

Let us see now how our results fit in with this Bernoulli-number representation. Substituting from (13a), Theorem I becomes, after a little algebra,

(15) $$\sum_{\nu=1}^{n-1} \binom{2n}{2\nu} B_{2\nu} B_{2n-2\nu} = -(2n + 1)B_{2n}.$$

From this point of view, then, the convolution is but a disguised recurrence equation for Bernoulli numbers. There is, of course, nothing to stop us from proving (15) directly from (13c), as follows. Differentiating,

$$\frac{1}{e^z-1} - \frac{ze^z}{(e^z-1)^2} = Be^{Bz},$$

$$\begin{aligned}
\frac{z^2}{(e^z - 1)^2} &= \frac{ze^{-z}}{e^z - 1} - Bze^{(B-1)z}, \\
e^{(B+B)z} &= e^{(B-1)z} - Bze^{(B-1)z}, \\
(B + B)^n &= (B - 1)^n - Bn(B - 1)^{n-1} \\
&= (B - 1)^n - n(B - 1)(B - 1)^{n-1} - n(B - 1)^{n-1} \\
&= -(n - 1)(B - 1)^n - n(B - 1)^{n-1},
\end{aligned}$$

and, by virtue of (14), we therefore have

(16) $$-(B + B)^n = (n - 1)B_n + nB_{n-1}.$$

When n is odd and greater than 1, this is trivial, since the right-hand side becomes simply nB_{n-1} and on the left all the terms in the binomial expansion vanish except the first ones in, from both ends, $i.e.$,

$$-\ (nB_1B_{n-1}+nB_{n-1}B_1).$$

When n is even, (16) is equivalent to (15). A similar reduction may be made of Theorem II by substituting for $\xi(2n+1)$; the latter is a rational multiple of π^{2n+1}, the coefficient involving an "Euler number."

In order to escape the imputation of producing nothing but new proofs of old facts, we conclude with a fresh result which flows from the method of this paper. By considering the convolution

$$\sum\sum\left\{\frac{1}{\mu^2}\cdot\frac{1}{\nu^{n-1}}+\frac{1}{\mu^3}\cdot\frac{1}{\nu^{n-2}}+\cdots+\frac{1}{\mu^{n-1}}\cdot\frac{1}{\nu^2}\right\},$$

one proves easily that

THEOREM III:

$$2\sum_{\nu=1}^{\infty}\frac{1}{\nu^n}\left(1+\frac{1}{2}+\cdots+\frac{1}{\nu-1}\right)$$
$$=n\zeta(n+1)-\{\zeta(2)\zeta(n-1)+\zeta(3)\zeta(n-2)+\cdots+\zeta(n-1)\zeta(2)\},$$
$$(n=2,3,4,\cdots).$$

It was his efforts to find a closed form for the left side of the foregoing that led the author to the present considerations.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

POWER SERIES AND ALGEBRAIC NUMBERS

R. M. REDHEFFER, University of California, Los Angeles

Introduction. The Lindemann theorem states that $ae^{\alpha}+be^{\beta}+\cdots+se^{\sigma}$ cannot vanish if $a, b, \cdots, s$ are algebraic numbers not all zero, and $\alpha, \beta, \cdots, \sigma$ are algebraic numbers no two of which are equal. In a recent paper [1] Dietrich and Rosenthal show how the Lindemann theorem can be used to prove the transcendence of certain power series:

THEOREM I (*Dietrich and Rosenthal*). *If $a_0, a_1, a_2, \cdots$ is a sequence of algebraic numbers, ultimately periodic but not ultimately all zero, then the value of the*

function $g(z)=\sum a_n z^n/n!$ *is transcendental when* z *has an algebraic value other than zero.*

A similar use of the Lindemann theorem is made in [2].

The theorem. Now, the full force of the Lindemann theorem is needed to prove Theorem I, and yet it seems that the Lindemann theorem cannot be deduced from Theorem I. Thus, Theorem I and the Lindemann theorem are not equivalent. Our purpose here is to prove a more general result along the same lines which, as we shall show, is equivalent to the Lindemann theorem:

THEOREM II. *Suppose* $f(z)=\sum a_n z^n$ *is rational, is regular at* $z=0$, *has algebraic coefficients, and is not a polynomial. Let* m *be the multiplicity of that root in the denominator of* f *which has maximum multiplicity. Then* (*apart from a set of at most* m *values of* z) *the value of the function* $g(z)=\sum a_n z^n/n!$ *is transcendental when the value of* z *is algebraic.*

In Theorem II we have replaced the periodicity condition of I by a requirement that the a_n's satisfy a linear recurrence relation. This rather minor alteration makes the result fit into the following more general context: Let $f(z)=\sum a_n z^n$, $h(z)=\sum b_n z^n$, and $g(z)=\sum a_n b_n z^n$, all with algebraic coefficients. If $f(z)$ is algebraic for algebraic values of z, and $h(z)$ is transcendental for algebraic $z\neq 0$, can anything be said about $g(z)$ for algebraic z? How must f and h be restricted to get a theorem in that direction? *Our result answers this question for* $f(z)$ *rational,* $h(z)=e^z$. The fact that this extremely special case is equivalent to the Lindemann theorem gives an idea of the depth of the general problem.

Proof. The proof of Theorem II has no mathematical depth, but it unites differential equations, difference equations, power series, and algebraic numbers in an entertaining manner.—To begin with, let $f(z)$ have denominator $z^p+bz^{p-1}+\cdots+rz+s$. Then multiplying and equating coefficients, as in [3], we see that $a_n+ba_{n+1}+\cdots+sa_{n+p}=0$ for all large n. Considering the derivatives of $g(z)$, as in [4], we find that $g(z)$ satisfies a differential equation $g^{(q)}+bg^{(q+1)}+\cdots+sg^{(q+p)}=0$ for some fixed q. Now, this is a linear differential equation with constant coefficients. Hence by the usual method of solution (which, one can prove, gives really *all* the solutions) we can write

$$g(z) = A(z) + B(z)e^{\beta z} + \cdots + S(z)e^{\sigma z} \tag{1}$$

where $\beta,\cdots,\sigma$ are the distinct roots of

$$1 + bz + \cdots + sz^p = 0 \tag{2}$$

and $A, B,\cdots, S$ are polynomials. The degree of $B,\cdots, S$ is one less than the multiplicity of the corresponding root.

As we see by the method given in [5], the algebraic character of the a_n's ensures that of $b,\cdots, s$, and hence that of $\beta,\cdots,\sigma$. Since the polynomials $A, B,\cdots, S$ are found from the initial conditions $g^{(k)}(0)$, they must have alge-

braic coefficients, and since $g(z)$ is not a polynomial by hypothesis, at least one number $\beta, \cdots, \sigma$ and the corresponding polynomial $A, B, \cdots, S$ are not zero. The Lindemann theorem applied to the equation $g(z)=w$ for algebraic $z\neq 0$ and w shows now that all of the coefficients $B, C, \cdots, S$ must vanish at this value of z. Since each of these polynomials has degree at most $m-1$, that situation can arise for at most $m-1$ points. Taking account of the case $z=0$, which is always exceptional, we obtain Theorem II.

Discussion. The fact that certain polynomial coefficients in (1) may be identically zero shows that *the m exceptional cases can actually occur.* If the coefficients ultimately form a periodic sequence, however, then the difference relation becomes simply $a_{n+p}=a_n$, so that (2) is $z^p=1$. Thus the exponents $\beta, \cdots, \sigma$ in (1) are the pth roots of unity, the polynomials $B, \cdots, S$ are constants, and $z=0$ is the only exceptional value. Alternatively, it is easy to compute $f(z)$ in this case, and the denominator is found to have no multiple roots; so $m=1$. Hence *Theorem II contains Theorem I.*

Again, the Lindemann theorem follows from Theorem II when we choose $f(z)=a/(1-\alpha z)+b/(1-\beta z)+\cdots+s/(1-\sigma z)$. Thus *Theorem II amounts to a rewording of the Lindemann theorem,* and an independent proof of the former would give the latter.

The theorem was suggested by [1], the proof by [3]–[5] and by conversation with Ernst Straus.

References

1. Dietrich and Rosenthal, Transcendence of Factorial Series with Algebraic Coefficients, Bull. Am. Math. Soc., v. 55, 1949, p. 55.
2. LeVeque, W. J., Note on the Transcendence of Certain Series, Proc. Am. Math. Soc., v. 2, 1951, No. 3, p. 401.
3. Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. II, Dover Publications, N. Y., Ch. VII, Prob. 17.
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ON A CLASS OF IRRATIONAL NUMBERS

M. R. Spiegel, Rensselaer Polytechnic Institute

The classical method of proving that e is an irrational number is well-known. The following theorem generalizes the approach in the classical method.

Theorem

1. *Let a_n, $n=1, 2, \cdots$ be positive or negative integers or zero, provided only that there be an infinite number of a_n not equal to zero.*

2. *Let a number S be defined by the series*

$$S=\sum_{n=1}^{\infty}\frac{a_n}{r^n(n!)^b}$$

where r and b are any positive integers.

3. *Suppose that a positive constant C and a positive constant $\alpha < b$ exist, independent of n, such that*

$$|a_n| < Cn^\alpha$$

for all n greater than some number N. Then the number S is irrational.

Proof. Suppose the number S represented by the series is rational and equal to p/q where p and q are taken to be relatively prime integers. Thus

$$\frac{p}{q} = \frac{a_1}{r(1!)^b} + \frac{a_2}{r^2(2!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq}[(Mq)!]^b} + \cdots .$$

Here M is an integer which we shall assume is larger than the number N described in 3.

Multiplying both sides by $r^{Mq}[(Mq)!]^b$ we obtain

$$F = G + H,$$

where

$$F = \frac{p}{q} r^{Mq}[(Mq)!]^b$$

$$G = \left\{\frac{a_1}{r(1!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq}[(Mq)!]^b}\right\} r^{Mq}[(Mq)!]^b$$

$$H = \frac{a_{Mq+1}}{r(Mq+1)^b} + \frac{a_{Mq+2}}{r^2(Mq+2)^b(Mq+1)^b} + \cdots .$$

Quantities F and G are integers. However

$$\begin{aligned} |H| &\leqq \frac{C}{r}\frac{(Mq+1)^\alpha}{(Mq+1)^b} + \frac{C}{r^2}\frac{(Mq+2)^\alpha}{(Mq+1)^b(Mq+2)^b} + \cdots \\ &< \frac{C}{r}\left[\frac{1}{(Mq+1)^{b-\alpha}} + \frac{1}{r(Mq+1)^{2b-\alpha}} + \cdots\right] \\ &\leqq \frac{C}{r}\frac{(Mq+1)^\alpha}{(Mq+1)^b - 1}, \end{aligned}$$

which can be made less than one by choosing M large enough. Hence our supposition that the series represents a rational number is contradicted and so the theorem is proved.

Remark. The above theorem may be used to prove that sin 1, cos 1, $J_0(1)$ among many others, as well as e, are irrational numbers.

A BRIEF PROOF OF A THEOREM ON T-TRANSFORMATIONS

C. F. MARTIN, London, England

The transformation of the sequence $\{z_n\}$ by the matrix $(a_{n,k})$ into the sequence $\{z'_n\} \equiv \{ \sum_{k=1}^{\infty} a_{n,k} z_k \}$ is called a T-transformation (and the matrix a T-matrix)* if it has the property that the transform of every convergent sequence also converges, and to the same limit. The well-known necessary and sufficient conditions for this are:†

$$\sum_{k=1}^{\infty} |a_{n,k}| < M \qquad \text{for } n > n_0 \text{ (fixed)}, \tag{1}$$

$$\lim_{n\to\infty} a_{n,k} = 0 \qquad \text{for each fixed } k, \tag{2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1. \tag{3}$$

The following theorem was first published by P. Dienes,‡ treating separately the case when L and U are (i) both finite, (ii) one infinite and (iii) both infinite. However, his treatment of the cases of infinite limit points was inadequate, and a proof of these cases was given by P. Vermes.§

The brief method given below treats all the cases simultaneously. The reader will notice its similarity to a construction of Agnew.||

THEOREM. *Let $\{x_k\}$ be a real sequence with (finite or infinite) upper and lower limits U and L, and let $L \leqq x \leqq U$. Then there is a non-negative T-transformation $(a_{i,j})$ such that* $\lim_{n\to\infty} x'_n \equiv \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k = x$ *(finite or infinite).*

If $L = U$ (finite or infinite), $x = L = U$ is the only limit point of the sequence, so that the identical transformation will serve.

Otherwise, let $\{y_r\}$ be a sequence such that $\lim_{r\to\infty} y_r = x$ and $L < y_r < U$.

Then, since L and U are limit points of $\{x_k\}$, there exists a sequence $\{x_{k_p}\}$ of points of $\{x_k\}$ such that $x_{k_{2r-1}} < y_r < x_{k_{2r}}$ $(r = 1, 2, \cdots)$.

The matrix $(a_{i,j})$ given by

$$a_{i,j} = \begin{cases} \dfrac{x_{k_{2i}} - y_i}{x_{k_{2i}} - x_{k_{2i-1}}} & \text{when } j = k_{2i-1}, \\[2ex] \dfrac{y_i - x_{k_{2i-1}}}{x_{k_{2i}} - x_{k_{2i-1}}} & \text{when } j = k_{2i}, \\[2ex] 0 & \text{otherwise,} \end{cases}$$

* In America such a matrix is called "regular."

† Cooke, (1), 64.

‡ Dienes, (2), 391–2.

§ See Cooke, (1), 77–79.

|| See Cooke, (1), 80.

is a non-negative T-matrix, and transforms the sequence $\{x_k\}$ into the sequence $\{y_r\}$ whose limit is x.

Thus any point in the closed interval $[L, U]$ is the generalized limit of the sequence by some non-negative T-matrix. On the other hand, no non-negative T-matrix can give a generalized limit outside this interval; in fact for every non-negative T-matrix, $L \leqq \underline{\lim}_{n\to\infty} x'_n \leqq \overline{\lim}_{n\to\infty} x'_n \leqq U$, by Knopp's core theorem.†

References

1. R. G. Cooke, *Infinite Matrices and Sequence Spaces* (Macmillan), 1950.
2. P. Dienes, *The Taylor Series* (Oxford), 1931.

† Cooke, (1), 138, (6.1, I).

CURVES ENCIRCLING A CYLINDER

J. W. GREEN, University of California, Los Angeles

Let C be a curve encircling a right circular cylinder of unit radius in such a manner that each element of the cylinder contains a single point of the curve. We further require that C have diameter 2, where by diameter we mean the least upper bound of the distance between points of C. It may not be immediately apparent that there exist curves other than the circular right cross-sections which have diameter equal to 2, but actually there is a considerable class of such curves, as appears shortly. It will also appear that C is rectifiable, and we denote its length by L. By the height H of C we mean the minimum distance between two planes perpendicular to the cylinder and containing C between them. We prove

$$2\pi \leqq L \leqq 2\sqrt{2}\,\pi, \tag{1}$$

$$0 \leqq H \leqq \sqrt{2}, \tag{2}$$

and the bounds given are the best possible ones.

In cylindrical coordinates, C may be described by the equations $r=1$, $z=f(\theta)$. If $(1, \theta, f(\theta))$ and $(1, \phi, f(\phi))$ are points of C, we require that $[f(\theta)-f(\phi)]^2 + [2-2\cos(\theta-\phi)] \leqq 4$, or

$$\left| f(\theta) - f(\phi) \right| \leqq 2 \left| \cos \tfrac{1}{2}(\theta - \phi) \right|. \tag{3}$$

We note that $f(\theta)-f(\theta+\pi)=0$; that is, f has period π. Setting $\phi=\alpha+\pi$, we have from (3).

$$\left| f(\theta) - f(\alpha) \right| \leqq 2 \left| \sin \tfrac{1}{2}(\theta - \alpha) \right|. \tag{4}$$

The equation (4) and the requirement that f have period π are sufficient as well as necessary conditions that C have diameter 2, since the steps by which

(4) was obtained can be reversed. From (4) we have for every θ, α,

$$\left|\frac{f(\theta) - f(\alpha)}{\theta - \alpha}\right| \leqq \left|\frac{\sin \frac{1}{2}(\theta - \alpha)}{\frac{1}{2}(\theta - \alpha)}\right| < 1.$$

Thus f is continuous, has its derivative quotients bounded by 1, and is in fact absolutely continuous, with $|f'| \leqq 1$ almost everywhere. Thus

$$2\pi = \int_0^{2\pi} d\theta \leqq L = \int_0^{2\pi} \sqrt{1 + (f'(\theta))^2}\, d\theta \leqq \int_0^{2\pi} \sqrt{2}\, d\theta = 2\sqrt{2}\,\pi,$$

and (1) is proved.

The lower bound of L is obviously attainable. To show that the upper bound given is the best possible, we construct an example by piecing together segments of the function $2 \sin \frac{1}{2}\theta$ as follows. Divide the interval $(0, 2\pi)$ into $4n$ equal parts. In the first part, $0 \leqq \theta \leqq \pi/2n$, we define $f(\theta) = 2 \sin \frac{1}{2}\theta$. In the second part, $\pi/2n \leqq \theta \leqq \pi/n$, we define f so as to make it an even function with respect to the point $\pi/2n$; that is, $f(\theta) = f((\pi/n) - \theta)$. In the remaining intervals, f is defined so as to have period π/n. It is now not difficult to show that f satisfies (4). In the first place, it is clear that we can restrict our attention to θ and α differing by less than π. By reducing angles modulo π/n and noting that $\sin \frac{1}{2}\theta$ is an increasing function between 0 and π, we can reduce the problem to the cases where θ and α are in the same or adjacent intervals. In each of the several special cases that arise, the result follows readily from the increasing and concave nature of the function $\sin \frac{1}{2}\theta$.

In the first interval, $|f'| = \cos \frac{1}{2}\theta \geqq \cos (\pi/4n)$. By the periodicity and symmetry, $|f'| \geqq \cos (\pi/4n)$ in every interval, and

$$L = \int_0^{2\pi} \sqrt{1 + (f')^2}\, d\theta \geqq 2\pi\sqrt{1 + \cos^2 (\pi/4n)}.$$

Thus L can be made arbitrarily close to $2\sqrt{2}\pi$. An interesting question is whether or not there exists a C whose length is exactly $2\sqrt{2}\pi$. If so, $f' = \pm 1$ almost everywhere and yet the derivative quotient is always less than 1, so that f' is discontinuous everywhere.

The height H is given by $\max f(\theta) - \min f(\theta)$. Assume that the minimum of f comes at 0. Then by (3) and (4),

$$f(\theta) - f(0) \leqq 2 \min \left[\, |\sin \tfrac{1}{2}\theta|,\ |\cos \tfrac{1}{2}\theta| \,\right]. \tag{5}$$

The largest value of the right-hand member of (5) is $\sqrt{2}$, which occurs when $\theta = \pi/2$. However, this same right-hand member is a function f satisfying (4) and having period π; in fact it is the same as the example constructed for length, with $n = 2$. Thus $\sqrt{2}$ is the best bound obtainable for H, and (2) is proved.

It would appear from the examples that it is impossible that simultaneously H be close to $\sqrt{2}$ and L close to $2\sqrt{2}\pi$. This suggests the interesting but perhaps difficult problem of finding the upper bound of $H + L$.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

INSIGHT AND UNDERSTANDING IN THE CALCULUS*

WALTER PRENOWITZ, Brooklyn College

Elementary calculus is, I think, one of the most difficult subjects to teach effectively. The concepts are unfamiliar to the student, hard to grasp and assimilate, and the techniques and applications often are involved and difficult. Even though he develops technical facility, often he feels he does not really understand the subject. His knowledge tends to be atomistic, consisting of individual topics and techniques loosely related to each other and individually unmotivated.

I think most teachers adopt one of two approaches to the situation. Some teachers, feeling the student can't really understand theory until he is more mature, concentrate on developing technique and problem-solving ability. Others, feeling that the student can't really understand the techniques of the subject without some knowledge of theory, and in order to avoid the need for unlearning in a later class in advanced calculus, put the emphasis on rigor of treatment.

I don't think either of these treatments is sufficient, in itself, to promote understanding of the calculus at the introductory level. The first tends to leave the student with particularistic knowledge of what to do in specific situations—but weak on general ideas and principles to guide him. As a result in a new or unfamiliar situation he clutches for a rule to apply rigidly. He achieves a feeling of security by setting up rules of thumb to be applied to the situations which ordinarily arise. Mathematics tends to degenerate into a mere tool for obtaining the correct answer to standard problems. The second approach, emphasizing rigor of treatment, tends to leave the student up in the air, since the abstractions studied are not rooted in concrete material. It is very doubtful that a student can appreciate a relatively rigorous treatment of the calculus if he does not first have a good understanding of the ideas at a more concrete and intuitive level.

I have a few points to make—they are not exhaustive—on how to foster understanding in elementary calculus. Most of them are not new—they are the sort of thing we piously agree upon as general principles but tend to disregard in the classroom under the pressure of achieving so-called practical results.

The first point I wish to make, and the most important, is that abstractions should be introduced on the basis of concrete experience. The student should see how the concepts arise from the attempt to cope with an aspect of experience, how they are suggested by concrete situations, so that he might almost feel he could have discovered them himself!

* Presented in a symposium on the teaching of calculus at the meeting of the Mathematical Association of America held at Brown University, December 29, 1951.

I think the most important application of this is to the introduction of the derivative. This is probably the most difficult notion the student encounters in the whole course—if he does not get some insight into this concept fairly early it will tend to color his reaction to the whole subject. I think it doubtful that a high order abstraction like derivative, which took some of the best minds of the human race centuries to develop, can be grasped on the basis of a brief introductory lecture on instantaneous rates which yields a four or five step method for calculating the result. If this is done, the derivative may seem to the student, not the culmination of years of thought on rates of change but the end result of a mysterious process of adding increments and dividing by Δx. In some texts we have, I think, an even worse situation—the derivative is defined without reference to instantaneous rates and the student discovers later, sometimes chapters later, that instantaneous rate is an "application" of the derivative. That is, the concrete materials which should motivate the abstraction appear later as applications of it. On the contrary, I think we should introduce the derivative as an abstract form of instantaneous rate on the basis of detailed study of the rate concept. I like to use the freely falling body as a central illustration, partly because the student thinks he understands the notion instantaneous speed in this case—he even knows the formula for it. I would have the student calculate average speeds in many specific intervals using the formula $s=16t^2$ so that he gets a good intuitive perception of the variation of average speed with the interval. Then I would bring up for discussion the instantaneous speed at $t=1$. We wrestle with the idea, analyze the "common sense" definitions suggested by the students, and show them inadequate. I would not give them the definition of instantaneous speed as a limit but try to elicit it from the physical situation. This can be done as follows. We do not know that there is such a thing as the speed of the falling body at $t=1$, but if there is such a speed v, it certainly must by physical intuition exceed the average speed in any interval terminating at $t=1$. Applying this to the interval $1-\Delta t$ to 1, we have by a simple calculation

$$32 - 16\Delta t < v.$$

Similarly physical intuition suggests v must be less than the average speed in any interval starting at $t=1$. Thus we get

$$v < 32 + 16\Delta t.$$

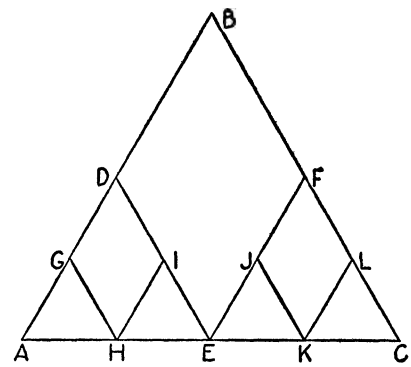
Considering these relations we see that v must be greater than all speeds below 32 ft./sec. and must be less than all speeds above 32 ft./sec. Thus we are forced to single out 32 ft./sec. as the only possible value for the speed at $t=1$. This is very important since it exhibits the unique character of 32 ft./sec.—it is singled out as the common boundary or limiting value of all speeds in intervals preceding $t=1$ and of all speeds in intervals succeeding $t=1$, and it is not itself an average speed in any such interval. This discussion I think, not merely gives some substance to the notion instantaneous speed, but shows that the concept

of limit or boundary value is essential for its definition and provides a good motivation for studying the limit concept. It is evident that instantaneous speed can not be gotten by algebra—that is by finite calculation—and that a new concept involving infinite collections or infinite sequences is essential.

Furthermore, I think these same ideas should be examined in detail in geometrical terms—in a discussion of slope of an arc of a curve, slope of a curve at a point, tangent line, *etc.* These notions belong not merely to theoretical geometry, but have an honest practical importance for the student, since they are applicable to graphs of scientific data. Thus if we plot a distance-time graph for a moving body, slope of an arc represents average speed *etc.* This gives the student a chance to rethink the ideas in graphical form and better assimilate them.

I have been emphasizing the need for concrete examples in motivating a theory. Now I want to point out the importance of counterexamples. In the attempt to clarify ideas and motivate theory, counterexamples are of the utmost importance. I would be tempted to say that often it is more important to give counter-illustrations of an idea than illustrations of it. Often the student feels a very simple formulation of an idea or a theorem is adequate, and sees no reason for the complex technical formulation. Or he sees no reason for introducing an idea at all, because he can't seriously imagine the opposite. Apparently his intuitive reactions based as they are on limited experience do not encompass the situation in all its generality. Of course he doesn't realize this. Here an appropriate counterexample can do wonders. Thus, for example, students do not appreciate the importance of continuity because they have no real experience with discontinuous functions. The examples given in the texts tend to be rather academic—some are so remote from the student's experience that they carry little weight, like the function which is 0 for rational values and 1 for irrational values of the argument. Many examples do not really illustrate the point. To be specific, we wish to convince the student that he may fall into error, if he obtains the limit of a function at a point by merely taking the functional value at that point. The familiar examples of discontinuous functions $1/x$ and $\tan x$ do not illustrate this, since these functions are not defined at the alleged points of discontinuity—in fact they are continuous throughout their domain of definition. A good example to illustrate discontinuous behavior is familiar as a sort of puzzle. Consider equilateral triangle ABC with each side of length 1. Join the midpoints of the legs to the midpoint of the base, forming two smaller triangles as in the diagram. Repeat the construction for the individual triangles formed and continue the process endlessly. Consider the sequence of paths: ABC, $ADEFC$, $AGHIEJKLC, \ldots$ and the corresponding sequence of their lengths. We ask for the limit of the sequence of lengths and the usual reply is that obviously the sequence of lengths approaches 1, the length of AC, as limit. Computing the lengths we see the first is 2, and the second equals the first since $DB=EF$ and $BF=DE$. Thus the second length is 2. Clearly all succeeding lengths equal 2 so that although the paths approach BC as a limit, their lengths

do not approach 1, the length of BC, as a limit. This strongly challenges the student's cherished belief that variation is always continuous—he is shaken, disturbed, begins to feel his intuition is not infallible and that perhaps there is something to the method of reasoning after all!



Once a basic concept has been introduced, its significance should be reinforced by having the student operate with it as a part of his basic intellectual equipment—not with a formalized manipulative substitute which enables him to get correct results without consciously applying the idea. The example I am most concerned about is the calculation of the derivative of a function from its definition. I fear the familiar 4-step process for doing this tends to degenerate into rote procedure which the student applies mechanically, without consciously having the definition of derivative in mind. Thus to obtain the derivative of $f(x)=x^2$ he writes $y=x^2$, replaces x by $x+\Delta x$, y by $y+\Delta y$ to get $y+\Delta y=(x+\Delta x)^2$, subtracts the first equation from the second, divides by Δx, *etc.* I'm afraid there is not apt to be much present in the student's mind when he does this except that he must add an increment to whatever he sees in the first equation—I sometimes think there must be a guardian angel who inhibits the student from adding an increment to the exponent 2 or to the equal sign. I don't mean that a student can't consciously think of the significance of the steps as he applies this technique—rather I think there is no need to—the process is motivated essentially by mechanical considerations. It has further disadvantages. It tends to enforce the impression that the basic idea is derivative of a function, not derivative of a function at a particular point. I like to give students a problem like this: Find the derivative of $f(x)=(x+1)/(x-1)$ at $x=2$ directly from the definition. I find many students who use the 4-step process can't do this. They must first get the derivative for x, which is more difficult, then substitute $x=2$. Apparently in the 4-step process after substituting 2 for x they are lost, since they do not have an x to replace by $x+\Delta x$. Furthermore I do not think this process prepares a student properly to cope with problems which arise later on, like finding the derivative of a function defined geometrically as an area or arc

length, or that of a function like $x^2 \sin(1/x)$, defined as 0 at $x=0$, which is not given by a single analytic expression.

The final point I wish to make is that given two alternative formulations of a theory, we should choose the one which brings the ideas to the fore—it is not sufficient for a treatment merely to be scientifically correct or to have the merit of facilitating calculation. This is completely disregarded in the treatment of Integral Calculus given in many current texts. We have in Integral Calculus two important ideas: integration as summation, integration as anti-differentiation. The first is an outgrowth of mensuration problems which the student encounters in plane and solid school geometry and which has an honorable history of over 2,000 years in our subject. Mathematicians are still studying it, in abstract form, in the modern theory of measure. It is of the utmost importance in applied mathematics. Yet in the conventional treatment no name is assigned to summation to signalize its importance, least of all the name integration. It is introduced after weeks of work on formal anti-differentiation as an application of anti-differentiation, and is merely referred to as the "limit of a sum." Thus the fundamental theorem of Integral Calculus is expressed

$$\int_a^b f(x)dx = \lim \sum_i f(x_i)\Delta x_i$$

where the left member is the variation in an anti-derivative of $f(x)$ from $x=a$ to $x=b$. This tends to give students the impression that anti-differentiation is the essential idea in Integral Calculus, and that the limit of a sum is just an interpretation or application of this idea. It appears to the student that limits of sums have no life of their own except in relation to formal anti-differentiation, since he never solves a single problem of summation in its own terms. This whole treatment impedes the unification of the student's past ideas concerning mensuration and those he will get in advanced calculus and graduate school where mensuration ideas or integration in the strict sense come into their own.

Surely we should prefer the alternative treatment, in which the concept limit of a sum is defined and studied in its own terms, as an important abstraction from problems of measuring magnitudes arising in geometry and physical science, and is represented by the historical symbol of summation $\int_a^b f(x)dx$. To give the student experience with the idea he should evaluate some simple definite integrals directly from the definition. Then he can better appreciate the importance of the fundamental theorem as relating the two apparently disparate ideas summation and anti-differentiation. The fundamental theorem might well be expressed in the form

$$\int_a^b f(x)dx = \left[D^{-1}f(x)\right]_a^b$$

where D^{-1} denotes the inverse of the differentiation operator. This then indicates the importance of anti-differentiation as a simple means of evaluating definite

integrals, and gives the student a motivation for weeks of study of formal anti-differentiation.

In conclusion I wish to emphasize that what I am advocating is not in the least opposed to raising standards of mathematical rigor or to developing technical facility. On the contrary, I think a good understanding of fundamental concepts and principles in concrete terms is a necessary core without which the student will not appreciate the need for raising standards of logical precision, and without which his technical work may degenerate into mere formalism.

NON-ABELIAN GROUPS OF ORDER pq

S. K. BERBERIAN, Southern Illinois University

This note is concerned with the construction of non-abelian groups of order pq. The point of departure is the theorem characterizing such groups:*

THEOREM: *If G is a non-abelian group of order pq, p and q primes, $p<q$, then:*

(1) $q\equiv 1 \pmod{p}$;
(2) *G is generated by elements P and Q, of orders p and q, respectively, such that*
(3) $P^{-1}QP=Q^{\beta}$, where
(4) *β is a primitive root of the congruence*
$x^p\equiv 1 \pmod{q}$.

It is evident that as soon as two elements P and Q satisfying (2), (3), and (4) are located in a known group, they generate a subgroup satisfying the requirements for G.

We therefore assume property (1) to hold, and attempt to locate suitable elements P and Q in a known group. For embedding purposes, we utilize the symmetric permutation group of degree q.

First, there are numbers β satisfying the congruence (4); for, by assumption (1), the prime p is a divisor of $q-1$, the order of the multiplicative group of the field of integers modulo q. Since such an element β has order p, the full list of distinct solutions modulo q of the congruence (4) is:

$$1, \beta, \beta^2, \cdots, \beta^{p-1}.$$

For Q, we choose the q-cycle

$$Q = (1, 2, \cdots, q).$$

Modulo q, we may write

$$Q^{\beta} = (1, 1+\beta, 1+2\beta, \cdots, 1+(q-1)\beta).$$

P will necessarily be the product of disjoint p-cycles. In assumption (1), suppose that $q=1+mp$; we will choose P to be the product of m disjoint p-cycles. With such a structure, P will have to leave one symbol of Q unmoved; if we agree to let this symbol be 1 (this choice is arbitrary), then by (3), and the

* Burnside, W.: Theory of Groups of Finite Order; p. 48.

properties of conjugacy for permutations, P must act on the remaining symbols of Q according to the scheme:

$$\begin{array}{lcccccccc} & Q = (1, & 2, & 3, & \cdots, & i, & \cdots, & q) \\ (5) & & \downarrow & \downarrow & & \downarrow & & \downarrow \\ & Q^\beta = (1, & 1+\beta, & 1+2\beta, & \cdots, & 1+(i-1)\beta, & \cdots, & 1+(q-1)\beta). \end{array}$$

Thus, P must send i to $1+(i-1)\beta$, which in turn must be sent to $1+[1+(i-1)\beta-1]\beta=1+(i-1)\beta^2$. A typical cycle of P would then be

$$\pi = (1 + (i - 1), 1 + (i - 1)\beta, 1 + (i - 1)\beta^2, \cdots, 1 + (i - 1)\beta^{p-1}). \tag{6}$$

The cycle π terminates after p entries, since $\beta^p \equiv 1 \pmod q$, and hence

$$1 + (i - 1)\beta^{p-1} \to 1 + (i - 1)\beta^p \equiv i \pmod q.$$

The application of this principle is very direct.

Example 1: $p=3$, $q=7$

A primitive solution of the congruence $x^3 \equiv 1 \pmod 7$ is $\beta=2$. Choose

$$Q = (1\ 2\ 3\ 4\ 5\ 6\ 7).$$

Then,

$$Q^\beta = (1\ 3\ 5\ 7\ 2\ 4\ 6).$$

Assuming again that P leaves the symbol 1 fixed, we simply force P to meet the requirements:

$$P = (2\ 3\ 5)(4\ 7\ 6).$$

Then,

$$P^{-1}QP = Q^2$$

and P and Q generate the required group G.

We next show that the construction of a P, satisfying (3), is always possible. Example 1 suggests a method of proof. For the first cycle of P, write

$$\pi_1 = (1 + 1, 1 + \beta, 1 + \beta^2, \cdots, 1 + \beta^{p-1}).$$

Since $1, \beta, \beta^2, \cdots, \beta^{p-1}$ are incongruent modulo q, so are the entries of π_1. The cycling is proper, since the symbol $1+\beta^{p-1}$ of Q must be sent to

$$1 + \beta^p \equiv 1 + 1 \pmod q.$$

At this point, if $q=3$, then $p=2$ and the construction is ended. Otherwise, there is a symbol $i \neq 1$ of Q which does not appear in π_1; for the next factor of P, construct

$$\pi_2 = (1 + (i - 1), 1 + (i - 1)\beta, 1 + (i - 1)\beta^2, \cdots, 1 + (i - 1)\beta^{p-1}).$$

It has been already noted in connection with (6) that the cycling of π_2 is proper.

The entries of π_2 are distinct modulo q, for, an equality

$$1 + (i - 1)\beta^r \equiv 1 + (i - 1)\beta^s \pmod{q}$$

would imply

$$\beta^{r-s} \equiv 1 \pmod{q},$$

and since $0 \leqq r,\ s < p$, and β has order p, it follows that $r = s$. Also, π_2 is disjoint from π_1; for suppose there were an equality

$$1 + (i - 1)\beta^r \equiv 1 + \beta^s \pmod{q}.$$

This would imply

$$i \equiv 1 + \beta^{s-r} \pmod{q}$$

contrary to the choice of i.

If there is a third step in the construction,

$$\pi_3 = (1 + (j - 1),\ 1 + (j - 1)\beta,\ \cdots,\ 1 + (j - 1)\beta^{p-1})$$

where $j \neq 1$ is neither in π_1 nor π_2, the preceding argument shows that π_3 is disjoint from π_1. To show that π_3 is disjoint from π_2, suppose there were an equality

$$1 + (j - 1)\beta^r \equiv 1 + (i - 1)\beta^s \pmod{q}.$$

Then,

$$j \equiv 1 + (i - 1)\beta^{s-r} \pmod{q}$$

whereas j is not in π_2.

Finally, $P = \pi_1\pi_2 \cdots \pi_m$, the construction terminating in exactly m steps, since there are $q-1$ symbols of Q to be moved, and $q-1 = mp$. (An explicit induction step has been omitted; the argument surrounding π_3 is nearly adequate.) We have done little more than construct the system of prime residues modulo q.*

Example 2: The case $p = 2$ is especially transparent, since we are then forced to choose $\beta = q-1$. If then $Q = (1, 2, \cdots, q)$, Q^β will be the inverse of Q:

$$Q^\beta = (1, q, q - 1, \cdots, 2),$$

and if 1 is the symbol unmoved by P, then

$$P = (2, q)(3, q - 1)(4, q - 2) \cdots.$$

For example, if $q = 7$, then

$$\begin{aligned} Q &= (1\ 2\ 3\ 4\ 5\ 6\ 7) \\ Q^\beta = Q^6 &= (1\ 7\ 6\ 5\ 4\ 3\ 2) \\ P &= (27)(36)(45). \end{aligned}$$

* Uspensky, J. V., and Heaslet, M. A.: Elementary Number Theory; p. 225.

This representation of the non-abelian group of order $2q$ naturally suggests the dihedral group. Let the vertices of a regular q-gon be numbered in cyclic order. In the dihedral group for this polygon, Q may be interpreted as a rotation of $360°/q$, and P as a reflection in the axis of symmetry passing through vertex 1. Since q is odd, every reflection symmetry leaves fixed exactly one vertex; our choice of 1, as the symbol unmoved by P, may therefore be interpreted as a choice of vertex.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 1046. *Proposed by E. P. Starke, Rutgers University*

Knowing that dy/dx and dx/dy are reciprocals, students sometimes jump to the conclusion that d^2x/dy^2 should be the reciprocal of d^2y/dx^2. Determine functions $y=f(x)$ for which this is indeed true.

E 1047. *Proposed by W. R. Ransom, Tufts College*

Find all Gaussian integral factorizations of $10i$, where $i=\sqrt{-1}$.

E 1048. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Let M be an $n\times\infty$ matrix. Show that if every $n\times n$ submatrix of M is singular, then the columns of M are linearly dependent.

E 1049. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If a uniform thin rod of length L is perpendicular to and touching the earth's surface at one end, find the distance between the center of gravity and the center of mass of the rod.

E 1050. *Proposed by S. H. Gould, Purdue University*

If $c_0=1$, $c_{i+1}=2^{c_i}$, prove that $\sum_{i=1}^{\infty} c_i^{-1}$ is transcendental.

For the symmetric quadrilateral $a=b=50$, $c=d=34$, $\cos B=7/25$ (Graph III), there are three solutions; the axis BD and two others, PP', for $AP=x=8\pm\sqrt{14}$.

For the case of the trapezoid we take $b=BC$ as the longer, and $d=DA$ as the shorter, of the parallel sides. Then the inequalities (1), (2), (3) hold, and all of the above treatment holds except that for the interval $D'A'$ the graph will be a straight line segment parallel to the x-axis. If the trapezoid is not isosceles this line segment will not coincide with the x-axis, and all solutions must occur in the three intervals from A to D'. Graph IV is for the trapezoid $a=12$, $b=130\sqrt{6}-310$, $c=13$, $d=130\sqrt{6}-315$, with A and B right angles; $s=130\sqrt{6}-300$ and $K=15(52\sqrt{6}-125)$. Again there are just two solutions, one for $x=13\sqrt{6}-24$, and the other for x approximately 13.7.

In the case of the isosceles trapezoid, $DAD'A'$ is a rectangle and we have solutions PP' for P at any point of the segment $D'A'$; but we may ask whether there are any other solutions in addition to these obvious ones. Graph V is for the isosceles trapezoid $a=c=41$, $b=30$, $d=12$, $\cos B=9/41$. In addition to the solutions for the interval $D'A'$ there are two other solutions for $x=(50\pm\sqrt{122})/2$.

Except for the cases of the parallelogram and the isosceles trapezoid the number of solutions is finite. The special cases show that there can be one, two, or three solutions. What is the maximum possible number of solutions? The maximum number is hardly four, because, given a case of four solutions, one could apparently alter it slightly to give a case of five. For reasons depending on the convexity upwards and downwards of the four parabolic arcs, I believe there could never be more than five solutions. Is the maximum five or three?

A Formula Yielding Palindromic Numbers

E 1013 [1952, 249]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Show that for $a=0, 1, 2, \cdots, 10$ and for any positive integral n except 1 and 3, the expression

$$(10^n - 1)(909a + 1)$$

yields palindromic numbers.

Solution by R. Z. Vause, University of North Carolina. For $a=0$,the expression is simply a sequence of 9's. For $a=1, 2, \cdots, 10$, the factor $909a+1$ can be written in the form

$$r(10^3) + s(10^2) + r(10) + (s + 1),$$

where r and s are integers such that $r+s=9$. Therefore, for $n\geqq 4$, the expression $(10^n-1)(909a+1)$ becomes

$$r(10^{n+3}) + s(10^{n+2}) + r(10^{n+1}) + s(10^n) + 9(10^{n-1}) + \cdots \\ + 9(10^4) + s(10^3) + r(10^2) + s(10) + r,$$

whereas for $n=2$ it is

$$r(10^5) + s(10^4) + s(10) + r.$$

Also solved by A. R. Hyde, H. Kaufman and J. S. Shipman (jointly), and the proposer.

Kaufman and Shipman pointed out two directions for generalization. In the first place the 909 may be replaced by any number of $2k+1$ digits of the form $9090 \cdots 909$. It can then be shown that

$$(10^n - 1)(9090 \cdots 909a + 1)$$

yields palindromic numbers for $a=0, 1, \cdots, 10$ and $n>2k+1$, and also for $n=2, 4, \cdots, 2k$. In the second place the problem, and the above generalization, can be extended in the obvious way to an arbitrary base B.

The proposer has neatly generalized his formula as follows. Let a_i, $i=1, 2, \cdots, n$, be n arbitrary digits, with $a_1 \neq 0$, and define $b_i = 9 - a_i$. Then, if we define R_n by the sequence of digits

$$R_n = a_1 a_2 \cdots a_n b_n b_{n-1} \cdots b_1,$$

the number

$$P_n = (10^m - 1)(R_n + 1)$$

is a palindrome of $m+4n$ digits for all $m \geqq 2n$.

An Impossible Arithmetic Triangle

E 1014 [1952, 249]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle having sides a, b, c and area S, if the ratio

$$(a^2 + b^2 + c^2)/4S$$

is a whole number $m>1$, then the sides a, b, c cannot all be integral.

Solution by C. V. Fronabarger, Southwest Missouri State College. Suppose a, b, c are integral. Since m is a constant for similar triangles, only three cases need be considered:

Case I, the three sides all odd;
Case II, only one side odd;
Case III, only one side even.

Now if

$$\begin{aligned}(a^2 + b^2 + c^2)/4S \\ &= (a^2 + b^2 + c^2)/\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\ &= m, \text{ an integer,}\end{aligned}$$

two conditions must be satisfied. First,

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16S^2$$

must be a perfect square, and second, 4 must be a divisor of $a^2+b^2+c^2$.

In Cases I and II, $16S^2 \equiv 3 \pmod 4$, which is impossible, since all perfect squares are congruent to 0 or 1 modulo 4.

In Case III, $a^2+b^2+c^2 \equiv 2 \pmod 4$, and is thus not divisible by 4.

It follows that a, b, c cannot all be integers.

Also solved by Louisa Grinstein, Vern Hoggatt, R. R. Phelps, and C. S. Venkataraman.

Editorial Note: There is nothing in the above proof which says that we cannot have $m=1$. Actually, however,

$$(a^2+b^2+c^2)/4S = \cot\omega,$$

where ω is the Brocard angle of the triangle. Since $\omega \leqq 30°$, $\cot\omega \geqq \sqrt{3}$, and it follows that $m \neq 1$. For the special case $m=2$ see problem 4401 [1952, 112].

To show, in Cases I and II, that

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) \equiv 3 \pmod 4$$

we observe, setting $d=a+b+c$, that

$$\begin{aligned}(a+b+c)&(-a+b+c)(a-b+c)(a+b-c)\\ &= d(d-2a)(d-2b)(d-2c) \equiv d^4 - d^3(2a+2b+2c) \equiv -d^4 \equiv 3 \pmod 4\end{aligned}$$

Cases I and II can be eliminated, as was done by Phelps, by using the fact that in a primitive Heronian triangle just one side must be even (see Wm. Fitch Cheney, Jr., *Heronian Triangles*, this MONTHLY [1929, 22–28]).

An Integral Involving a Bessel Function

E 1015 [1952, 249]. *Proposed by C. A. Shook and Albert Wilansky, Lehigh University*

Evaluate

$$\int \frac{dx}{x[J_n(x)]^2},$$

where J_n is the usual Bessel function.

Solution by W. J. Klimczak, Trinity College. Let Y_n denote the Bessel function of the second kind. Then, for all values of n,

$$\frac{d}{dx}\left[\frac{Y_n(x)}{J_n(x)}\right] = \frac{J_n(x)Y_n'(x) - Y_n(x)J_n'(x)}{[J_n(x)]^2} = \frac{2}{\pi x[J_n(x)]^2}\cdot$$

Hence

$$\int \frac{dx}{x[J_n(x)]^2} = \frac{\pi}{2} \frac{Y_n(x)}{J_n(x)} + C.$$

This formula is derived on p. 133 of G. N. Watson's *A Treatise on the Theory of Bessel Functions*. It also occurs as prob. 9, p. 184, of Ince's *Ordinary Differential Equations*.

Also solved by A. E. Danese, Vern Hoggatt, H. Kaufman and J. S. Shipman (jointly), M. S. Klamkin, David Mandelbaum, O. E. Stanaitis, and the proposers.

Editorial Note: If n is not an integer it can be shown that

$$\int \frac{dx}{x[J_n(x)]^2} = -\frac{\pi}{2 \sin n\pi} \frac{J_{-n}(x)}{J_n(x)}.$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4518. *Proposed by Paul Erdös, National Bureau of Standards, and Mark Kac, Cornell University*

For each integer n let $\sigma_k(n)$ be the sum of the kth powers of its divisors. It is a natural conjecture that

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational. Prove this for the case $k=2$. (The case $k=1$ is problem 4493 [1952, 412].)

4519. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$1 + \frac{1-n}{1+n}\cdot\frac{1}{3} + \frac{(1-n)(2-n)}{(1+n)(2+n)}\cdot\frac{1}{5} + \cdots = \frac{1}{4n}\cdot\left\{\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)}\right\}^2.$$

4520. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $\phi(x)$ be a Lebesgue measurable function in the interval $0 \leqq x \leqq 1$ such that

$$\int_0^1 \phi(x)dx = 0, \qquad \int_0^1 x\phi(x)dx = 1.$$

Show that $|\phi(x)| \geqq 4$ in some set of measure greater than zero.

4521. *Proposed by I. A. Barnett, University of Cincinnati*

Find the complete solution of the diophantine equation

$$x^4 + y^4 + 64 = z^4.$$

4522. *Proposed by O. E. Stanaitis, St. Olaf College, Northfield, Minn.*

Evaluate

$$\lim_{k\to\infty} \int_{2k\pi}^{(2k+1)\pi} \phi(t^\alpha) \sin t dt,$$

where $\phi(x) = x - [x]$, $\alpha > 1$, and $[x]$ denotes the greatest integer in x.

SOLUTIONS

A Minimum Property of the Eigenvalues of a Hermitian Transformation

4429 [1951, 194]. *Proposed by Ky Fan, University of Notre Dame*

Let H be a non-negative (positive semi-definite) Hermitian transformation in the n-dimensional unitary space. If the eigenvalues $\lambda_i (1 \leqq i \leqq n)$ of H are arranged in ascending order: $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, show that, for any positive integer $k \leqq n$, the product $\lambda_1\lambda_2 \cdots \lambda_k$ of the first k eigenvalues is equal to the minimum of the expression

$$\prod_{i=1}^{k} (Hx_i, x_i),$$

when k orthonormal vectors $x_1, x_2, \cdots, x_k$ vary in space. Here (Hx_i, x_i) denotes the inner product of the vector Hx_i with x_i. (This result is related to the Proposer's paper, *On a theorem of Weyl concerning eigenvalues of linear transformations, I*, Proceedings of the National Academy of Sciences, U.S.A., v. 35 (1949), pp. 652–655.)

Solution by Hans Schneider, Edinburgh, Scotland. Let the unitary space be spanned by the n orthonormal vectors $x_1, x_2, \cdots, x_n$. Let X be the unitary matrix having these n vectors as columns, and let X^* be its conjugate transposed matrix. Then the matrix X^*HX has the same eigenvalues as H. Now let H_k denote the matrix obtained by deleting the last $n-k$ rows and columns of X^*HX for $k = 1, 2, \cdots, n$. It is clear that for $k = 1, 2, \cdots, n$, H_k is also a non-

negative definite Hermitian matrix.

Denote the eigenvalues of H_k by $\lambda_1^{(k)}, \cdots, \lambda_k^{(k)}$. It is known that the eigenvalues of H_{k+1} are separated by those of H_k and thus it follows that

$$\lambda_1 \cdots \lambda_k \leqq \lambda_1^{(k)} \cdots \lambda_k^{(k)} = \det (H_k).$$

But the determinant of a non-negative definite Hermitian matrix is less than the product of its diagonal elements. The diagonal elements of H_k are precisely the numbers (Hx_i, x_i), $i=1, 2, \cdots, k$, and thus we have

$$\prod_{i=1}^{k} \lambda_1 \cdots \lambda_k \leqq \prod_{i=1}^{k} (Hx_i, x_i). \tag{1}$$

If the x_i are chosen to be the eigenvalues of H, the equality holds in (1) and the proposed result therefore follows.

Also solved by Richard Arens, Bernard Vinograde, Burton Wendroff, K. G. Wolfson, and the Proposer.

Note by the Proposer. Using the theorem stated in problem 4429, one can easily prove the following theorem: *Let A be a linear transformation in the n-dimensional unitary space. Let the eigenvalues $\{\lambda_i\}$, $\{\rho_i\}$ of A and $(A+A^*)/2$ respectively be so arranged that*

$$R\lambda_1 \leqq R\lambda_2 \leqq \cdots \leqq R\lambda_n, \qquad \rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_n.$$

If the Hermitian transformation $(A+A^*)/2$ is non-negative (i.e. if all $\rho_i \geqq 0$), then for any positive integer $k \leqq n$:

$$\prod_{i=1}^{k} \rho_i \leqq \prod_{i=1}^{k} R\lambda_i. \tag{2}$$

This result is a completion of a recent theorem by Ostrowski-Taussky (A. M. Ostrowski and O. Taussky, *On the variation of the determinant of a positive definite matrix*, Proc. Kon. Nederl. Akad. v. Wetensch. Amsterdam, Series A, v. 54 (1951), pp. 383–385). Under the same hypothesis as in the above theorem, these authors proved the inequality

$$\det \frac{A + A^*}{2} \leqq |\det A|, \tag{3}$$

which may be written $\rho_1\rho_2 \cdots \rho_n \leqq |\lambda_1\lambda_2 \cdots \lambda_n|$ and is therefore a consequence of the case $k=n$ of (2).

Proof. By Schur's superdiagonal form of matrices, there exist n orthonormal vectors $x_1, x_2, \cdots, x_n$ such that $(Ax_i, x_i)=\lambda_i$, $1 \leqq i \leqq n$. Then we have

$$\left(\frac{A + A^*}{2} x_i, x_i\right) = \frac{\lambda_i + \overline{\lambda_i}}{2} = R\lambda_i, \qquad (1 \leqq i \leqq n),$$

and

$$\prod_{i=1}^{k}\left(\frac{A+A^*}{2}x_i, x_i\right) = \prod_{i=1}^{k} R\lambda_i, \qquad (1 \leqq k \leqq n).$$

Since $(A+A^*)/2$ is non-negative, according to the theorem of problem 4429, the left side is $\geqq \rho_1\rho_2 \cdots \rho_k$. Thus (2) is proved.

Eigenvalues of a Sum of Hermitian Matrices

4430 [1951, 194]. *Proposed by Ky Fan, University of Notre Dame*

Let H', H'' be two n-rowed non-negative Hermitian matrices, and let $H=\alpha H'+\beta H''$, where α, β are two non-negative numbers with $\alpha+\beta=1$. Let the eigenvalues of H', H'' and H be denoted by λ'_i, λ''_i and $\lambda_i (1\leqq i\leqq n)$ respectively and so arranged that

$$\lambda_i' \leqq \lambda_{i+1}', \qquad \lambda_i'' \leqq \lambda_{i+1}'', \qquad \lambda_i \leqq \lambda_{i+1}.$$

Prove that for any positive integer $k\leqq n$:

$$(1) \qquad \lambda_1\lambda_2 \cdots \lambda_k \geqq (\lambda_1'\lambda_2' \cdots \lambda_k')^\alpha(\lambda_1''\lambda_2'' \cdots \lambda_k'')^\beta,$$

and in particular:

$$(2) \qquad \det H \geqq (\det H')^\alpha(\det H'')^\beta.$$

Solution by the Proposer. Let $\phi_i (1\leqq i\leqq n)$ be an orthonormal set of eigenvectors of H: $H\phi_i=\lambda_i\phi_i$. Then

$$\lambda_1\lambda_2 \cdots \lambda_k = \prod_{i=1}^{k} (H\phi_i, \phi_i) = \prod_{i=1}^{k} \left[\alpha(H'\phi_i, \phi_i) + \beta(H''\phi_i, \phi_i)\right]$$

and therefore

$$\lambda_1\lambda_2 \cdots \lambda_k \geqq \prod_{i=1}^{k} (H'\phi_i, \phi_i)^\alpha(H''\phi_i, \phi_i)^\beta.$$

Now by a minimum property of the eigenvalues of non-negative Hermitian transformations (see problem 4429 above), the right side of the last inequality is not less than

$$(\lambda_1'\lambda_2' \cdots \lambda_k')^\alpha(\lambda_1''\lambda_2'' \cdots \lambda_k'')^\beta.$$

(1) is thus proved. For the particular case (2), another proof can be found in K. Fan, *On a theorem of Weyl concerning eigenvalues of linear transformations, II*, Proc. Nat. Acad. Sci. U.S.A., 36 (1950), pp. 31–34.

Also solved by Hans Schneider, and Bernard Vinograde.

Words Having No Part Repeated *n* Times

4454 [1951, 569]. *Proposed by R. C. Lyndon, Princeton University*

The "Burnside problem" for semigroups without cancellation can be given the following form. Let $S(n, m)$ be the set of all words, or finite sequences, formed from an alphabet of m letters, which have the property that no (non-empty) part is repeated as many as n consecutive times. Show that, for $m, n > 1$, with the single exception $m = n = 2$, the set $S(n, m)$ is infinite.

Comment by William Gustin, Indiana University. The set $S(2, 2)$ in two letters a, b obviously consists of the following six words: a, ab, aba, b, ba, bab; so is finite. That all other sets $S(n, m)$ are infinite has been shown by Morse and Hedlund (*Unending chess, symbolic dynamics, and a problem in semi-groups*, Duke Mathematical Journal, v. 11 (1944), pp. 1–7).

They exhibit an infinite word B in two letters a, b by defining its initial word intervals B_k of length 2^k inductively as follows. Let $B_0 = a$; having defined B_k, define $B_{k+1} = B_k\overline{B}_k$ where $\overline{B}_k$ is obtained from B_k by interchanging the letters a and b. Thus

$$B = abbabaabbaababba \cdots .$$

It is shown that B contains no word interval of the form $pQpQp$ where p is a letter and Q is a (possibly null) word. In particular B is cube free, that is, contains no non-null word interval of the form EEE. Therefore each of the infinitely many initial word intervals of B belongs to $S(n, m)$ for $n \geqq 3$, $m \geqq 2$.

The infinite word B in two letters a, b is transformed into an infinite word C in three letters a, b, c by destroying all letter squares aa, bb in B in the following way. Each letter in B which is not followed by the same letter is left unchanged; each letter in B which is followed by the same letter is changed to c. Thus

$$C = acbabcacbcabacb \cdots .$$

It is shown that C is square free, that is, contains no non-null word interval of the form EE. Therefore each of the infinitely many initial word intervals of C belongs to $S(n, m)$ for $n \geqq 2$, $m \geqq 3$.

Also solved by R. P. Dilworth, W. R. Scott, F. B. Thompson, and the Proposer.

Editorial Note. The Proposer wishes to correct the impression that the systems $S(n, m)$ are semigroups. In fact, the present problem contributes little to the solution of the Burnside problem for semigroups. (Burnside's conjecture is that every group on a finite number, m, of generators, in which every element except the unit element is of the same finite order, n, is necessarily a finite group. This conjecture has been confirmed only for a handful of cases.)

Tetrahedron—Sum of Analogous Ratios

4456 [1951, 570]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If through the vertices A, B, C, D of a tetrahedron parallel planes are drawn cutting a given line L in points A_2, B_2, C_2, D_2, and if A_1, B_1, C_1, D_1 are the points in which the lines AA_2, BB_2, CC_2, DD_2 cut the planes BCD, CDA, DAB, ABC, then

$$\frac{AA_2}{AA_1}+\frac{BB_2}{BB_1}+\frac{CC_2}{CC_1}+\frac{DD_2}{DD_1}=2.$$

Solution by W. E. Byrne, Virginia Military Institute. Taking a system of cartesian coordinates with oblique axes we may put $D(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$. Let line L be given by

$$x = x_0 + \lambda l, \qquad y = y_0 + \lambda m, \qquad z = z_0 + \lambda n,$$

and let plane Π_D through D be $ux+vy+wz=0$. Then D_2 is the point on L for which the parameter λ satisfies

$$ux_0 + vy_0 + wz_0 + \lambda(lu + mv + nw) = 0.$$

Since plane ABC has equation $x/a+y/b+z/c=1$, we have

$$\frac{1}{\mu}\left[\frac{x_0}{a}+\frac{y_0}{b}+\frac{z_0}{c}-\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)\left(\frac{ux_0+vy_0+wz_0}{lu+mv+nw}\right)\right]=1$$

where $\mu=DD_2/DD_1$. Thus

$$(1)\qquad \frac{DD_2}{DD_1}=\frac{x_0}{a}+\frac{y_0}{b}+\frac{z_0}{c}-\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)\left(\frac{ux_0+vy_0+wz_0}{lu+mv+nw}\right).$$

Plane Π_A is $u(x-a)+vy+wz=0$ and plane BCD is $x=0$. Hence for A_2

$$u(x_0 - a) + vy_0 + wz_0 + \lambda(lu + mv + nw) = 0$$

so that

$$(2)\qquad \frac{AA_2}{AA_1}=\frac{-(x_0-a)}{a}+\left(\frac{l}{a}\right)\frac{u(x_0+a)+vy_0+wz_0}{lu+mv+nw},$$

and similarly

$$(3)\qquad \frac{BB_2}{BB_1}=\frac{-(y_0-b)}{b}+\left(\frac{m}{b}\right)\frac{ux_0+v(y_0-b)+wz_0}{lu+mv+nw},$$

$$(4)\qquad \frac{CC_2}{CC_1}=\frac{-(z_0-c)}{c}+\left(\frac{n}{c}\right)\frac{ux_0+vy_0+w(z_0-c)}{lu+mv+nw}.$$

Addition of equations (1)–(4) gives at once

$$\sum \frac{AA_2}{AA_1} = 3 - \left(\frac{lu + mv + nw}{lu + mv + nw}\right) = 2.$$

Also solved by Josef Langr.

Coefficients in a Series of Cosines

4457 [1951, 570]. *Proposed by Israel Halperin, Queens College, Kingston, Ontario*

If

$$\sum_{m=1}^{\infty} | b_m | < \infty \quad \text{and} \quad \lim_{t\to 0} \sum_{m=1}^{\infty} b_m \cos\frac{1}{mt} = 0,$$

show that every $b_m = 0$.

Solution by the Proposer. By actual integration

$$\int_{r\pi}^{(r+1)\pi} \left| \sum_{m=1}^{N} b_m \cos n_m u \right|^2 du = \frac{1}{2}\pi \sum_{m=1}^{N} | b_m |^2,$$

where N, r are any positive integers and the n_m are different integers. Hence

$$\underset{r\pi \leqq u \leqq (r+1)\pi}{\text{l.u.b.}} \left| \sum_{m=1}^{N} b_m \cos n_m u \right| \geqq \sqrt{\frac{1}{2} \sum_{m=1}^{N} | b_m |^2}.$$

Choosing $n_m = N!/m$ and setting $t = (N!)u$ gives

$$\underset{r(N!)\pi \leqq t \leqq (r+1)(N!)\pi}{\text{l.u.b.}} \left| \sum_{m=1}^{N} b_m \cos\frac{t}{m} \right| \geqq \sqrt{\frac{1}{2} \sum_{m=1}^{N} | b_m |^2}.$$

Since $\sum | b_m |^2$ is convergent and $\sum b_m \cos (t/m)$ is uniformly convergent, this implies that

$$\overline{\lim_{t\to\infty}} \left| \sum_{m=1}^{\infty} b_m \cos\frac{t}{m} \right| \geqq \sqrt{\frac{1}{2} \sum_{m=1}^{\infty} | b_m |^2}$$

and hence $\sum_{m=1}^{\infty} b_m \cos (1/mt) \to 0$ as $t \to 0$ only if all b_m are zero.

An Expansion Related to the Lagrange Interpolation Formula

4458 [1951, 636]. *Proposed by Z. A. Melzak, McGill University, Montreal*

Let $f(x)$ be a polynomial of degree n. Then

$$f(x + y) = \frac{y(y + 1)(y + 2) \cdots (y + n)}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x - k)}{y + k}.$$

I. *Solution by V. D. Gokhale, Atlanta University.* Consider the fraction

$$\frac{f(x+y)}{y(y+1)(y+2)\cdots(y+n)}.$$

Since the degree of the denominator is at least one higher than that of the numerator, this fraction decomposes into partial fractions of the form

$$\frac{A_k}{y+k},$$

$$A_k = \left[\frac{f(x+y)}{g'(y)}\right]_{y=-k}, \qquad g(y) = y(y+1)\cdots(y+n).$$

Thus

$$A_k = \frac{f(x-k)}{(-k)(-k+1)\cdots(-1)\cdot 1\cdot 2\cdots(n-k)} = \frac{(-1)^k f(x-k)}{k!(n-k)!}$$
$$= \frac{(-1)^k}{n!}\binom{n}{k} f(x-k),$$

so that

$$\frac{f(x+y)}{y(y+1)\cdots(y+n)} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}.$$

Multiplying both sides by $y(y+1)\cdots(y+n)$ we get the desired result.

II. *Solution by W. V. Parker, Alabama Polytechnic Institute.* Each member is a polynomial in y of degree n. It is readily seen that they are equal for $y=-k$, $k=0, 1, 2, \cdots, n$. Hence they are identical polynomials in y.

The right member can be obtained immediately from Lagrange's interpolation formula. In fact, for $f(x)$ a polynomial of degree n, $g(y)= \Pi_{i=0}^{n}(y-y_i)$, where the y_i are distinct, we have

$$f(x+y) = g(y)\sum_{k=0}^{n}\frac{f(x+y_k)}{(y-y_k)g'(y_k)}.$$

Also solved by H. L. Alder, T. M. Apostol, Gerald Berman, Leonard Carlitz, W. C. Foreman, Emil Grossman, H. W. Gould, M. S. Klamkin, Walter Littman, A. E. Livingston, George Millman, F. D. Parker, M. Perisastri, H. N. Shapiro, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

Squarefull Integers

4459 [1951, 636]. *Proposed by D. J. Newman, Harvard University.*

Find an asymptotic expression for the number of integers, not exceeding x, each of which has the property that each of its prime divisors divides it to the second power at least.

Solution by Abe Sklar, University of Chicago. We call such integers squarefull and denote the number of squarefull integers not exceeding x by $A(x)$. If n is squarefull and m^2 is the largest square divisor of n, then $d=n/m^2$ divides m and d is squarefree. Conversely, any squarefull integer can be written as a product of a square m^2 and a squarefree divisor d of m. Hence:

$$A(x) = \sum_{m \leqq x^{1/2}} \sum_{d} | \mu(d) |,$$

where the inner sum extends over all divisors of m for which the product m^2d does not exceed x; and $\mu(d)$ is the Möbius function, whose absolute value is 1 for squarefree arguments, zero otherwise. Changing the order of summation, we have:

$$A(x) = \sum_{d \leqq x^{1/3}} \sum_{m} | \mu(d) |,$$

where the inner sum now extends over all multiples of d for which the product m^2d does not exceed x. There are $[(x/d^3)^{1/2}]$ such multiples. Hence:

$$\begin{aligned} A(x) &= \sum_{d \leqq x^{1/3}} | \mu(d) | \, [(x/d^3)^{1/2}] \\ &= x^{1/2} \sum_{d \leqq x^{1/3}} | \mu(d) | \, d^{-3/2} + O(x^{1/3}) \\ &= x^{1/2} \sum_{d=1}^{\infty} | \mu(d) | \, d^{-3/2} \{ 1 + O(x^{-1/6}) \} + O(x^{1/3}) \\ &= x^{1/2} \sum_{d=1}^{\infty} | \mu(d) | \, d^{-3/2} + O(x^{1/3}). \end{aligned}$$

The coefficient of $x^{1/2}$ may be written in several forms, since

$$| \mu(d) | = \prod_{p} (1 + p^{-3/2}) = \frac{\zeta(3/2)}{\zeta(3)}.$$

Also solved by Robert Breusch, Leonard Carlitz, H. N. Shapiro, Abe Sklar (a second solution), and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Advanced Calculus. By Wilfred Kaplan. Cambridge, Mass., Addison-Wesley Press, Inc., 1952. 13+679 pages. $8.50.

This book which covers the subject matter generally included in a course in advanced calculus contains sufficient material for a course of two, three, or even four semesters. After a brief review of algebra, analytic geometry and calculus, the author devotes chapters 1, 3 and 5 to a notable treatment of vectors, and chapters 2 and 4 to the differential and the integral calculus respectively of functions of several variables. Chapter 6 contains a particularly fine treatment of series, chapter 7 is concerned with Fourier series and orthogonal functions, chapters 8 and 10 with ordinary and with partial differential equations respectively, while chapter 9 contains a long (131 pages) and excellent treatment of complex variable theory. Scattered through the book are many problems, ranging from the simple and straightforward to those which will tax the best students.

The author has an excellent style: definitions are clearly stated, new concepts are discussed and well motivated, important results are generally presented in the form of theorems, proofs are lucid and contain a sufficient amount of detail, theorems are stated formally in italics before their proofs are given, and the hypotheses of a theorem are never buried in the body of the proof.

Aside from a few typographical errors (pp. 23, 221, 276, 429) and a misleading drawing (fig. 1–16), the reviewer found only three statements in the book to which he can object, and these are trivial matters of phraseology. They are the statements on p. 56: "the vector $d\mathbf{u}/dt$ represents the tangent to the curve . . . ", on p. 195: "R_1 and R_2 overlapping only at boundary points," and on p. 307: "In general, the lower limit cannot exceed the upper limit". The dearth of errors of all kinds and the excellence of the format redound to the credit of both author and publisher and should greatly enhance the popularity of the book.

For the student planning to specialize in mathematics the book has a serious deficiency. He receives no training in the ϵ, δ technique, learns nothing about the real number system or the fundamentals of the Riemann integral, and does not encounter the Heine-Borel theorem or the Bolzano-Weierstrass theorem. The statement in the preface that "A competent teacher can easily fill in these gaps, if so desired, and thereby present a complete course in real analysis" should not go unchallenged. A typical engineering student, on the other hand,

who is seeking some mathematics beyond the calculus will probably regard the book as too theoretical in spite of its applications, its numerical methods, and its extensive treatment of vectors.

L. J. GREEN
Case Institute of Technology

Advanced Statistical Methods in Biometric Research. C. R. Rao. New York. John Wiley & Sons, Inc. 1952. xvii+390 pp. $7.50.

Many books dealing with statistics emphasize either statistical theory (for those who are interested in the mathematical and logical bases of the subject) or statistical methods (for those who are interested primarily in the applications of statistics to some particular field). This cleavage of the subject is partly due to the training and interests of the authors, partly due to the backgrounds and interests of the two groups of readers, and partly due to the fact that a thorough development of the theory, an adequate explanation of its application, and a suitable demonstration of appropriate computations are apt to result in a work which is bulky and which lacks the desired unity of style.

The purpose of the author, as indicated in the preface and as revealed throughout the volume, was "to present a number of statistical techniques, keeping in view the requirements of both the student who questions the basis of a particular method employed and the practical worker who seeks a recipe for the reduction of data." He has hence provided whatever theoretical material (usually of a mathematical nature) was necessary for a precise statement and mathematical solution of each problem, and, in many cases, detailed computations illustrating the various steps in the numerical solution.

The book, as its title suggests, is primarily in the "methods" class rather than in the "theory" class, though the theory is by no means neglected. The first 85 pages are devoted to necessary mathematical and statistical material with little reference to "methods." The rest of the book is in a general way devoted to "methods" though additional suitable theoretical material (including several appendices) is included as needed. The "methods" part of the book features historical background, with lists of appropriate references and generous discussions of the nature of the problems under consideration, as well as the recommended solution.

The book is written primarily for workers in biometric research but will be of value to workers in other fields who are interested in the general area of multivariate analysis and the particular area of discriminatory analysis.

The material of the last two chapters dealing with the use of multiple measurements in problems of biological classification is of especial interest. An objective method which minimizes the errors of classification is introduced in the first of these chapters. The development leads to a problem which is similar to such linear programming problems as the Hitchcock-Koopmans transportation problem and the problem of differential prediction. The final chapter features discriminatory topology, which is useful in ordering statistical groups, and in

general the interrelationships between a number of populations or groups of individuals. Especially featured is the concept of group distance, introduced by Mahalanobis.

Each chapter has its own list of references. There are no problems in each chapter, except for the author's illustrations, but a list of 20 miscellaneous problems, most of which are of a theoretical nature, is given as an appendix.

The mathematician may be especially interested in the first 85 pages where the basic multivariate material through the derivation of the Wishart distribution is presented in concise form.

This book seems to the reviewer to be a valuable addition to the literature of statistical methods and statistical theory. The author has achieved considerable success in unifying the study of mathematical distributions, statistical inference, and computational methods and in presenting the result in a single volume.

P. S. Dwyer
University of Michigan

Elementary Differential Equations. By E. D. Rainville. The Macmillan Company, 1952. xii+392 pp. $5.00.

The text "A Short Course in Differential Equations" by the same author and publisher has been extended and supplemented to form the new "Elementary Differential Equations." This was done by using all of the plates from the Short Course, with no revision, to form the first thirteen chapters of the new text. The author, no doubt, saw a need for some revision and supplementing in this material and at least partially accomplished this by the appendage of three supplements. This procedure was a desirable economy from the publisher's point of view.

The introduction of elementary concepts in differential equations to the average, or lower-ability, student is accomplished in a remarkable manner. Necessary theory to give this student a proper concept is very clearly illustrated, avoiding the many fine points that would add to his confusion. Plenty of exercises are available to assure proficiency in the mechanics of solving differential equations. The inclusion of the Miscellaneous Exercises, all word problems, at the end of Supplement A is certainly a welcome addition.

The new chapters, fourteen through twenty-one, are written with the same care and viewpoint. An average student should be able to continue through this text and end with as many "right ideas" as could be expected for an introductory course.

Teachers will, in any course, want to introduce their own peculiar "likes" in the presentation of theory or new subject matter, and some are at times inclined to mention their "dislikes." I find the latter urge less frequent in this text. The answer to practically every problem is given with the problem. Some would prefer that the answers be listed at the end of the book; others that only

half of the answers be given. We all agree, I think, that where the answers are given, they be correct. To find a wrong answer in this text should almost be considered an accomplishment.

K. H. STAHL
University of Colorado

Econometrics. By Gerhard Tintner. New York, John Wiley and Sons. 1952. 370 pages. $3.75.

Econometrics, as the empirical counterpart of economic theory, bears about the same relation to mathematical economics as does experimental physics to mathematical physics. But controlled experiment is replaced by the collection and analysis of economic data, and statistical techniques rather than apparatus are the tools of observation. Whereas mathematical economics is over a century old, econometrics is a development of the last few decades and is now mushrooming with the encouragement of business, government, and the military. It is, inevitably, in a somewhat chaotic state and in need of summary and synthesis. Professor Tintner's book is a welcome contribution to answering this need and a very useful survey of the development and current status of econometrics.

Without attempting to be exhaustive, Prof. Tintner carefully discusses a considerable number of important methods, illustrates them with examples chosen from the literature or worked out by himself from real data, and gives extensive bibliographic references. The first part, which occupies only about one-fifth of the book, gives an introduction that presupposes only some acquaintance with economic theory, and, except for a short sketch of regression methods in Chapter 2, only a minimal mathematical background. The remainder of the book, devoted to multivariate analysis and time series, requires background in modern economics, calculus, and mathematical statistics. There is an appendix on elementary matrix theory and related numerical methods.

One of the best features of the book is that the author frankly recognizes and explicitly mentions the difficulties that arise from lack of data and of mathematical tools to handle even such data as are available. He points to the deviations of assumptions from actual conditions, indicates the resulting approximations and drawbacks, and suggests the required cautions. Particularly in the field of time series analysis there are numerous interesting indications of work to be done. This mature approach coupled with the worked out examples and references makes the book unusually and commendably alive and closely related to current significant work.

Although the writing is generally quite clear, there are a few places where terms are defined only by context and others where the reader is left to puzzle out important distinctions. For example, "identification" and "overidentification" are not explicitly defined in their technical senses in Chapter 7, and the purposes of discriminant analysis have to be distinguished and inferred from examples in Chapter 6. There are unsatisfactory statements about the solution

of linear homogeneous equations on pages 104, 116, 127, and 338—the first and last resulting from misplacing "only." The definition of a scalar as "a quantity which is not a matrix" on page 333 is unfortunate. Calculations to as many as nine significant digits (p. 211) seem inappropriate where both data and method are crude. Readers with strong views on the foundations of probability may be irritated by the author's ambiguous position based on a leaning toward the ideas of Keynes and Jeffreys coupled with a recognition that the frequency approach is at present the only usable tool. The complete lack of diagrams is surprising since there are many places where they would help the reader. These, however, are minor weaknesses and do not alter this reviewer's opinion that the book is a fine job, the best available introductory survey of econometrics, and an essential reference for anyone working in the field.

K. O. May
Carleton College

NEW BOOKS RECEIVED

Sir Isaac Newton, Opticks. Foreword by Albert Einstein. Introduction by Sir Edmund Whittaker, Preface by I. B. Cohen. New York, Dover Publications. $1.90.

Algebraic Projective Geometry. By J. G. Semple and G. T. Kneebone. New York, Oxford University Press, 1952. $7.00.

Linear Algebra and Matrix Theory. International Series in Pure and Applied Mathematics. By R. R. Stoll. New York, McGraw-Hill Book Co., 1952. $6.00.

Practical Calculus, Revised Second Edition. By C. I. Palmer and C. E. Stout. New York, McGraw-Hill Book Company, 1952. $6.00.

A Guide to Tables of the Normal Probability Integral (Applied Mathematics Series 21). By National Bureau of Standards. August 1952, 16 pages, 15 cents.

Differential Equations. By Robert C. Yates. New York, McGraw-Hill Book Co., 1952. vii+215 pages. $3.75.

Philosophy of Natural and Mathematical Sciences. By Sister M. Helen Sullivan. New York, Vantage Press, Inc., 230 W. 41 Street. 1952. $3.75.

Mental Prodigies. By Fred Barlow. New York, Philosophical Library, 1952. 256 pages. $4.75.

Description of a Magnetic Drum Calculator (Annals, vol. 25). By the Computation Laboratory of Harvard University, 1952. 318 pages. $8.00.

General College Mathematics, First Edition. By W. L. Ayres, C. G. Fry and H. F. S. Jonah. New York, McGraw-Hill Book Co., 1952. xii+283 pages. $3.75.

Methods of Applied Mathematics. By F. B. Hildebrand. New York, Prentice Hall, 1952. xi+523 pages. $7.75.

Fundamental Procedures of Financial Mathematics. By Merrill and Irene Rassweiler. New York, The MacMillan Company, 1952. 8+260 pages. $3.25.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

SUMMER WORK FOR MATHEMATICS STUDENTS

The Board of U. S. Civil Service Examiners for Scientific and Technical Personnel of the Potomac River Naval Command announced on October 28, 1952, an examination for Student Aid Trainee in Chemistry, Metallurgy, Physics, Mathematics, or Engineering. This examination will be used to recruit college students for both a cooperative education plan and a summer employment plan. In the summer employment plan, a student is employed only during the summer vacation period, and attends college during the entire regular college term. For further information, see Announcement No. 4-34-4 (1952) available at any first or second-class post office, except in cities where a U. S. Civil Service Regional Office is located.

CLUB NEWS

(Clubs are invited to contribute newsworthy items.)

Kappa Mu Epsilon, national honorary mathematics society, reports the following chapter activities:

Alabama Beta, State Teachers College, is making plans to be co-hosts with the College for a meeting of the Alabama Teachers of College Mathematics.

Indiana Alpha, Manchester College, prepared an exhibit of mathematical equipment and models.

Kansas Alpha at State Teachers College, Pittsburg, and Missouri Alpha at Southwest Missouri State College, Springfield, have arranged an exchange of programs. On April 17, four student papers were given at Springfield by members of Kansas Alpha. This Fall a similar program will be given at Pittsburg by members of Missouri Alpha.

Michigan Gamma, Wayne University, announces the following Mathematics Awards on the basis of a competitive examination: Richard Pauley, first place; Joseph Garrity, second place; Duane Morrow, third place; Robert Reibel and David Morrison, honorable mention. William Shulevitz and Max Krolik were awarded student memberships in the Mathematical Association of America.

Miss Zelia Zulauf of Missouri Beta, Central Missouri State College, Warrensburg, presented a paper, *Mathematics and music*, at the St. Louis meeting of the Missouri Academy of Science.

Morris Rosen was awarded the prize given by New York Alpha, Hofstra College, to the student ranking highest in the first year of mathematics.

Ruth Rickloff of Pennsylvania Alpha, Westminster College, was awarded the K.M.E. prize. This prize is presented annually to a second-year student in mathematics on the basis of his entire academic record.

Tennessee Alpha, Polytechnic Institute, Cookeville, awarded the Mathematics Medal to Ira F. Grissom at the 1952 graduation exercises.

O. U. MATH LETTER

The University of Oklahoma Chapter of Pi Mu Epsilon is sponsoring the *O. U. Mathematics Letter*, a four-page mimeographed publication designed particularly for high-school students. It is sent free about twice a semester to mathematics teachers who request it. The issue of September, 1952, contains brief discussions of "Angle trisection" and "What mathematics to take in high school," as well as a "Problem Box" and other items of interest.

Many other clubs sponsor similar publications. This department would be pleased to receive sample copies so that a check list of the various club periodicals can be made.

LIGHT—BUT NOT TRIVIAL—PROBLEM

The following problem proposed by Professor Norman Anning may be of interest: Find where these two curves intersect.

$$y = x(x-1)(x-2)(x-3), \qquad y = (x^2 - 3x + 1)^2.$$

"The student who cheerfully eliminates y will at the same time eliminate x and get a surprise. The curves do not intersect at any finite point. At infinity on $x=0$, both curves have the same sort of super-singularity which $y^3=x^4$ has at the origin."

ON SOLVING CUBIC EQUATIONS

NORMAN ANNING, University of Michigan

We teach two methods for solving cubic equations. It is the purpose of this note to smooth the transition from one method to the other. What Bombelli in 1572 called *casus irreducibilis* remains irreducible in the sense that a cubic which has three real roots cannot be solved by algebra alone.

It is suggested that, after a cubic has been prepared for solution, it be identified with

$$x^3 - 3m^2x - 2n^3 = 0.$$

The roots of this are $p+q$, $p\omega+q\omega^2$, $p\omega^2+q\omega$, where

$$\omega^2 + \omega + 1 = 0 \tag{1}$$

$$p^3 = n^3 + \sqrt{n^6 - m^6} \tag{2}$$

$$q = m^2/p. \tag{3}$$

There are three real roots if $n^6 < m^6$. In this "case" let us put $n^3 = m^3 \cos 3A$. The equation becomes the trisection equation and the student by a simple application of DeMoivre's theorem can transform the roots given by "Cardan" into

$$x_1 = 2m \cos A, \qquad x_2 = 2m \cos (A + 120°), \qquad x_3 = 2m \cos (A + 240°).$$

For instance,

$$x = \sqrt[3]{n^3 + \sqrt{n^6 - m^6}} + \sqrt[3]{n^3 - \sqrt{n^6 - m^6}}$$

becomes

$$x/m = \sqrt[3]{\cos 3A + i \sin 3A} + \sqrt[3]{\cos 3A - i \sin 3A} = 2 \cos A.$$

Please note that, if we put $x = ny$ in our chosen form of cubic, we shall get the equation treated on pages 20–30 of the addenda to the Jahnke-Emde *Tables*.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MICHIGAN MATHEMATICAL JOURNAL

A new medium for publication of mathematical research, the *Michigan Mathematical Journal*, has been initiated. The editorial board, formed of members of the faculty of the University of Michigan, is as follows: R. Brauer, W. Kaplan, E. Moise, G. Y. Rainich, R. L. Wilder. The Journal is published by the University of Michigan Press and will appear in lithoprinted form. Inasmuch as this necessitates close cooperation between author and typist, publication will in general be restricted to articles of authors living in or near Ann Arbor.

The Journal will appear semi-annually, two issues making one volume. Volume 1, No. 1, dated January, 1952, is now ready. The subscription price is $4.00 per year, with the special rate of $2.00 per year for individual research mathematicians. All inquiries and subscriptions should be addressed to Michigan Mathematical Journal, Mathematics Department, 3012 Angell Hall, University of Michigan, Ann Arbor, Michigan. Checks should be made payable to University of Michigan Press.

PERSONAL ITEMS

Professor H. J. Ettlinger of the University of Texas was the representative of the Association at the inauguration of President J. W. Laurie of Trinity University on October 8, 1952.

Professor J. C. Polley of Wabash College represented the Association at the inauguration of President R. J. Humbert of DePauw University on October 18, 1952.

Professor H. W. Syer of Boston University was appointed to represent the

Association at the One Hundredth Anniversary of the Founding of Tufts College on October 11, 1952.

Professor A. E. Pitcher has been awarded a John Simon Guggenheim Memorial Foundation Fellowship and is on leave of absence from Lehigh University during the current academic year.

Duke University makes the following announcements: Associate Professor F. G. Dressel has been promoted to a professorship; Mr. W. L. Gordon and Mr. J. R. Shoenfield have been appointed to instructorships; Professor W. W. Rankin has retired with the title of Professor Emeritus.

At Illinois Institute of Technology: Associate Professor Haim Reingold has been appointed Acting Chairman of the Department of Mathematics; Dr. Allen Devinatz has been granted a leave of absence to accept a National Research Fellowship and will spend a year at the Institute for Advanced Study.

Michigan State College reports the following: Assistant Professor K. J. Arnold of the University of Wisconsin has been appointed to an associate professorship; Dr. Henry Parkus, recently consultant in aerodynamics with the National Pneumatic Company, Boston, has been appointed to an assistant professorship; Professor H. L. Harter has resigned to accept a position at the Wright-Patterson Air Force Base.

Montana State University announces: Professor A. S. Merrill, formerly chairman of the Department of Mathematics, has been appointed Dean of the Faculty and Dean of the College of Arts and Sciences; Professor Harold Chatland has been named Chairman of the Department; Dr. W. M. Meyers, Jr., formerly an assistant instructor at Ohio State University, has been appointed to an assistant professorship; during the 1952 Summer Session Professor Paul Reichelderfer of Ohio State University was a visiting professor in the Department.

Rutgers University makes the following announcements: Assistant Professor M. G. Galbraith has been promoted to an associate professorship; Teaching Assistant John Bender has been promoted to an instructorship; Dr. Solomon Leader, formerly a research assistant at Princeton University, Mr. B. H. McCandless, previously a teaching assistant at Indiana University, Mr. R. E. Montgomery who has been with the Atomic Energy Commission, Albuquerque, New Mexico, Dr. V. L. Shapiro, formerly a teaching assistant at the University of Chicago, Mr. H. G. Tucker, a teaching assistant at the University of California, Berkeley, and Dr. K. G. Wolfson, formerly a university fellow at the University of Illinois, have been appointed to instructorships; Dr. R. K. Brown, formerly a teaching assistant at the University, is now a mathematician with the United States Signal Corps, Belmar, New Jersey.

University of Illinois announces: Professor Reinhold Baer is on sabbatical leave during 1952–53 and is traveling in Europe; Assistant Professors E. J. Scott and H. E. Vaughan have been on sabbatical leaves during the first semester of 1952–53; Assistant Professors P. T. Bateman and Josephine Mitchell have received research grants from the National Science Foundation for 1952–53;

Assistant Professor M. K. Fort, Jr. has received a Ford Foundation grant for 1952–53 and is on leave of absence for the year; Dr. Arno Jaeger of Goettingen, Germany, who has been teaching recently in Ibadan, Africa, and Dr. Chung-Tao Yang, recently at Tulane University, have been appointed Research Associates during 1952–53.

University of Mississippi announces the following: With the help of a grant from the General Education Board, the Division of Arts and Sciences of the University of Mississippi is beginning a five-year program for the improvement of the undergraduate work in this division. One of the five points of the program is concerned with special attention to the superior student. The work will be distinctly supplementary to the regular routine program. To help implement this program in mathematics, the University announces the appointment of Professor Lester R. Ford as Professor of Mathematics for the second semester of the 1952–53 session.

At the University of New Hampshire: Dr. R. M. Conkling of the University of Florida has been appointed to an assistant professorship; Mr. S. B. Hobbs, Mr. R. E. Hux, Mr. A. R. Lamontagne, and Miss Elizabeth A. Stone have been appointed to part-time instructorships.

Wayne University reports the following: Dr. H. D. Huskey, formerly an assistant director at the Institute for Numerical Analysis, Los Angeles, California, has been appointed to a professorship; Associate Professor Yu Why Chen of the University of Oklahoma and Professor Casper Goffman, University of Oklahoma, have been appointed to associate professorships; Miss Winifred Burroughs has been appointed to an instructorship.

Mr. J. B. Bartoo of State University of Iowa has been appointed to an assistant professorship at Pennsylvania State College.

Assistant Professor W. N. Birchby of the California Institute of Technology has retired.

Mrs. Barbara B. Blair, formerly a part-time instructor at State University of Iowa, has been appointed to an instructorship at Smith College.

Professor A. H. Blue of Culver-Stockton College has been appointed to an associate professorship at Cornell College.

Mr. L. F. Boron, previously a geophysicist with the Navy Department, Washington, D. C., has been appointed to an assistant professorship at Norwich University.

Dr. H. D. Brunk who has been with the Sandia Corporation, Albuquerque, New Mexico, has been appointed to an associate professorship at the University of Missouri.

Assistant Professor V. W. Burrows of Northeastern State College, Oklahoma, has resigned to become Coordinator of Secondary Education of City Schools, Tahlequah, Oklahoma.

Mr. R. G. Buschman, formerly a part-time instructor at the University of Colorado, has been appointed to an instructorship at McNeese State College.

Mr. L. G. Campbell is now on active duty with the United States Air Force and has been assigned to an instructorship at the United States Naval Academy.

Mr. W. L. Carter of Western Illinois State College has accepted a position with the Department of Education, Guam, Mariana Islands.

Professor P. F. Cauffman, formerly Head of the Department of Mathematics of Maryland State Teachers College, Salisbury, has been appointed Chairman of the Department of Mathematics of State Teachers College, Shippensburg, Pennsylvania.

Assistant Professor R. S. Christian of Georgia Institute of Technology has been appointed to an assistant professorship in the Atlanta Division of the University of Georgia.

Mr. D. E. Coffey who has been a graduate student at Montana State University is now Head of the Department of Mathematics of Saline High School, Michigan.

Assistant Professor D. A. Darling is on leave of absence from the University of Michigan during 1952–53 and has a position as Visiting Assistant Professor in the Department of Mathematical Statistics, Columbia University.

Mr. B. C. DeLoach, previously a student at Alabama Polytechnic Institute, has been appointed to a teaching fellowship at Ohio State University.

Professor W. L. Duren, Jr. of Tulane University has been serving as Acting Program Director for Mathematics for the National Science Foundation.

Dr. M. P. Emerson, previously a graduate assistant at the University of Illinois, has been appointed to an assistant professorship at Harpur College of the State University of New York.

Mr. J. E. Forbes of Bradley University has been appointed to a graduate assistantship at Purdue University.

Mrs. Virginia Forbes who has been teaching at Community Unit High School, Heyworth, Illinois, has been appointed to an assistantship at Purdue University.

Mr. H. H. Fox, formerly an assistant at the University of Illinois, has accepted a position as Research Physicist with Mound Laboratory, Miamisburg, Ohio.

Assistant Professor Abraham Franck of Kansas State College has a position as Senior Mathematician with Engineering Research Associates, St. Paul, Minnesota.

Mr. E. T. Frankel who has been on the staff of the Health and Welfare Federation of Allegheny County, Pennsylvania, has accepted a position as Regional Research Analyst with the Bureau of Public Assistance, Federal Security Agency, New York City.

Mr. N. S. Free, previously a lecturer at the University of California, Berkeley, has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Professor H. M. Gehman, chairman of the Department of Mathematics of

the University of Buffalo, has been serving as Acting Dean of the Graduate School of Arts and Sciences during the first semester of 1952–53.

Mr. H. H. Goode has been appointed Director of the Willow Run Research Center, University of Michigan, Ypsilanti, Michigan.

Dr. L. W. Green, previously an assistant at Yale University, has been appointed to an instructorship at Princeton University.

Professor Emeritus F. L. Griffin of Reed College has a position as Visiting Professor at Wesleyan University.

Assistant Professor Simon Gruenzweig of Lincoln University, Pennsylvania, has been appointed Professor and Head of the Department of Mathematics and Physics of Philander Smith College.

Dr. B. F. Hadnot has been appointed to an instructorship at Florida State University.

Mr. D. K. Hartman of the University of Minnesota has accepted a position with General Electric Company, Syracuse, New York.

Mr. J. R. Hatcher, formerly an instructor at Howard University and more recently a graduate student at Brown University, has been appointed to an assistant professorship at Fisk University.

Mr. A. T. Hind has been appointed to an associate professorship at Clemson College.

Mr. T. R. Horton, formerly an instructor at Bolles School, Jacksonville, Florida, has been appointed to a graduate assistantship at the University of Florida.

Mr. S. L. Hull who was a research participant at Oak Ridge National Laboratory has been appointed to an assistant professorship at The Citadel.

Mr. H. F. Hunter who has been with the Northrop Aircraft Company, Hawthorne, California, has accepted a position as a mathematician at the United States Naval Radiological Defense Laboratory, San Francisco, California.

Assistant Professor L. S. Laws of the University of Minnesota has been appointed to a graduate assistantship in the School of Education, Michigan State College.

Associate Professor Joseph Lehner of the University of Pennsylvania is on leave of absence during the current academic year and is engaged as Staff Member at the Los Alamos Laboratory, New Mexico.

Mr. Stanislaw Leja has accepted a position with the Ford Motor Company, Buffalo, New York.

Associate Professor Caroline A. Lester of New York State College for Teachers, Albany, has been promoted to a professorship.

Dr. Octave Levenspiel, formerly a research engineer at the University of California, has been appointed to an assistant professorship in the Department of Chemical Engineering, Oregon State College.

Assistant Professor P. T. Mielke of Wabash College has a position as Stress Analyst with Boeing Airplane Company, Seattle, Washington.

Mr. D. G. Miller, previously a graduate student at the University of Illinois,

has been appointed to an assistant professorship in the Department of Chemistry of the University of Louisville.

Dr. Morris Ostrofsky has resigned from his position as Chairman of the Department of Mathematics of Duquesne University to accept a position as Advisory Scientist to the Atomic Power Division of the Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Professor W. R. Ransom of Tufts College has retired with the title of Professor Emeritus after fifty years of service.

Mr. O. M. Rasmussen of the University of Kansas has been appointed to an assistant professorship at the University of Denver.

Mr. J. G. Renno, Jr., formerly a graduate assistant at the University of Wisconsin, has been appointed to an instructorship at the Milwaukee Extension of the University.

Dr. E. K. Ritter, formerly of the Willow Run Aeronautical Research Center, Ypsilanti, Michigan, has accepted a position as Director of Computation and Ballistics, United States Naval Proving Ground, Dahlgren, Virginia.

Associate Professor G. G. Roberts of Berea College has been promoted to a professorship.

Mr. P. C. Rogers, previously a graduate assistant at the University of Maryland, has accepted a position as a mathematician with the United States Air Force, Washington, D. C.

Professor Helen G. Russell of Wellesley College has been named Chairman of the Department of Mathematics.

Sister Mary Esther, formerly of Mundelein College, is now at St. Vincent Convent, Petaluma, California.

Mr. W. L. Stamey of the University of Missouri has been appointed to an assistant professorship in the Atlantic Division of the University of Georgia.

Professor C. G. Stipe of the Michigan College of Mining and Technology was engaged during the summer of 1952 by the Calumet and Hecla Consolidated Copper Company, Michigan.

Dr. D. L. Thomsen, previously of the Jet Propulsion Laboratory, Pasadena, California, has been appointed to an assistant professorship at Pennsylvania State College.

Dr. Peter Thullen has accepted a position in the International Labor Office, Geneva, Switzerland.

Mr. R. F. Tidd of Canisius College has been promoted to an assistant professorship.

Mr. C. H. Tross, previously a graduate assistant at the University of Illinois, has a position as a mathematician with the Wright Air Development Center, Dayton, Ohio.

Dr. W. R. Van Voorhis of Fenn College has been appointed Visiting Professor of Engineering Administration at Case Institute of Technology; he is also a member of the Case Operations Research Group.

Mr. C. R. Wampole, formerly a student at St. John's University, has been

appointed to an instructorship at Eastern Military Academy, Cold Spring Harbor, New York.

Mr. Chih-Yi Wang who has been a teaching assistant at the University of Minnesota has been appointed to an associate professorship at Hampton Institute.

Dr. Albert Wolinsky of New York University has accepted a position as Assistant Physicist with Farrand Optical Company, New York City.

Associate Professor Emeritus E. F. A. Carey of Montana State University died on May 19, 1952. He was a charter member of the Association.

Professor Emeritus E. V. Huntington of Harvard University died on November 25, 1952. He was a charter member and third President of the Association.

Mr. D. F. Peterson of Utah State Agricultural College died in September, 1952.

Associate Professor J. H. Pitman of the Department of Mathematics and Astronomy, Swarthmore College, died on September 23, 1952.

Professor Emeritus T. R. Running of the University of Michigan died on October 10, 1952. He was a charter member of the Association.

Professor Emeritus W. J. Rusk of Grinnell College died on September 10, 1952. Professor Rusk was a charter member of the Association.

Professor Otto Szasz of the University of Cincinnati died in Switzerland on September 19, 1952.

Professor Marie J. Weiss of Newcomb College, Tulane University, died on August 19, 1952. She was a member of the Board of Governors and served as an associate editor of the MONTHLY during 1940–46.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

COMBINED MEMBERSHIP LIST

The Mathematical Association of America and the American Mathematical Society are planning hereafter to publish a *Combined Membership List* in place of the separate lists which have been issued in the past. The 1952 edition will be sent only to members of the American Mathematical Society.

Members of the Association who are not members of the Society and who wish to purchase copies of the list may do so at $2.00 per copy from the office of the American Mathematical Society, 80 Waterman Street, Providence 6, Rhode Island.

HARRY M. GEHMAN
Secretary-Treasurer

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 74 persons have been elected to membership by the Board of Governors on applications duly certified.

T. A. ABOUHALKAH, Bacc. in Math. (France) Grad. Student, Petroleum Engineering, University of Texas.

W. P. ANDERSON, Student, Macalester College.

M. A. ARREOLA, M.S. (Philippines) Instr., Mapua Institute of Technology, Manila, Philippines.

J. L. BAGG, M.S. (Illinois) Grad. Assistant, Michigan State College.

R. E. BAYLES, B.A. (Brown) Grad. Student, Harvard University.

J. S. BERGEN, B.A. (New Jersey S.T.C., Montclair) 626 Joralemon Street, Belleville, N. J.

C. R. BERNDTSON, B.A. (Bridgeport) Mathematician, Nuclear Development Associates, White Plains, N. Y.

R. H. BOYER, Student, Carnegie Institute of Technology.

E. R. BRENTZEL, B.A. (Minnesota) Cartographer, Army Map Service, Washington, D. C.

R. A. BROMAN, M.A. (Indiana) Chairman, Department of Mathematics, Mishawaka High School, Ind.

J. C. BROOKS, M.A. (Georgia) Asst. Professor, Georgia Institute of Technology.

AZELLE BROWN, M.A. (Columbia) Instr., Hofstra College.

S. H. COHN, M.A. (Toronto) Instr., Fournier Institute of Technology, Lemont, Ill.

C. C. CRELL, M.S. (Miami) Instr., Miami University, Oxford, Ohio.

L. J. DIXON, M.S. (Oklahoma A. & M.) Asst. Professor, Arkansas State College.

MARTHA E. EDWARDS, Ph.D. (North Carolina) Teacher, Campbell College, Buies Creek N. C.

F. M. ELLIS, M.Ed. (Pittsburgh) Acting Dean, Engineering School, Youngstown College.

L. A. ELROD, M.A. (Kansas City) Mathematician, White Sands Proving Ground, Las Cruces, N. M.

M. A. FAMIGLIETTI, M.A. (Oklahoma) Mathematician, Aberdeen Proving Ground, Md.

D. L. FRIED, Student, Rutgers University.

A. S. GREGORY, Computer, Sandia Corporation, Albuquerque, N. M.

W. R. GRUBB, B.A. (Buffalo) Grad. Student, University of Buffalo.

FRED GRUENBERGER, M.S. (Wisconsin) Computing Service Project Supervisor, University of Wisconsin.

O. G. HARROLD, JR., Ph.D. (Stanford) Professor, University of Tennessee.

J. W. HOLLINGSWORTH, M.S. (Wisconsin) Assistant in Computing, University of Wisconsin.

R. W. HUFF, Student, College of Wooster.

JACK INDRITZ, M.S. (Chicago) Instr., University of Minnesota.

J. A. JACOBS, M.Ed. (Duke) Registrar; Head, Department of Mathematics, Pembroke State College, N. C.

BROTHER CYPRIAN JOHN, Ph.D. (Catholic) Instr., Manhattan College.

H. J. JOHNSON, B.S. (Oklahoma A. & M.) Engineer, American Telephone and Telegraph Company, St. Louis, Mo.

C. E. JONES, M.S. (Michigan) Asso. Professor, Engineering, Tennessee Agricultural and Industrial State College.

R. H. JONES, JR., M.S. (Union C.) Head Engineer, Navy Department, Bureau of Ships, Washington, D. C.

S. T. KAO, Ph.D. (Catholic) Asst. Professor, St. Joseph's College, Albuquerque, N. M.

A. A. KARWATH, M.S. (Iowa S. C.) Instr., St. Ambrose College.

M. L. KEEDY, M.A. (Nebraska) Asst. Professor, Physics, North Dakota State College.

GEORGE KLEIN, Ph.D. (Chicago) Asst. Professor, Mount Holyoke College.

T. R. KNAPP, B.A. (Rochester) Grad. Student, University of Rochester.

L. C. LABOWITZ, Student, University of Maryland.

H. S. LEONARD, JR., B.S. (Michigan S. C.) Grad. Student, Harvard University.

OCTAVE LEVENSPIEL, Ph.D. (Oregon S. C.) Asst. Professor, Chemical Engineering, Oregon State College.

T. A. LOVE, Ph.D. (N.Y.U.) Professor and Head, Department of Mathematics, Tennessee Agricultural and Industrial State College.

E. G. MACE, Toolmaker, Chevrolet Division of General Motors, Tonawanda, N. Y.

B. J. MARKS, M.S. (Iowa S. C.) Research Assistant, Merrill Project, University of Illinois.

MRS. DELEPHARE B. MAYHALL, M.A. (Mississippi) Head, Department of Mathematics, Itawamba Junior College, Fulton, Miss.

J. G. MCGOUGH, Student, School of General Studies, Columbia University.

BROTHER J. G. MCKENNA, M.A. (Columbia) Dean, Iona College.

R. J. MERCER, B.A. (California) Ensign, United States Navy, San Francisco, Calif.

W. K. MOORE, Ph.D. (Kansas) Asst. Professor, Albion College.

MRS. VERA T. MORRIS, B.A. (DePauw) Instr., Purdue University.

LOIS M. NEFF, M.A. (Minnesota) Teacher, Rochester Senior High School, Minn.

LOIS NICKCHEN, Student, Marquette University.

R. C. NICKERSON, Student, Brown University.

REV. C. F. O'CALLAGHAN, B.A. (St. Bonaventure) Instr., Siena College.

R. F. PAVLEY, Research Technician, Wayne University.

G. A. PAXSON, Student, University of Michigan.

GILBERT PUENTE, B.S. (Arizona) Mathematician, White Sands Proving Grounds, Las Cruces, N. M.

LOIS J. ROPER, M.A. (Missouri) Instr., Trenton Junior College, Mo.

F. M. SANDS, Student, Gonzaga University.

J. H. SARTAIN, M.S. (Chicago) Edwin Shields Hewitt and Associates, Libertyville, Ill.

MARLOW SHOLANDER, Ph.D. (Brown) Asso. Professor, Washington University.

WILLIAM SHULEVITZ, Student, Wayne University.

SISTER ANNA CONCILIO O'NEILL, M.A. (Columbia) Professor, College of St. Elizabeth.

S. E. SKLAR, M.S. (N.Y.U.) Instr., Brooklyn Technical High School; Instr., Engineering Cooper Union.

D. G. STECHERT, M.S. (Cincinnati) Research Engineer, Gates Rubber Company, Denver, Colo.

J. E. STEINKRAUS, Student, Marquette University.

R. L. STERNBERG, Ph.D. (Northwestern) Mathematician, Laboratory for Electronics, Boston, Mass.

E. H. STONE, B.A. (Denver) Teacher, Smiley Junior High School, Denver, Colo.

LISTON TATUM, M.A. (Northwestern) Administrative Assistant, International Business Machines Corporation, New York, N. Y.

TUNG TSANG, M.S. (Minnesota) Grad. Student, University of Minnesota.

W. T. TUTTE, Ph.D. (Cambridge) Asst. Professor, University of Toronto.

FLORENT VENNE, St. Jacques, Quebec Province, Canada.

MARY E. WILSON, B.A. (Vassar) Grad. Student, University of Chicago.

F. L. WOLF, M.A. (Washington) Instr., Carleton College.

MRS. GERALDYNE P. ZIMMERMAN, M.A. (South Carolina S. C.) Asst. Professor, South Carolina State College.

THE THIRTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The thirteenth annual William Lowell Putnam Mathematical Competition will be held on Monday, March 23, 1953. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to undergraduate students in universities and colleges of the United States and Canada who have not received a degree. The examination will consist of two parts of three hours each.

The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, 112 Albertus Magnus Hall, College of St. Thomas, St. Paul 1, Minnesota, by a postcard request. All applications must be filed with the Director not later than March 1, 1953. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to a contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are \$400, \$300, \$200, and \$100, in the order of their rank. In addition, there will be prizes of \$40, \$30, \$20 and \$10 awarded to the members of these teams according to the rank of the team; a prize of \$50 to each of the five highest contestants and a prize of \$20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual \$2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the twelve previous competitions and examinations will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, August-September, 1947, December, 1948, August–September, 1949, 1950, 1951, and October 1952.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The October meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Scholastica in Duluth, Minnesota on October 11, 1952. Sessions were held in the forenoon, at luncheon and in the afternoon. Sister M. Mercedes, Professor W. R. McEwen and Professor A. G. Swanson, Chairman of the Section, presided at the respective sessions.

Thirty-seven persons attended the meeting including the following twenty-four members of the Association:

H. M. Anderson, F. J. Arena, L. E. Bush, R. H. Cameron, H. D. Colson, K. L. Hankerson, Ruth E. Henning, J. S. Hill, R. T. John, W. C. Kalinowski, Karlis Kaufmanis, P. G. Kirmser, C. B. Lindquist, K. O. May, W. H. McBride, W. R. McEwen, Sister M. Mercedes, E. O. Nelson, J. C. Peterson, L. W. Sheridan, F. C. Smith, O. E. Stanaitis, A. G. Swanson, Takashi Terami.

By invitation of the Executive Committee, Mr. J. S. Hill, Actuary of the Minnesota Mutual Life Insurance Co., delivered an address at the morning session entitled "Changing Horizons in Certain Areas of Applied Mathematics." Abstract of this address follows:

Modern scientific development has introduced new areas of applied mathematics. The address did not consider these new areas so much as it did the changes in actual mathematical methods themselves being wrought by modern computational and data-handling equipment. The nature of modern computational equipment was reviewed. Specific applications of the equipment were reviewed and the general nature of the resulting changes in mathematical method were considered.

The traditional mathematical approach to life insurance reserves was reviewed and the changing concepts produced by the introduction of modern computational equipment was demonstrated.

It was concluded that professional mathematicians (1) must recognize the resurgence of the empirical approach to problem-solving, (2) must decide whether to leave the teaching emphasis on the classic analysis methods or to train more computation-minded students, and (3) may still retain one of the greatest satisfactions of the teaching profession, namely, the enriching experience of seeing young minds develop into useful contributors to our modern society.

The following short papers were presented:

1. *A problem in arrangements*, by Professor W. R. McEwen, University of Minnesota, Duluth Branch.

There are many problems in which one wishes to count a number of arrangements in which certain arrangements are excluded by the conditions of the problem. The *problème des ménages* is an example. By virtue of their definition, determinants are peculiarly adapted to this kind of counting. This was illustrated by solving a special case of the above problem and two others in one of which the exclusion was not systematic.

2. *An estimate of variations in amplitude of forced vibrations*, by Professor P. G. Kirmser, University of Minnesota.

In investigating the stability of a machine for testing metals in vibration, it was found necessary to estimate the variations in magnitude of steady state forced vibrations caused by slight hunting of the driving motor.

The system was idealized to

$$m\ddot{x} + \beta\dot{x} + kx = a \sin (\omega t + \gamma \sin pt)$$

which has the steady state solution

$$x = \sum_{n=-\infty}^{+\infty} A_n J_n(\gamma) \sin \left[(\omega + np)t + \phi_n\right].$$

Variations in amplitude are estimated by

$$M \cong \frac{|J_{-1}(\gamma)|A_{-1} + |J_{+1}(\gamma)|A_{+1}}{|J_0(\gamma)|A_0}$$

where A_n is the amplitude of the steady state vibration for the system

$$m\ddot{x} + \beta\dot{x} + kx = a\sin(\omega + np)t.$$

3. *On the history of the problem of the isochrone*, by Professor F. J. Arena, North Dakota State College.

In this paper, the author discusses the circumstances which led Leibnitz to propose the problem of the isochrone and James Bernoulli's analytical solution of the problem.

4. *Remarks on convergence of certain series*, by Professor O. E. Stanaitis, St. Olaf College.

The series

$$\sum_{n=1}^{\infty}\frac{\sin n^2\theta\sin n\theta}{n^\alpha},\quad \sum_{n=1}^{\infty}\frac{\cos n^2\theta\sin n\theta}{n^\alpha},\quad \sum_{n=1}^{\infty}\frac{\sin n^2\theta\cos n\theta}{n^\alpha},\quad \sum_{n=1}^{\infty}\frac{\cos n^2\theta\cos n\theta}{n^\alpha}$$

for $0<\alpha<1$ were considered. It was shown that the first two series converge uniformly in every interval of real values of θ. However, the third and fourth series are divergent.

5. *Three pathological functions*, by Mr. D. C. McGarvey, Carleton College, introduced by Professor K. O. May.

Three functions discontinuous on an everywhere dense set of points are presented which can be handled in an elementary and intuitively satisfying manner that lends them to classroom discussion. For rational x, $f_1(x)$ is the reciprocal of the decimal place in which x begins the first repeating cycle, $f_2(x)$ is the reciprocal of the length of the cycle and $f_3(x)$ is the reciprocal of the decimal place in which x begins the second cycle. For irrational x, all three functions are 0. All three functions are discontinuous for a rational r since arbitrarily close to r is an irrational number at which the functions are 0 yet at r they are positive. At any irrational point i, f_1 is proved discontinuous by constructing a number arbitrarily close to i at which $f_1=1$. Another number arbitrarily close to i can be constructed such that $f_2=1$ so that f_2 is also everywhere discontinuous. That f_3 is continuous on the irrationals is established by proving that in an interval there are but a finite number of points at which $f_3>\epsilon$ so that a δ can always be found excluding these points.

6. *Direct currents in infinitely long parallel wires*, by Professor P. C. Rosenbloom and Mr. T. T. Wu, University of Minnesota.

Let $D_1, D_2, \cdots, D_n$ be simply connected bounded domains with disjoint closures in the xy-plane, and let D_0 be the exterior of the union. We suppose the boundaries to be smooth. We imagine erected on $D_1, \cdots, D_n$ cylinders parallel to the z-axis representing conductors of given magnetic permeabilities μ_i and conductivities, and we assume that the exterior is a medium of permeability μ_0 and conductivity 0. We suppose that currents are running parallel to the z-axis in the conductors, that the system is in steady state, and that the total current in each cross section D_i is given. We set up integral equations for determining the field produced. The necessary and sufficient condition for the magnetic energy in each cross section to be finite is that the total current be zero. We compute the inductance coefficients and show that for non-ferromagnetic materials Maxwell's formulae in terms of the geometrical mean distance are correct to within errors of the order 10^{-5},

$$\lambda_i - 1 = 10^{-6}.$$

F. C. SMITH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring, 1953.

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA, San Francisco State College, January 31, 1953.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.

No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.

No. 4. *Projective Geometry* by J. W. Young, ix+185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii+216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 2

CONTENTS

FEBRUARY 1953

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

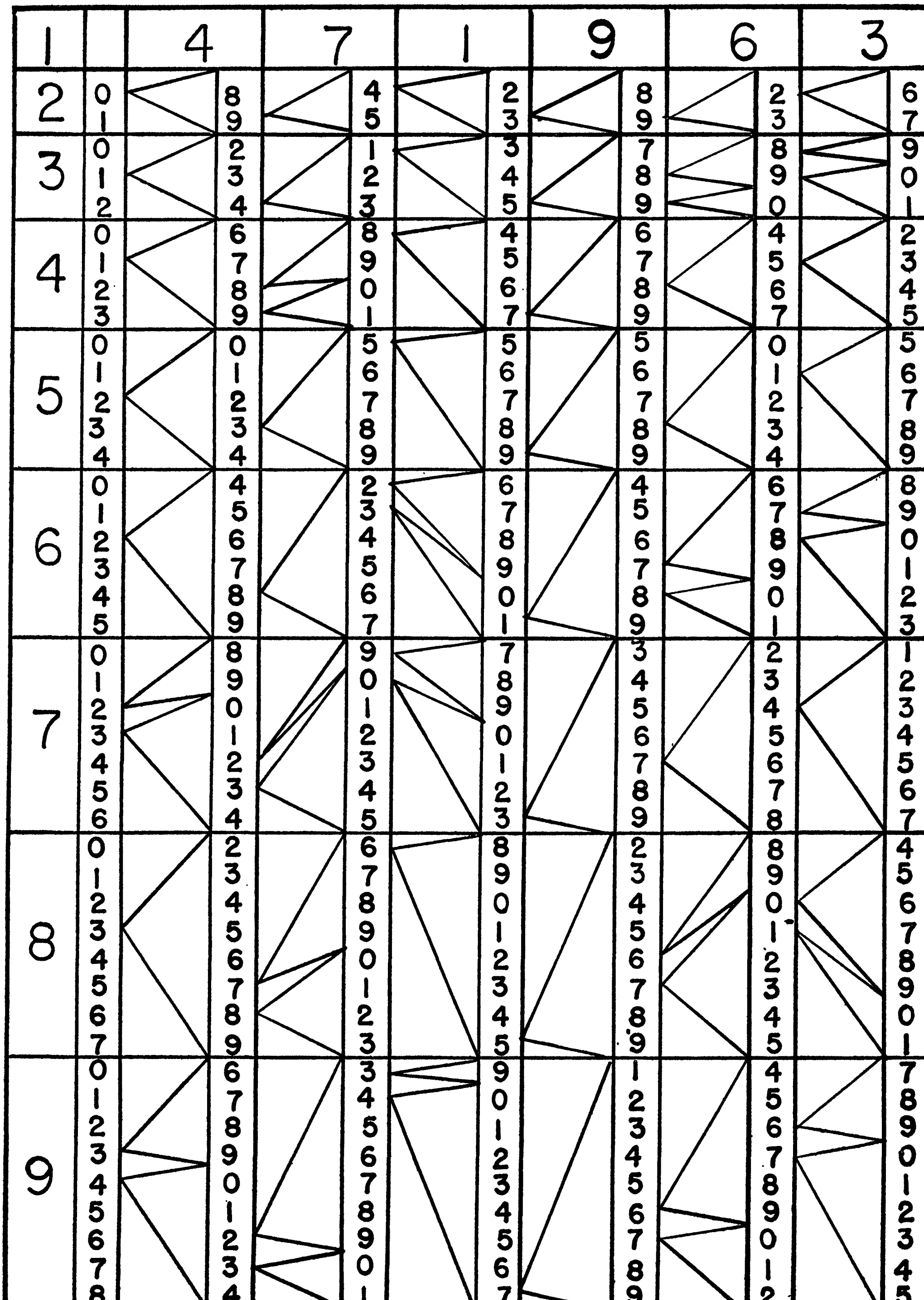
(Devoted to the Interests of College Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA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CANONICAL FORMS FOR MAPPINGS OF VECTOR SPACES

W. G. LEAVITT, University of Nebraska

In a recent paper [this MONTHLY, vol. 59, 1952, pp. 219–222]* some results were presented illustrating the way in which certain parts of the theory of matrices may be simplified by regarding them as mappings of vector spaces. In the present paper these methods are to be applied to the construction of certain canonical forms to which matrices may be reduced by transformations of type $T^{-1}AT$ (*similarity* transformations). The only added information needed here is some knowledge of the properties of the set $F[x]$ of all polynomials with coefficients in a field F; namely, (1) every two polynomials $\alpha, \beta \in F[x]$ have a greatest common divisor (α, β), (2) there exist polynomials $\gamma, \delta \in F[x]$ such that $\gamma\alpha + \delta\beta = (\alpha, \beta)$, and (3) every polynomial is uniquely factorable into a finite set of prime (irreducible in $F[x]$) polynomials.

Let A be a linear mapping of an n-dimensional vector space S over a field F. Define $A^0 = I$ (the identity mapping), and consider the successive powers $A^i(i=0, 1, \cdots)$. If $V \in S$ for which $VA^{i-1}=0$, then $VA^i=0$, so that the null space $N(A^{i-1}) \subseteq N(A^i)$. If $\nu(A)$ designates the dimension $d[N(A)]$, this implies that $\nu(A^{i-1}) \leqq \nu(A^i)$. Define $w_i = \nu(A^i) - \nu(A^{i-1})$.

LEMMA 1. *$w_i \leqq w_{i-1}$ for all $i > 1$.*

Proof. The image of S under the mapping A^{i-1} is SA^{i-1}. Thus $SA^i = SA(A^{i-1}) \subseteq SA^{i-1}$. The null space $N(A^i)$ maps under A^{i-1} into a subspace of SA^{i-1}; in fact, into that part of SA^{i-1} which goes into zero under an additional mapping by A. The image of $N(A^i)$ is thus $N(A) \cap SA^{i-1}$. It follows [Lemma 1, p. 220] that $\nu(A^i) = \nu(A^{i-1}) + d[N(A) \cap SA^{i-1}]$. But $N(A) \cap SA^{i-1} \subseteq N(A) \cap SA^{i-2}$, and hence $\nu(A^i) - \nu(A^{i-1}) \leqq \nu(A^{i-1}) - \nu(A^{i-2})$.

A set of vectors $\{V_i\} \in N(A^j)$ is said to be *j-independent* if it forms the basis of a subspace S' of $N(A^j)$ which has zero intersection with $N(A^{j-1})$. (That is, S' is a subspace of the complement of $N(A^{j-1})$ in $N(A^j)$.) Alternatively, j-independence might be defined by requiring that the image of the set $\{V_i\}$ under the mapping A^{j-1} shall be independent. (In the language of cosets, this is the same as saying that the cosets of the $\{V_i\}$ modulo $N(A^{j-1})$ are independent.)

From either form of the definition, a j-independent set must also be independent in the usual sense. In fact, it is clear that if $\{V_i\}$ and $\{W_i\}$ are respectively j-independent and k-independent ($j \neq k$), then the whole set $\{V_i, W_i\}$ is independent.

* References to this paper will be enclosed in brackets.

LEMMA 2. *For each $j>0$ there exists a set containing w_j j-independent vectors, and no such set contains more than w_j vectors.*

Proof. The maximum number of j-independent vectors is clearly the dimension of the image of $N(A^j)$ under the mapping A^{j-1}. But this is precisely w_j [Lemma 1, p. 220].

Note that an easy way to obtain a j-independent set is first to start with a basis for $N(A^{j-1})$. Then the extra vectors needed to form a basis for $N(A^j)$are automatically j-independent.

The sum of two mappings A_1 and A_2 is defined in a natural way: for any $V\in S$, $V(A_1+A_2)=VA_1+VA_2$. For a mapping A, let $F[A]$ be the set of all polynomials in A with coefficients in F. It is easily seen that $F[A]$ is isomorphic with the set $F[x]$ of all polynomials in an indeterminate x, under the correspondence $A\rightarrow x$. Thus in the following, we may adopt the nomenclature of $F[x]$, saying, for example, that a polynomial $\alpha=p(A)$ is prime if the corresponding polynomial $p(x)$ is prime (irreducible in $F[x]$), or that one member of $F[A]$ divides another if the corresponding polynomials have that relation in $F[x]$, and so on. Now let $\alpha\in F[A]$ be a prime polynomial of degree h which is also a singular mapping (it will be shown later that such a singular polynomial always exists). Since $N(\alpha)\neq 0$, the w_i relative to α are not all zero. Also, since $\nu(\alpha^j)=\sum_1^j w_i\leqq n$, the set of non-zero w_i is finite. There must thus exist some $w_j\neq 0$, with $w_{j+1}=0$. Then, by Lemma 1, $w_i=0$ for all $i>j$.

To illustrate the process to be used in constructing a canonical form for A, and to avoid complications, we will consider a special case: assuming that, relative to α, $w_3\neq 0$, $w_i=0 (i>3)$. The extension of the method to the general case will be obvious.* Without loss of generality, it may be assumed that α is monic (leading coefficient 1), so that

$$\alpha^3=A^{3h}+a_{3h}A^{3h-1}+\cdots+a_1I. \tag{1}$$

Since $w_3\neq 0$, there exists a vector $V\in N(\alpha^3)$, $V\notin N(\alpha^2)$. Suppose $\beta\in F[A]$ such that $V\beta=0$. Let α^t be the largest power of α contained in β, then $\beta=\alpha^t\beta'$. But since α is prime, this implies that $(\beta',\alpha)=1$, and so β' is relatively prime to any power of α. There thus exist $\gamma,\delta\in F[A]$ for which $\beta'\gamma+\alpha^3\delta=1$. (In the language of congruences, $\beta'\gamma\equiv 1 \pmod{\alpha^3}$.) The mapping γ applied to $V\beta$, using the fact that any two members of $F[A]$ are commutative, thus yields $V\alpha^t=0$. Since $V\notin N(\alpha^2)$, it follows that $t\geqq 3$, and so β is of degree $\geqq 3h$. It follows that the set $V_i=VA^{i-1}$ $(i=1,\cdots,3h)$ is independent, and is therefore the basis of a $3h$-dimensional subspace S_1. Now $V_iA=V_{i+1}(i=1,\cdots,3h-1)$, and using (1), $V_{3h}A=-\sum_1^{3h}a_iV_i$. Accordingly S_1 is *invariant* under A (that is, A maps S_1 into itself), the matrix of the mapping, relative to the basis $\{V_i\}$, being

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* Some of the ideas used here may also be found in B. L. van der Waerden, *Moderne Algebra*, vol. 2, Springer Berlin, 1931, pp. 135–139. See also C. C. MacDuffee, *Vectors and Matrices*, Carus Mathematical Monographs, No. 7, 1943, Chapter VI; and Paul R. Halmos, *Finite Dimensional Vector Spaces*, Princeton University Press 1942, Appendix I, pp. 159–169.

$$(2) \qquad \begin{bmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 1 \\ -a_1 & \cdot & \cdot & \cdot & -a_{3h} \end{bmatrix}$$

This is the so-called *companion matrix* of the polynomial α^3. A space for which such a basis exists will be said to be *associated* with α^3.

If for some $\beta \in F[A]$, $V\beta$ (with $V \in S_1$) maps into zero under α^2, then clearly α divides β, and so the set $\{V_i\}$ $(i=1, \cdots, h)$ is 3-independent. By Lemma 2, this means that $w_3 \geqq h$. Similarly $\{V_i\alpha\}$ and $\{V_i\alpha^2\}$ $(i=1, \cdots, h)$ are respectively 2-independent and 1-independent. Accordingly, the whole set $\{V_i, V_i\alpha, V_i\alpha^2\}$ is independent and is therefore a basis for S_1. Further, the space $S_1\alpha^2$ has dimension h, and thus at most h members of S_1 are 3-independent. It may be of interest to note that relative to the basis $\{V_i, V_i\alpha, V_i\alpha^2\}$, the matrix of the mapping of S_1 assumes another well-known canonical form.*

If $w_3 > h$, then by Lemma 2 there exists a vector W such that the set $\{V_i, W\}$ $(i=1, \cdots, h)$ is 3-independent. As in the above discussion, a vector space S_2 may then be set up with basis $W_i = WA^{i-1}$ $(i=1, \cdots, 3h)$, again mapping under A with companion matrix (2). This space also has alternative basis $\{W_i, W_i\alpha, W_i\alpha^2\}$ $(i=1, \cdots, h)$.

Now suppose it were possible to have $V\beta_1 + W\beta_2 \in N(\alpha^2)$ for some $\beta_1, \beta_2 \in F[A]$ each of degree $<h$. Since α is prime there exists $\gamma \in F[A]$ such that $\beta_2\gamma \equiv 1$ (mod α^3). Since $N(\alpha^2)$ is closed under $F[A]$ and $W \in N(\alpha^3)$, mapping by γ yields the relation $V\beta_1\gamma + W \in N(\alpha^2)$. But $V\beta_1\gamma \in S_1$ and so is expressible in terms of $\{V_i\}$ $(i=1, \cdots, h)$ together with a vector in $N(\alpha^2)$. The set $\{V_i, W\}$ would thus not be 3-independent, contrary to the choice of W. Accordingly the whole set $\{V_i, W_i\}$ $(i=1, \cdots, h)$ is 3-independent, and so also $w_3 \geqq 2h$. Similarly the sets $\{V_i\alpha, W_i\alpha\}$ and $\{V_i\alpha^2, W_i\alpha^2\}$ are respectively 2-independent and 1-independent. The whole set made up of these $6h$ vectors is therefore independent, and since it spans $S_1 + S_2$, it is a basis for that space. Further, its dimension $6h = d[S_1] + d[S_2]$, and so [Lemma 2, p. 221] $S_1 \cap S_2 = 0$.

The above process may evidently be continued as long as w_3 holds out, from which it follows that $w_3 = w_3'h$ for some integer w_3'. In the course of the process *disjoint* spaces (that is, spaces each having zero intersection with the sum of all the rest) S_k $(k=1, \cdots, w_3')$ are defined, each associated with α^3. The sum $S' = \sum_1^{w_3'} S_k$ is then a $3w_3$-dimensional space one of whose bases is made up of w_3 3-independent vectors, and a similar number of 2-independent and 1-independent vectors. Further, each S_k is invariant under A (each associated with α^3), and thus the matrix of the mapping of S' consists of blocks of submatrices down the main diagonal (direct sum), each the companion matrix of α^3.

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* Cf. MacDuffee, *loc. cit.*, p. 127.

Now the space S' constructed above contains just w_3 2-independent vectors. Thus if $w_2 > w_3$, there must by Lemma 2 exist at least one vector which forms with them a larger 2-independent set. In a manner similar to the above, this vector then determines a $2h$-dimensional subspace associated with α^2. By a proof parallel to that above, the new space has zero intersection with S', and adds h vectors each to the sets of 2-independent and 1-independent vectors. This process may be continued in turn, showing that $w_2 = w_2'h$ for some integer w_2', and defining in the process $w_2' - w_3'$ disjoint spaces each associated with α^2. Finally, if $w_1 > w_2$, then $w_1 = w_1'h$ and there exist an additional $w_1' - w_2'$ h-dimensional spaces each associated with α. The whole set of spaces defined above is disjoint and so their sum is a space whose basis may be chosen to be the combined set of their individual bases. Its dimension is $\sum_1^3 w_i = \nu(\alpha^3)$, and so it is the null space $N(\alpha^3)$. Each of its constituent subspaces is invariant under A, with matrix the companion matrix of the appropriate power of α, and so, relative to the basis chosen, the matrix of the mapping for $N(\alpha^3)$ is the direct sum of these companion matrices.

A set of disjoint subspaces of a given space, whose sum is equal to the given space is called a *decomposition* of the space. The above construction, as applied to the general case, may be formalized as

THEOREM 1. *Let A be a linear mapping of a finite dimensional vector space, and α a singular prime polynomial in A of degree h. Relative to α, each $w_i = w_i'h$ for some integer w_i'. For some integer j, $w_j \neq 0$ and $w_i = 0$ $(i > j)$; then $N(\alpha^j)$ has a decomposition into invariant subspaces, $w_i' - w_{i+1}'$ of which are associated with α^i. The totality of the bases of its constituent subspaces is a basis for $N(\alpha^j)$, and relative to this basis, $N(\alpha^j)$ maps under A with matrix the direct sum of the matrices of the subspaces.*

Now suppose $\alpha_1, \alpha_2 \in F[A]$ with $(\alpha_1, \alpha_2) = 1$. If $V \in N(\alpha_1) \cap N(\alpha_2)$, then $V\alpha_2 = 0$, while $\alpha_2\beta \equiv 1 \pmod{\alpha_1}$ for some mapping β. Since also $V\alpha_1 = 0$, it follows that $V = 0$, and so $N(\alpha_1) \cap N(\alpha_2) = 0$. If α_1 and α_2 are singular prime polynomials, the construction of Theorem 1 relative to α_1 and α_2 thus results in disjoint subspaces whose sum maps under A with matrix which is the direct sum of the companion matrices of its constituent subspaces, and whose dimension is the sum of their dimensions. The set of all prime singular polynomials is therefore finite, for each contributes at least one to the dimension of the sum of the associated subspaces, and the whole dimension cannot exceed n.

To show that the set of singular prime polynomials is non-empty, consider first the set of all matrices. This is itself a vector space with basis $\{A_{ij}\}$, where A_{ij} is a matrix with 1 in the i, j position and 0 elsewhere. Since this space has dimension n^2, it follows that no more than n^2 of the matrices $A^i (i = 0, 1, \cdots)$ may be independent. There must thus exist some polynomial (of degree $\leqq n^2$) $\gamma = 0$, so that $\nu(\gamma) = n$. But γ may be factored into a finite set of prime factors at least one of which must have nullity greater than zero, since the nullity of their product is not zero.

Let $\{\alpha_i\}$ be the set of all distinct singular prime polynomials. For each α_i a power exists such that no higher power of α_i has greater nullity; let β_i be the least such power of α_i. Since $N(\beta_i)\cap N(\beta_j)=0(i\neq j)$, it follows [Lemma 1, p. 220] that $\nu(\prod\beta_i)=\sum\nu(\beta_i)$. The β_i are commutative, so the sum $\sum N(\beta_i) \subseteq N(\prod\beta_i)$, and the equality of their dimensions implies that $\sum N(\beta_i)=N(\prod\beta_i)$. The sum of the subspaces constructed for the various α_i in accordance with Theorem 1 thus form a basis for $N(\prod\beta_i)$. Now let $\{\gamma_i\}$ be the set of all singular prime power factors of γ. But $\{\alpha_i\}$ is the complete set of all distinct singular prime polynomials, and thus each γ_i must be a power of some one of them, say α_i. It follows that $\nu(\gamma_i)\leqq\nu(\beta_i)$, and hence the result $n=\nu(\gamma)=\sum\nu(\gamma_i)\leqq\sum\nu(\beta_i)$. Thus $\nu(\prod\beta_i)=n$. This establishes

THEOREM 2. *The subspaces constructed in accordance with Theorem* 1 *for the various singular prime polynomials of a mapping A define a decomposition of the whole space S into invariant subspaces each associated with the appropriate power of the polynomial. The totality of bases defined in the construction form a basis for S, and relative to this basis the matrix of A is the direct sum of the companion matrices of the constituent subspaces.*

Now if A' is the matrix of the mapping expressed relative to some other basis, it is well known that there is a non-singular matrix T such that $T^{-1}A'T = A$. Theorem 2 may therefore be rephrased to state that every matrix is similar to a direct sum of companion matrices. Further, the set of companion matrices depends only on the various sets of $\{w_j\}$, and these depend only on the nullities of the powers of the corresponding α_i. Since the nullity of a mapping is invariant under change of basis, it follows that this form is unique (except for the order in which the constituent companion matrices are chosen).

It may be remarked that if $\alpha=A-cI$ is singular (in which case c is called a *characteristic value*), then the set $\{w_i\}$ relative to α is called the *Weyr characteristic* of A relative to c. It is easy to show that a slight variation of the above construction then leads to the so-called Jordan matrix. For example, suppose $w_j\neq 0$ while $w_i=0(i>j)$. The basis of the first subspace for α^j may then be chosen as follows: Let $V_1\in N(\alpha^j)$ such that $V_1\notin N(\alpha^{j-1})$, and then define $V_{i+1}=V_i\alpha$ $(i=1,\cdots,j-1)$. Thus $V_iA=cV_i+V_{i+1}(i=1,\cdots,j-1)$ and $V_jA=cV_j$. Relative to the basis $\{V_i\}$ the mapping thus has the matrix (Jordan Matrix)

$$\begin{bmatrix} c & 1 & 0 & \cdot & \cdot & 0 \\ 0 & c & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & c & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & c \end{bmatrix}.$$

If all the singular prime polynomials are linear, then A is evidently similar to the direct sum of the appropriate Jordan matrices (Jordan normal form).

ONE-SIDED MAXIMA AND MINIMA OF FUNCTIONS OF TWO OR MORE VARIABLES

J. D. MANCILL, University of Alabama and Redstone Arsenal

1. Introduction. We shall first consider a real single-valued function

$$z = f(x, y)$$

of the two real variables x and y, which is continuous on a closed region R bounded by an ordinary* curve C. It is well known that such a function has an absolute maximum value and an absolute minimum value in the region R. These absolute extremes, as well as any relative extremes, of the function $f(x, y)$ may be at points interior to the region R or at points of its boundary C. In the latter case, the extreme is said to be *one-sided* (unilateral). So far as the author knows there is no reference in the literature to one-sided extremes of functions of two or more variables.

Before treating one-sided extremes, we shall determine necessary conditions and sufficient conditions in order that $f(x, y)$ have an extreme value at a point (x_0, y_0) interior to R in terms of cylindrical coordinates. The methods used are simplifications of those of Van Dantscher.§ Next, we shall determine necessary conditions and sufficient conditions in order that $f(x, y)$ have an extreme value at a point (x_0, y_0) on the boundary C of the region R of admissible points, under the assumption that appropriate continuity and differentiability properties of $f(x, y)$ hold on an extended region R' containing R in its interior.

Finally, we shall show how the results of the general case of one-sided extremes are easily extended to functions of three or more variables. One reason for treating first the case of two variables in detail is that certain special cases can be fully developed for functions of two variables which do not seem to extend, by the methods employed, to functions of three or more variables.

2. Free relative extremes. Suppose that the point to be investigated has the coordinates (x_0, y_0) and lies interior to the region R. Let

$$x - x_0 = r \cos \theta, \qquad y - y_0 = r \sin \theta,$$

where r and θ are restricted to the region

$$0 \leqq r \leqq d, \qquad 0 \leqq \theta \leqq 2\pi,$$

for $d > 0$. Then, we have

$$z = f(x_0 + r \cos \theta,\ y_0 + r \sin \theta) \equiv z(r, \theta).$$

If $f(x, y)$ has a minimum at (x_0, y_0), then by definition there exists a value of d

* By an "ordinary curve" is meant one which is either of class C' or is made up of a finite number of arcs of class C'.

§ *Cf.* Math. Ann., vol. 42, p. 89. Also, Hancock, Theory of Maxima and Minima, Ginn and Co., 1917, pp. 42–69.

such that

$$z(r, \theta) - z(0, \theta) \geqq 0,$$

on the above region. Therefore, we may state

THEOREM 1. *If $f(x, y)$ is continuous in R and assumes a minimum at (x_0, y_0) interior to R, and if*

$$z_{r+}(0, \theta) = \lim_{r=0+} [z(r, \theta) - z(0, \theta)]/r$$

exists for $0 \leqq \theta \leqq 2\pi$, then

$$z_{r+}(0, \theta) \geqq 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

If $f(x_0, y_0)$ is a maximum then

$$z_{r+}(0, \theta) \leqq 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

THEOREM 2. *A continuous function $f(x, y)$ assumes a minimum at (x_0, y_0) in R if*

$$\lim_{r=0+} [z(r, \theta) - z(0, \theta)]/r = z_{r+}(0, \theta) \geqq \rho > 0, \qquad 0 \leqq \theta \leqq 2\pi,$$

uniformly in θ. If the first two inequalities in the preceding line are reversed, then $f(x_0, y_0)$ is a maximum.

As an example, let us consider the function

$$z = \sqrt{x^2 + 2y^2}$$

and the point $(0, 0)$. In this case we have

$$z(r, \theta) = |r| \sqrt{1 + \sin^2 \theta},$$

and

$$z_{r+}(0, \theta) = \sqrt{1 + \sin^2 \theta} \geqq 1 > 0$$

uniformly for all values of θ. Therefore, $f(0, 0)$ is a minimum which is also an absolute minimum for all values of x and y.

If $f(x, y)$ is of class C' in R, then

$$z_r(r, \theta) = f_x \cos \theta + f_y \sin \theta$$

in the region

$$N: \quad |r| \leqq d, \qquad 0 \leqq \theta \leqq 2\pi.$$

Consequently, $z_r(0, \theta) = 0$, $0 \leqq \theta \leqq 2\pi$ if and only if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, and we may state

THEOREM 3. *If $f(x, y)$ is of class C' in R and assumes a maximum or a minimum*

at (x_0, y_0) *in* R, *then*

$$z_r(0, \theta) = 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

If $f(x, y)$ is of class C^{n+1} in R, then

$$z_r^{(k)}(r, \theta) = (\cos\theta\, \partial/\partial x + \sin\theta\, \partial/\partial y)^k f(x, y), \qquad k = 1, 2, \cdots, n+1,$$

and $z_r^{(k)}(0, \theta) = 0$, $0 \leqq \theta \leqq 2\pi$, if and only if all derivatives of $f(x, y)$ of order k are zero at (x_0, y_0). If all derivatives of order less than n vanish at (x_0, y_0) while not all of order n vanish there, it follows from Taylor's development that

$$\begin{aligned} \Delta z = f(x_0 + r\cos\theta, y_0 + r\sin\theta) - f(x_0, y_0) &= r^n [z_r^{(n)}(0, \theta) + rL_n]/n! \\ &= r^n \phi(r, \theta)/n! \end{aligned} \tag{1}$$

where L_n is finite for $|r|$ sufficiently small, since $f(x, y)$ is assumed to be of class C^{n+1}, and where

$$z_r^{(n)}(0, \theta) = (\cos\theta\, \partial/\partial x + \sin\theta\, \partial/\partial y)^n f(x_0, y_0). \tag{2}$$

We seek conditions which insure the existence of a value of d such that Δz is of one sign in the region N. Obviously the factor $r^n/n!$ may be omitted, since $r = 0$ corresponds to (x_0, y_0). We are interested in the lower limit of the absolute values of the real roots of the equation

$$\phi(r, \theta) = 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

If this lower limit is different from zero, then $f(x_0, y_0)$ is an extreme, but if it is zero then $f(x_0, x_0)$ may or may not be an extreme. There are three cases to be considered.

Case I. If the function (2) is different from zero on $0 \leqq \theta \leqq 2\pi$, which can happen only if n is even, then it follows from (1) that $f(x_0, y_0)$ is an extreme. Consequently, we may state

THEOREM 4. *If* (x, y) *is of class* C^{2n+1} *in* R *and at* (x_0, y_0)

$$z_r^{(k)}(0, \theta) = 0, \quad k = 1, 2, \cdots, 2n-1, \qquad z_r^{(2n)}(0, \theta) \neq 0, \quad 0 \leqq \theta \leqq 2\pi,$$

then $f(x_0, y_0)$ *is a minimum if* $z_r^{(2n)}(0, \theta) > 0$ *and a maximum if* $z_r^{(2n)}(0, \theta) < 0$.

The reader's attention is called to the close analogy between this theorem and the corresponding one for functions of a single variable.

Case II. If the function (2) assumes both positive and negative values on $0 \leqq \theta \leqq 2\pi$, as for example when n is odd, then it follows from (1) that Δz assumes both positive and negative values in every sufficiently small neighborhood of (x_0, y_0). Thus, we have

THEOREM 5. *If* $f(x, y)$ *is of class* C^{n+1} *in* R *and*

$$z_r^{(k)}(0, \theta) = 0, \qquad k = 1, 2, \cdots, n-1, 0 \leqq \theta \leqq 2\pi,$$

while $z_r^{(n)}(0, \theta)$ *assumes both positive and negative values on* $0 \leqq \theta \leqq 2\pi$, *then* $f(x_0, y_0)$ *is neither a maximum nor a minimum.*

Case III (*Ambiguous case*). If the function (2) vanishes for certain values of θ but does not change sign, which can happen only if n is even, then we are not able to decide from (2) whether $f(x_0, y_0)$ is an extreme or not. The ambiguity can usually be removed by the following procedure.

Suppose that the function $z_r^{(n)}(0, \theta)$ in this case vanishes for $\theta = \theta_t$, $t = 1, 2, \cdots, m$, and expand

$$z(r, \theta) = f(x_0 + r\cos\theta, y_0 + r\sin\theta)$$

in powers of r and $\theta - \theta_t$. It follows that if $z_r^{(k)}(0, \theta) = 0$, $0 \leqq \theta \leqq 2\pi$, $k = 1, 2, \cdots, 2n-1$,

$$\Delta_t z = z(r, \theta) - z(0, \theta_t) = (r, \theta - \theta_t)_{2n} + L_{2n},$$

where $(r, \theta - \theta_t)_{2n}$ denotes the sum of terms of degree $2n$ in r and $\theta - \theta_t$, and L_{2n} is finite near $(0, \theta_t)$. Consequently, in this ambiguous case, there are three possibilities:

1) *If* $(r, \theta - \theta_t)_{2n}$ *is a definite form for each value of* t, *then* $f(x_0, y_0)$ *is an extreme; a minimum or a maximum according as* $z_r^{(2n)}(0, \theta)$ *is positive or negative for* $\theta \neq \theta_t$, *respectively.*

2) *If* $(r, \theta - \theta_t)_{2n}$ *is an indefinite form for at least one value of* t, *then* $f(x_0, y_0)$ *is not an extreme.*

3) *If* $(r, \theta - \theta_t)_{2n}$ *is indefinite for no value of* t *but is semi-definite for at least one value of* t, *then we cannot say whether or not* $f(x_0, y_0)$ *is an extreme.*

The first coefficient $z_r^{(2n)}(0, \theta_t)$ in the form $(r, \theta - \theta_t)_{2n}$ is zero for each value of t. It is not difficult to show by direct differentiation that the *last two* coefficients are also zero for each value of t. Consequently, in the ambiguous case the resulting form $(r, \theta - \theta_t)_{2n}$ *always has its first and its last two terms zero.*

It follows that if $n = 1$, then

$$(r, \theta - \theta_t)_2 \equiv 0.$$

In this case it is necessary to proceed to the next term in $\Delta_t z$ to determine which of the three possibilities arises.

As a simple illustration, let us consider Peano's example

$$z = f(x, y) = y^2 - (p^2 + q^2)x^2y + p^2q^2x^4.$$

The point (0, 0) is the only possible extreme for this function. Thus, we have

$$z(r, \theta) = r^2\sin^2\theta - r^3(p^2 + q^2)\sin\theta\cos^2\theta + r^4p^2q^2\cos^4\theta$$

and

$$z_r^{(2)}(0, \theta) = 2\sin^2\theta,$$

which vanishes for $\theta_t = 0$ (not necessary to consider π since r is positive or negative) and we have Case III. However, we find that

$$\Delta_0 z = p^2q^2r^2 - (p^2 + q^2)r^3\theta + r^2\theta^2 + L_4,$$

and the form

$$(r, \theta - 0)_4 = r^2(p^2r - \theta)(q^2r - \theta)$$

is indefinite if $p \neq q$ and $f(0, 0)$ is not an extreme. If $p = q$, then this form is semidefinite and no conclusion can be drawn. However, it is easily seen directly that in this case $f(0, 0)$ is an improper minimum and the surface is tangent to the xy-plane along the parabola $y = p^2x^2$.

3. One-sided extremes. We shall derive necessary conditions and sufficient conditions in order that a function

$$z = f(x, y)$$

shall have a relative extreme at a point (x_0, y_0) on the boundary of the region R of admissible points. In the general case to be treated first, not both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ vanish at (x_0, y_0).

Let

$$C: \quad y = g(x)$$

represent the equation of the boundary of R near the point (x_0, y_0). The function

$$z = f[x, g(x)] \equiv Z(x)$$

has the derivative

$$Z^{(1)}(x) = f_x + f_y g^{(1)}(x) \tag{3}$$

near the point (x_0, y_0) provided $f(x, y)$ is of class C' in an extended region R' containing R in its interior and C is of class C' near (x_0, y_0). Throughout this paper R' denotes a region containing the admissible region R in its interior.

Now the equation

$$z_0 = f(x, y)$$

has the solution (x_0, y_0) at which we shall assume that $f_y(x_0, y_0) \neq 0$. Therefore, there exists a function

$$C_0: \quad y = g_0(x)$$

which satisfies this equation near (x_0, y_0) and has the same continuity class as $f(x, y)$ in R' near (x_0, y_0). Thus, we have the identity

$$z_0 = f[x, g_0(x)],$$

from which it follows that

$$0 = f_x + f_y g_0^{(1)}(x). \tag{4}$$

From equations (3) and (4), we have

$$Z^{(1)}(x_0) = f_y(x_0, y_0)[g^{(1)}(x_0) - g_0^{(1)}(x_0)].$$

THEOREM 6. *If $f(x, y)$ is of class C' in R' and $g(x)$ is of class C' near the point (x_0, y_0) on the boundary C at which $f(x, y)$ has a relative extreme with respect to the admissible region R, and if $f_y(x_0, y_0) \neq 0$, then*

$$g^{(1)}(x_0) = g_0^{(1)}(x_0).$$

It can also be shown that

$$Z^{(n)}(x_0) = f_y(x_0, y_0)[g^{(n)}(x_0) - g_0^{(n)}(x_0)]$$

by differentiating the relations (3) and (4) $n-1$ times and subtracting the resulting equations member by member, provided

$$g^{(k)}(x_0) = g_0^{(k)}(x_0), \qquad k = 1, 2, 3, \cdots, n-1.$$

THEOREM 7. *If $f(x, y)$ is of class C^{2n} in R' and $g(x)$ is of class C^{2n} near the point (x_0, y_0) on the boundary C at which $f_y(x_0, y_0) \neq 0$ and*

$$\begin{gathered} g^{(k)}(x_0) = g_0^{(k)}(x_0), \qquad k = 1, 2, \cdots, 2n-1, \\ f_y(x_0, y_0)[g^{(2n)}(x_0) - g_0^{(2n)}(x_0)] > 0, \end{gathered} \tag{5}$$

then $f(x_0, y_0)$ is a one-sided relative minimum
1) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

If the inequality in (5) is reversed, then $f(x_0, y_0)$ is a one-sided relative maximum
1) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

We shall consider the condition for a minimum where $f_y(x_0, y_0) > 0$, in which case the admissible region is $R[y \geqq g(x)]$. Then $f_y(x, y) > 0$ in an admissible neighborhood of (x_0, y_0) and $f(x, y)$ is increasing in y for each x in this neighborhood sufficiently near (x_0, y_0). But $f(x, y) = z_0$ at points of C_0 near (x_0, y_0). Therefore, $f(x, y) > z_0$ beyond C_0 along lines $x = x_k$ near (x_0, y_0). Also, since $f_y(x_0, y_0) > 0$, it follows that

$$g^{(2n)}(x_0) > g_0^{(2n)}(x_0)$$

and C_0 does not cross C at (x_0, y_0). Also, C is beyond C_0 along the lines $x = x_k$. Therefore, for all admissible points near (x_0, y_0) but distinct from it, $f(x, y) > z_0$ and $f(x_0, y_0) = z_0$. Consequently, $f(x_0, y_0)$ is a one-sided relative minimum. The other parts of the theorem can be proved in a similar manner.

THEOREM 8. *If $f(x, y)$ is of class C^{2n+1} in R' and $g(x)$ is of class C^{2n+1} near the point (x_0, y_0) on the boundary of R at which*

$$g^{(k)}(x_0) = g_0^{(k)}(x_0), \qquad k = 1, 2, \cdots, 2n,$$

$$f_y(x_0, y_0)[g^{(2n+1)}(x_0) - g_0^{(2n+1)}(x_0)] \neq 0,$$

then (x_0, y_0) is not an extreme of $f(x, y)$ with respect to the region R.

This follows from the fact that under these conditions the function $Z(x)$ does not have an extreme at $x = x_0$, that is, $f(x, y)$ does not have a *constrained* relative extreme at (x_0, y_0). In this connection it should be pointed out that Theorem 6 contains a necessary condition and Theorem 7 sufficient conditions in order that $f(x, y)$ have a constrained extreme at (x_0, y_0) along the curve C.

THEOREM 9. *If $f(x, y)$ is of class C' in R' and $g(x)$ is of class D' near a corner (x_0, y_0) at which $f_y(x_0, y_0) \neq 0$ and $f(x_0, y_0)$ is a one-sided relative minimum with respect to R, then*

$$f_y(x_0, y_0)[g^{(1)}(x_0+) - g_0^{(1)}(x_0)] \geqq 0, \qquad f_y(x_0, y_0)[g^{(1)}(x_0-) - g_0^{(1)}(x_0)] \leqq 0.$$

These inequalities are reversed if $f(x_0, y_0)$ is a one-sided relative maximum.

This theorem follows from an application of necessary conditions for a one-sided minimum of the function $Z(x)$ above to each branch of C at (x_0, y_0), at which the derivatives of $g(x)$ on the left and right are assumed to exist.*

THEOREM 10. *If $g(x)$ consists of two functions near (x_0, y_0), one of class C^N preceding and one of class C^n following the point (x_0, y_0); $f(x, y)$ is of class C^m in R', where m is the larger of N and n; and*

$$\begin{aligned}
&g^{(k)}(x_0+) = g_0^{(k)}(x_0), && k = 1, 2, \cdots, n-1,\\
&f_y(x_0, y_0)[g^{(n)}(x_0+) - g_0^{(n)}(x_0)] > 0, &&\\
&g^{(s)}(x_0-) = g_0^{(s)}(x_0), && s = 1, 2, \cdots, N-1,\\
&f_y(x_0, y_0)[g^{(N)}(x_0-) - g_0^{(N)}(x_0)] \begin{matrix} > 0, \\ < 0, \end{matrix} && \begin{matrix} N \text{ even}, \\ N \text{ odd}, \end{matrix}
\end{aligned} \tag{6}$$

then $f(x_0, y_0)$ is a one-sided relative minimum
1) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

If the inequalities in (6) are reversed, then $f(x_0, y_0)$ is a one-sided relative maximum
1) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

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* *Cf.* Mancill, One-sided Maxima and Minima, this MONTHLY, vol. 55, 1948, p. 313.

These results follow from an argument§ similar to that for Theorem 7, applied to each branch of C near (x_0, y_0).

This concludes the study of the general case of functions of two variables in which not both f_x and f_y are zero at (x_0, y_0), for if $f_y=0$ and $f_x\neq 0$, then all the above conditions hold with the roles of x and y interchanged.

In the special case of one-sided extremes in which both f_x and f_y vanish at the point (x_0, y_0) on the boundary C, one can apply the results of the preceding section and, in general, determine whether (x_0, y_0) is a one-sided extreme of $f(x, y)$ with respect to the admissible region R except possibly in the ambiguous case.

4. One-sided extremes of functions of three or more variables. We shall consider the function

$$w = f(x, y, z)$$

for (x, y, z) in a closed region R and derive conditions analogous to theorems 6, 7, and 8 of the preceding section. The generalization of these conditions to functions of more than three variables is a mere routine involving only a problem of notation.

Let

$$S: \quad z = g(x, y)$$

represent the equation of the boundary surface of R near the point (x_0, y_0, z_0), denoted by P_0, on S. The function

$$w = f[x, y, g(x, y)] \equiv W(x, y)$$

has the derivative

$$W_x(x, y) = f_x + f_z g_x(x, y)$$

near the point P_0 in R provided $f(x, y, z)$ is of class C' in an extended region R' containing R in its interior and S is of class C' near P_0.

Now the equation

$$w_0 = f(x, y, z)$$

has the solution (x_0, y_0, z_0) at which we shall assume that $f_z(x_0, y_0, z_0)\neq 0$, since we are to consider the general case where not all the derivatives f_x, f_y, and f_z are zero. Therefore, there exists a function

$$S_0: \quad z = g_0(x, y)$$

which satisfies this equation near P_0 and has the same continuity class as $f(x, y, z)$ in R'. Thus, we have the identity

$$w_0 = f[x, y, g_0(x, y)]$$

§ See also, Mancill, *loc. cit.*, p. 314.

from which it follows that

$$0 = f_x + f_z g_{0x}(x, y). \tag{8}$$

From equations (7) and (8), we have

$$W_x(x_0, y_0) = f_z(x_0, y_0, z_0)[g_x(x_0, y_0) - g_{0x}(x_0, y_0).]$$

In a similar way we can show that

$$W_y(x_0, y_0) = f_z(x_0, y_0, z_0)[g_y(x_0, y_0) - g_{0y}(x_0, y_0)],$$

and generally

$$\frac{\partial^n W(x_0, y_0)}{\partial x^h \partial y^k} = f_z(x_0, y_0, z_0)\left(\frac{\partial^n g(x_0, y_0)}{\partial x^h \partial y^k} - \frac{\partial^n g_0(x_0, y_0)}{\partial x^h \partial y^k}\right), \qquad h + k = n.$$

THEOREM 11. *If $f(x, y, z)$ is of class C' in R' and $g(x, y)$ is of class C' near P_0 on the boundary S at which $f(x, y, z)$ has a one-sided relative extreme with respect to R and $f_z(x_0, y_0, z_0) \neq 0$, then*

$$g_x(x_0, y_0) = g_{0x}(x_0, y_0), \qquad g_y(x_0, y_0) = g_{0y}(x_0, y_0).$$

THEOREM 12. *If $f(x, y, z)$ is of class C^{n+1} in R' and $g(x, y)$ is of class C^{n+1} near P_0 on the boundary S, where $f_z(x_0, y_0, z_0) \neq 0$; if*

$$\frac{\partial^m W(x_0, y_0)}{\partial x^h \partial y^k} = 0, \qquad h + k = m, m = 1, 2, \cdots, n - 1,$$

and if the form

$$\left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^n W(x, y) \tag{9}$$

at P_0 is positive definite in dx and dy, then $f(x_0, y_0, z_0)$ is a one-sided relative minimum

1) *with respect to the region $R[z \geqq g(x, y)]$ if $f_z(x_0, y_0, z_0) > 0$,*
2) *with respect to the region $R[z \leqq g(x, y)]$ if $f_z(x_0, y_0, z_0) < 0$.*

If the form (9) is negative definite, then $f(x_0, y_0, z_0)$ is a one-sided relative maximum

1) *with respect to the region $R[z \leqq g(x, y)]$ if $f_z(x_0, y_0, z_0) > 0$,*
2) *with respect to the region $R[z \geqq g(x, y)]$ if $f_z(x_0, y_0, z_0) < 0$.*

If the form (9) is indefinite, then $f(x_0, y_0, z_0)$ is not an extreme.

If the form (9) is semi-definite, then no conclusion can be drawn.

These conclusions can be proved similarly to those in Theorem 7.

It should be pointed out that theorems 11 and 12 contain necessary conditions and sufficient conditions, respectively, for $f(x_0, y_0, z_0)$ to be a constrained extreme of $f(x, y, z)$ on the surface S.

THE LENGTHS OF CURVES

J. L. COOLIDGE, Harvard University

1. The problem stated. There can be no doubt that the first persons who interested themselves at all in the study of curves had very definitely in mind that a curve had a definite length. A curve appeared as a portion of a thin cord which might always be pulled out to its greatest extent, at which time the length would be evident. The question of how lengths were to be compared on different curves must have presented almost insoluble difficulties to reflective geometers, but to the uninitiated the matter was quite simple. We see in [1] that some 2500 years before Christ the Babylonians assumed that the circumference of a circle was three times its diameter; whether this was thought to be strictly accurate or to be a sufficiently correct approximation to the fundamental number of mathematics is not clear. We have no information as to the early Egyptian views on the matter but we know from [2] that some eighteen hundred years before the Christian era they had for π the excellent approximation 256/81. Euclid gives in XII [3] an excellent proof that circles are to one another as the squares of their radii, the proof being based on Eudoxus [3]. Euclid's proof was founded on the theorem that if from the greater of two magnitudes we subtract a quantity greater than or equal to one half itself, and continue to do this indefinitely, we shall eventually have a remainder which is less than the smaller of the two given magnitudes. The root idea is suggested by Eudoxus and his fundamental method of exhaustion.

We must turn to Archimedes who made the first great steps in the matter of finding the lengths of curves. In his *Sphere and Cylinder* we have the fundamental assumption:

A curve is concave in one direction if every line which connects two of its points has all points of the curve lying on one side of the line or on the line itself and none on the other side. It is then assumed (as axioms):

(A) Of all lines connecting two points the straight line is the shortest.

(B) Of all lines in the plane having the same extremities two are unequal when both are concave in the same direction, and that one of them included between the other and the straight line connecting the two points is the lesser.

(C) The length of a curve is a magnitude under the definition of Euclid X [1].

From these assumptions Archimedes proceeds to build up the measurement of circular arcs. We learn from *Sphere and Cylinder* that the perimeter of a circumscribed polygon is greater than that of a circle, and that we can circumscribe and inscribe two polygons so that the ratio of the circumscribed to the inscribed is less than any number greater than unity. In his book on the circle we learn [5] that its area is equal to that of a triangle one of whose lengths is equal to the circumference and the other to a radius. Finally he shows that π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$.

It is worth noting that this truly astounding mathematician in his treatise *On Spirals* studied totally different curves and developed a totally different technique for finding their lengths. Here a curve is looked on kinematically; it is traced by a point which moves by two different impetuses, and the length depends upon the relative strength of the two. This approach we shall meet later when we come to Roberval, to whom I shall return presently. It is worth noticing that Hippocrates of Chios had studied the transformation of other curves into combinations of circular arcs, but gave no indication of the metrical relations involved. The determination of the number π I leave entirely aside, referring to Cantor. A geometrical approach will be found in [6] and to the whole subject in [7]. The earliest infinite process leading to π was Vieta's

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}} \cdot \sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}} \cdot \sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}}} \cdots .$$

Let us return to Cavalieri [9] and Roberval, who lived at a time when the study of curves and their tangents was much in evidence. Roberval had very much in mind Archimedes' kinematical idea of resolving a motion into two parts, and taking the geometrical sum. A curve is traced by a moving point, the tangent anywhere is the line of instantaneous motion of that point. The real philosophical difficulty, to define just what is meant by instantaneous motion, was veiled in the future to bedevil those successors who occupied themselves with the foundations of the calculus. To the unspoiled eye of common sense there was no difficulty. Here is the *Axiome* or *principe d'invention*; "*La direction du mouvement d'un point qui décrit une ligne courbe est la touchante de la ligne courbe en chaque position de ce point là* [8]." This seems somewhat tautological; a tangent is a line which touches, but he goes on to explain: "*Par les proprietés spécifiques (qui vous seront données) examinez les divers mouvements qu'a le point à l'endroit ou vous voulez mener la touchante; de tous ces mouvements composez en un seul et tirez la ligne de direction du mouvement composé, vous aurez la touchante a la courbe.*"

The meaning is this. Determine two measurements which connect the moving point with two fixed elements. Determine the vector velocities of the change of the two; their vector sum will be the instantaneous velocity. The easiest example is the tangent to the parabola. Here the distances to the focus and directrix are equal, hence the tangent to the curve makes equal angles with the focal radius and with a parallel to the axis.

2. Actual Lengths. The question of actually rectifying a curve with a continuously moving tangent occupied the attention of a number of contemporaneous geometers of the seventeenth century. There is a good deal to be said for the thesis that the first writer actually to do this was Torricelli [10]. He claims the discovery of the curve which we should write $\rho = e^{l\phi}$ and of which the subtangent is always equal in length to the preceding arch of the curve. There is some

doubt about the originality of this discovery, for it is usually ascribed to Descartes who communicated it to Mersenne and who in turn may have communicated it to Torricelli. I think, however, that on the whole the Italian should receive the credit for the discovery. In any case he did not long enjoy any primacy, for in 1657 a not very distinguished English mathematician (*cf.* Wallis in [**11**]), William Neil, showed how you could rectify a semi-cubical parabola by comparing its length with another known curve. In 1658 a very easy solution to the problem of rectifying the cycloid was published by Christopher Wren and was published by Wallis [**11**, p. 533]. This came presently to the notice of Fermat who in 1660 published [**12**]. He acknowledged Wren's rectification of the cycloid, and his editors, Tannery and Henry, acknowledge that he may have been familiar with other writers on the same subject, but in his own method there was certainly a novel and interesting element. Archimedes had laid down the method of exhaustion that the length of a curve differed by less than any assigned quantity from that of either an inscribed or a circumscribed curve; Fermat's idea was to use two circumscribed curves.

Let us take along the arc of a curve (which is concave downward) tangents at points whose abscissas are evenly spaced. Consider first the portion of each tangent from its point of contact to the intersection with the ordinate at the preceding point, the ordinates being supposed to increase. The length of this backward section of the tangent is less than the chord and so less than the section of the arc. On the other hand the length of the section of each tangent from its point of contact to the intersection with the ordinate of the following point is greater than the corresponding section of arc. Adding the backward sections of the tangents together and then the forward sections, we have two circumscribed "polygons" whose difference can be made as small as we please. We are then back to Archimedes. Fermat takes the curve $y^3 = kx^2$ and shows that its rectification can be thrown back on $y^2 = lx$, and this was done by Archimedes.

Another writer who should be considered in this connection was Barrow whose studies were elaborately discussed in [**13**]. He reverts to Archimedes' study of the motion of a point by considering this as composed of two separate motions, and finding their resultant. In the first examples a straight line moves parallel to itself, let us say uniformly, while a point moves down that line according to some known law. We study the motion of that point. On p. 120 of [**13**] he discusses briefly a method of drawing tangents "by calculation frequently used by us. Although I hardly know after so many well known and well worn methods of the kind above whether there is any advantage in doing so." The method so introduced was the formula of de Sluse [**14**] who as early as 1652 had some sort of a method for drawing tangents. The subtangent to $f(x,y) = 0$ may be written $a = -yf_y/f_x$.

Newton himself made no contribution to our general question, but he states as a well known theorem that the fluxion of an element of an arc is the square root of the sum of the squares of the fluxions of the two ends. This tacitly assumes that an element has a determinate fluxion.

3. The fundamental theorem. Let us return again to our fundamental question which is carefully studied in [15]. The first person to take this up seems to have been Dirksen [16]. He points out that a mathematical statement is either a definition, or an axiom or a theorem. Which category should we assign to Archimedes' statements about lengths? His statement, (B), about unequal lengths cannot be counted as an axiom, for we have no immediate way of comparing lengths on lines of different curvature. If we treat it as a theorem, a proof would involve a discussion of the infinitesimal domain that would have surely been beyond the reach of Archimedes' contemporaries and might call for the use of essentially equivalent statements. There remains only the possibility of its being a definition, but it must also be put into a more usable form.

Take a section of a curve AB which is concave in one sense and is cut by n equally spaced parallel lines in n points, each connected with the preceding and the succeeding by straight lines. The limit of the sum of these line segments as n increases indefinitely shall be defined as the length of the arc AB.

Dirksen proceeds to show that the length of the sum of two such arcs is the sum of their lengths. This seems to me to have been a real attempt to put the whole subject on a sound basis, but was still lacking in detail. The introduction of the equally spaced lines was unfortunate, and we do not know that the length of an arc is independent of the method of dividing it up. There was no direct method suggested for comparing different arcs. The suggestion was put forth once or twice that we might define as the length of an arc the lower limit of the lengths of circumscribed polygons as the number of sections increased indefinitely, but this ran straight into the difficulty that since there were an infinite number of circumscribed polygons of different lengths, how could we be sure that the lower limit existed? True, none of these could be less than the straight line, but that did not solve the problem completely. An ambitious attempt appeared in [17] but this was better done in [18].

Following Rouché and Comberouse, let us suppose that we have an arc of a curve concave in one sense. We take any number of small sections which together make up the arc, and inscribe and circumscribe polygons, the circumscribed being tangent at the vertices of the inscribed. We define as the length of the arc the limit which the lengths of the inscribed polygons approach as the length of each chord approaches zero. It remains to prove that this length is independent of the method of subdivision.

Let L be the length which is obtained from a particular method of subdivision. Also let the circumscribed vertices be $M'N'I' \cdots$, the inscribed being $ABC, \cdots$, while $MNI \cdots$ are the projections of the circumscribed vertices on the inscribed polygon.

Consider the fraction $AM'+M'B+BN'+ \cdots /AM+MB+BN+ \cdots$. The ratio of each term of the numerator to the corresponding term of the denominator, say AM'/AM, is greater than 1, but tends to 1 as the lengths of the

subarcs tend to zero. Hence the limit of the fraction above will be unity. Let p be the length of one of the inscribed polygons, and P that of a circumscribed polygon. Then

$$p \leqq L \leqq P; \qquad 1 \leqq \frac{L}{p} \leqq \frac{P}{p}.$$

Since $\lim P/p = 1$, it follows that $\lim L/p = 1$, or $L = \lim p$ regardless of the method of subdivision.

We have successfully avoided the difficulty of Dirksen of using a particular division, but this proof is certainly more dependent on intuition.

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NORMAL AND DIAGONALIZABLE MATRICES

B. E. MITCHELL, Louisiana State University

Normal matrices and matrices which can be diagonalized by similarity transformation are closely connected. While this is to be expected, since the former is a subset of the latter, the connection is not usually pointed out. It might be interesting then to consider various criteria for normal matrices and to note the corresponding analogues for diagonalizable matrices. The relation between the two classes of matrices is given by the following theorem.

THEOREM 1. *The matrix A is diagonalizable if and only if there exists a positive definite hermitian matrix H such that $H^{-1}AH=N$ is normal.*

The sufficiency is clear since a normal matrix may be unitarily diagonalized.

For the necessity suppose A may be diagonalized, say $P^{-1}AP=D$. By a theorem of Autonne [1], $P=HU$ where H is positive definite hermitian and U is unitary. Hence $H^{-1}AH=UDU^*=N$ is normal.

Thus in general any necessary and sufficient condition for a matrix A to be normal becomes a necessary and sufficient condition for A to be diagonalizable when applied to $H^{-1}AH$ for a positive definite hermitian matrix H.

The Toeplitz condition [7] on unitary diagonalization of a normal matrix may be stated in the equivalent form: A is normal if and only if it has n linearly independent orthogonal characteristic vectors. For A to be diagonalizable we merely require that it have n linearly independent characteristic vectors since if $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent there always exists a positive definite hermitian matrix H such that $H^{-1}\phi_1, H^{-1}\phi_2, \cdots, H^{-1}\phi_n$ are orthogonal.

Williamson [8] proved that A is normal if and only if A^* is a polynomial in A. The analogue is: A is diagonalizable if and only if $H^{-1}A^*H$ is a polynomial in A for H positive definite hermitian.

For if A is diagonalizable $H^{-1}AH=N$ is normal and $N^*=f(N)=H^{-1}f(A)H$. Hence $HA^*H^{-1}=H^{-1}f(A)H$ and $H^2A^*H^{-2}=f(A)$.

Conversely if $H^{-1}A^*H=f(A)$ then since H is positive definite hermitian it has a unique positive definite hermitian square root H_1. Then $H_1^{-1}A^*H_1=H_1f(A)H_1^{-1}=f(H_1AH_1^{-1})$, and so $H_1AH_1^{-1}$ is normal.

Since the analogues for the remaining criteria are obtained easily, we shall no longer explicitly state them.

THEOREM 2 (Parker [6]). *Let the characteristic roots of A be $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then A is normal if and only if the characteristic roots of A^*A (or of AA^*) are $\lambda_1\bar{\lambda}_1, \cdots, \lambda_n\bar{\lambda}_n$.*

The necessity is obvious.

For the sufficiency let A be unitarily triangularized, say $U^*AU=D=(d_{ij})$. Then $\operatorname{tr}(D^*D)=\operatorname{tr}(U^*A^*AU)=\sum_{i=1}^n\lambda_i\bar{\lambda}_i+\sum_{i\neq j}d_{ij}\bar{d}_{ij}=\operatorname{tr}(A^*A)=\sum_{i=1}^n\lambda_i\bar{\lambda}_i$. Hence $\sum_{i\neq j}d_{ij}\bar{d}_{ij}=0$ and D is diagonal.

Parker also proves in this paper that A is normal if and only if the roots of

$A+A^*$ (or of A^*+A) are $\lambda_1+\bar{\lambda}_1, \cdots, \lambda_n+\bar{\lambda}_n$. This theorem is a consequence of a theorem of Goldhaber [3]. For it is shown there that any two matrices A_1, A_2 with the property that the roots of A_1+A_2 are the sum of the roots of A_1 and A_2 satisfy Frobenius' Theorem and hence (McCoy [5]) may be simultaneously unitarily triangularized. But the assertion that A and A^* may be simultaneously unitarily triangularized implies that A may be unitarily diagonalized, and hence is normal.

This same principle will yield other criteria for normal matrices. For instance it is well known that any matrix A may be uniquely expressed in the form $A=H+K$ where H is hermitian and K is skew-hermitian, and that A is normal if and only if $HK=KH$. Application of the above-mentioned theorem of McCoy gives the result that A is normal if and only if H, K satisfy Frobenius' Theorem, or if and only if any two of A, H, K may be simultaneously unitarily triangularized (and hence diagonalized).

THEOREM 3. *A necessary and sufficient condition for A to be normal is that any characteristic vector of A be also a characteristic vector of A^*.*

The necessity follows from the fact that A^* is a polynomial in A.

For the sufficiency suppose $A\xi=\lambda\xi$ implies $A^*\xi=\beta\xi$. Then $\xi^*A^*\xi=\bar{\lambda}=\beta$. Hence if $U=(\xi, *, \cdots, *)$ is unitary,

$$U^*AU = \begin{bmatrix} \lambda & 0 \\ 0 & A_1 \end{bmatrix} \quad \text{and} \quad U^*A^*U = \begin{bmatrix} \bar{\lambda} & 0 \\ 0 & A_1^* \end{bmatrix}.$$

The theorem is obviously true for $n=1$. Assume it is true for $n-1$. Then, since A_1 is of order $n-1$ and any characteristic vector of A_1 is also one of A_1^*, A_1 may be unitarily diagonalized by the induction hypothesis. So therefore may A, and hence A is normal.

COROLLARY 1. *A necessary and sufficient condition for A to be normal is that if ξ is any characteristic column vector of A then ξ^* is a characteristic row vector.*

COROLLARY 2. *A necessary and sufficient condition for A to be normal is that any space invariant under A also be invariant under A^*.*

The necessity follows from the fact that A^* is a polynomial in A.

The sufficiency follows from the theorem.

COROLLARY 3 (Halmos [4], page 132). *A necessary and sufficient condition for A to be normal is that if S is any space invariant under A then S', the orthogonal complement of S, is also invariant under A.*

Since, for any matrix A, S invariant under A implies S' invariant under A^*, this corollary is equivalent to Corollary 2.

Note. The writer was unable to obtain Drazin's paper [2] until some months after completion of the present paper. Despite the fact that titles and topics are nearly identical, there is very little overlap in content, and almost none in

method.

His Theorem 1 (i) is an immediate consequence of the Jordan normal form of A. It is equivalent to each of the following statements: For every characteristic root α of the matrix A,

(1) nullity $(A-\alpha I)^k$ = nullity $(A-\alpha I)$ for all positive integers k,

(2) The Weyr characteristic of A relative to α is $r, 0, \cdots, 0$ where r is the nullity of $(A-\alpha I)$,

(3) The Segré characteristic of A relative to α is $1, 1, \cdots, 1$ to r terms.

Statements (i), (1), (2), and (3) are all corollaries to the theorem of Weierstrass (1861) that a matrix is diagonalizable if and only if it has only simple elementary divisors.

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

SOME NECESSARY CONDITIONS FOR CONVERGENCE OF INFINITE SERIES AND IMPROPER INTEGRALS

M. J. NORRIS, College of St. Thomas

If the terms of a convergent infinite series, $\sum a_n$, are non-increasing, then $\lim_{n\to\infty} na_n$ is zero. Prompted by this fact and the logarithmic comparison tests, we present a chain of necessary conditions for convergence. Analogous tests are first developed for improper integrals, and the results for series are then easily obtained. Throughout the discussion we assume, for convenience, that $a>0$

and that a and m are large enough so that any required logarithms are positive.

Our starting point is the known result for integrals. A simple proof may be modeled after the one for series as given in K. Knopp, *Theory and Application of Infinite Series*, p. 124.

THEOREM. *If $f(x)$ is non-increasing and $\int_a^\infty f(x)dx$ is convergent, then* $\lim_{x\to\infty} f(x)x$ *is zero.*

In general we have the following result, where $\log^{(k)} x$ stands for the logarithm iterated k times.

THEOREM 1. *If $f(x)\ x \log x \log\log x \cdots \log^{(k-1)}x$ is non-increasing and $\int_a^\infty f(x)dx$ is convergent, then*

$$\lim_{x\to\infty} f(x)x \log x \log\log x \cdots \log^{(k-1)} x \log^{(k)} x \text{ is zero.}$$

Proof. For the case that $k=1$, the substitution, $x=e^y$, yields the convergent integral $\int_{\log a}^\infty f(e^y)e^y dy$. Since $f(x)x$ is non-increasing by hypothesis, $f(e^y)e^y$ cannot increase as y increases. By the previous theorem $\lim_{y\to\infty} f(e^y)e^y y$ is zero. Therefore $\lim_{x\to\infty} f(x)x \log x$ is zero.

The general case is treated by making the substitution, $x=\exp^{(k)}y$.

We now prove the corresponding theorem for series.

THEOREM 2. *If $\sum_{n=m}^\infty a_n$ is a convergent infinite series such that $a_n n \log n \cdots \log^{(k-1)}n$ is non-increasing, then* $\lim_{n\to\infty} a_n n \log n \cdots \log^{(k-1)}n \log^{(k)}n$ *is zero.*

Proof. Theorem 2 will follow from an application of Theorem 1 to the function $f(x)$ defined for $n \leqq x < n+1 (n=m, m+1, \cdots)$ by the formula $f(x) = a_n n \log n \cdots \log^{(k-1)}n/(x \log x \cdots \log^{(k-1)}x)$. The second hypothesis of Theorem 2 implies that $f(x)$ has the first property required in Theorem 1. From the hypotheses of Theorem 2 it is easy to show that $a_n \geqq 0$, whence $0 \leqq f(x) \leqq a_n$ for $n \leqq x < n+1 (n=m, m+1, \cdots)$. Thus $\int_m^\infty f(x)dx$ is dominated by $\sum_{n=m}^\infty a_n$, $f(x)$ meets the second requirement of Theorem 1, and Theorem 2 is proved.

GENERALIZATION OF CLAIRAUT'S DIFFERENTIAL EQUATION AND THE ANALOGOUS DIFFERENCE EQUATION

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In a recent note (this MONTHLY, vol. 59, page 100) W. H. Witty generalizes Clairaut's equation to

$$y - xy' + \frac{x^2y''}{2!} + \cdots + \frac{(-1)^{n-1}x^{n-1}y^{(n-1)}}{(n-1)!} + f(y^{(n)}) = 0$$

and solves it. It is the purpose of this note to point out a still wider generalization which is given with references, as a problem in Goursat-Hedrick, vol. II, part II, page 44, and also its analogue in difference equations.

The problem is to integrate

$$F\left(y'', y' - xy'', y - xy' + \frac{x^2y''}{2}\right) = 0$$

and it is noted that there exist equations of analogous form and of any order.

Consider the equation $F(z_0, z_1, \cdots, z_{n-1}) = 0$ where

$$z_r = y^{(r)} - xy^{(r+1)} + \frac{x^2y^{(r+2)}}{2!} + \cdots + \frac{(-1)^{n-r-1}x^{n-r-1}y^{(n-1)}}{(n-r-1)!}$$

$$= \sum_{s=0}^{n-r-1} \frac{(-1)^s x^s y^{(r+s)}}{s!} \quad \text{where} \quad y^{(r)} = \frac{d^r y}{dx^r}.$$

Thus

$$\begin{aligned} z_0 &= y - xy' + \frac{x^2y''}{2!} + \cdots + \frac{(-1)^{n-1}x^{n-1}y^{(n-1)}}{(n-1)!} \\ &\vdots \\ z_{n-2} &= y^{(n-2)} - xy^{(n-1)} \\ z_{n-1} &= y^{(n-1)}. \end{aligned}$$

It follows by induction that

$$\frac{dz_r}{dx} = y^{(n)}\phi_r(x) \tag{1}$$

where $\phi_r(x)$ are some functions of x. This suggests that a solution $y(x)$ of $F(z_0, \cdots, z_{n-1}) = 0$ may have $y^{(n)}(x) = 0$, or that it may be of the form:

$$y(x) = a_0 + a_1x + \frac{a_2x^2}{2} + \cdots + \frac{a_{n-1}x^{n-1}}{(n-1)!} = \sum_0^{n-1} \frac{a_r x^r}{r!}, \tag{2}$$

where the a's are yet to be determined. In this case the z_r are constants because of (1); and these constants must satisfy $F(z_0, \cdots, z_{n-1}) = 0$. From the definition of z_r, $z_r(0) = y^r(0) = a_r$. Hence $z_r = a_r$, and so $F(a_0, \cdots, a_{n-1})$ must be zero. Therefore the function (2) is a solution of $F(z_0, \cdots, z_{n-1}) = 0$ whenever $F(a_0, \cdots, a_{n-1}) = 0$. As this solution has $n-1$ arbitrary constants, it is the general solution.

The analogue of the generalized differential equation as a difference equation is

$$F(z_0, z_1, z_2, \cdots, z_{n-1}) = 0$$

where

$$z_0 \;= u_x - x\Delta u_x + \frac{x(x+1)\Delta^2 u_x}{2!} + \cdots + \frac{(-1)^{n-1}x(x+1)\cdots(x+n-2)\Delta^{n-1}u_x}{(n-1)!}$$

$$z_1 = \quad \Delta u_x - x\Delta^2 u_x + \frac{x(x+1)\Delta^3 u_x}{2!} + \cdots + \frac{(-1)^{n-1}x(x+1)\cdots(x+n-3)\Delta^{n-1}u_x}{(n-2)!}$$

$$\vdots$$

$$z_{n-2} = \Delta^{n-2}u_x - x\Delta^{n-1}u_x$$

$$z_{n-1} = \Delta^{n-1}u_x.$$

It follows by induction that

$$\Delta z_r = \phi_r(x)\Delta^n u_x.$$

As before, this suggests a solution of the form:

$$(3) \qquad u_x = a_0 + a_1 x + a_2 \frac{x(x-1)}{2!} + \cdots + a_{n-1}\frac{x(x-1)\cdots(x-n+1)}{(n-1)!}$$

$$= \sum_0^{n-1} a_r \binom{x}{r}.$$

Also z_r are constants; $z_r(0) = \Delta^{(r)}u_0 = a_r$. So $z_r = a_r$, and $F(a_0, \cdots, a_{n-1}) = 0$. Therefore (3) is a solution of $F(z_0, \cdots, z_{n-1})$ whenever $F(a_0, \cdots, a_{n-1}) = 0$.

A MATRIC PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA FOR REAL QUATERNIONS

MARK LEUM and M. F. SMILEY, State University of Iowa

In her doctoral thesis [7], Louise A. Wolf showed how one may discuss the theory of matrices whose elements are in the ring Q of real quaternions.* She made essential use of the isomorphism of Q_n onto a ring of matrices with complex elements which is accomplished by replacing each real quaternion by its matric form. (See, for example, [3, p. 60].) We observe here that a very minor consequence of the results of Wolf, combined with a general lemma of Hamilton-Cayley type (which applies in an arbitrary ring with unit) will yield a very simple proof of a† "fundamental theorem of algebra" for real quaternions.

Let R be a ring with unit 1 and let $f(x)$ be a non-constant monic polynomial with coefficients in R. We define $C_L(f)$, the *left* companion matrix of $f(x)$, as does MacDuffee [5, pp. 81–82] for the case in which R is a field. For example, if $f(x) = x^2 + a_1 x + a_0$, then

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* N. Jacobson [2, pp. 50–51] derived the results of Wolf as a special case of a more general theory of non-commutative principal ideal domains. Very recently, independent work by H. C. Lee [4] and by J. L. Brenner [1] on this topic has appeared.

† See our concluding remark.

$$C_L(f) = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}.$$

In this special case a simple computation shows that

$$f_L(C_L(f)) = (C_L(f))^2 + (C_L(f))a_1 + a_0 I = 0.$$

By means of an inductive proof, which is straightforward and will be omitted, we have the following theorem.

THEOREM 1. *If R is a ring with unit element and $f(x)$ is a non-constant monic polynomial with coefficients in R, then $f_L(C_L(f))=0$.*

Now let $R=Q$ and let $f(x)$ be a non-constant monic polynomial with coefficients in Q. The fact that there is a nonzero vector v of real quaternions such that $vC_L(f)=qv$, for some q in Q, is a consequence of the results of Wolf and is very easy to prove directly using her isomorphism. Since $vf_L(C_L(f))=0$, we easily find that $f_L(t^{-1}qt)=0$ for each nonzero component t of v. If $g(x)$ is a non-constant polynomial with coefficients in Q and with leading coefficient a_n, then $f(x)=g(x)a_n^{-1}$ is monic and has a left-hand root r in Q. Since $g(x)=f(x)a_n$, r is also a left-hand root of $g(x)$.

THEOREM 2. *Every non-constant polynomial with real quaternion coefficients has a left-hand root which is a real quaternion.*

Concluding remark. We have been unable to find a correspondingly simple proof of the more general form of the "fundamental theorem of algebra" for real quaternions which is due to I. Niven and S. Eilenberg [6].

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UNIQUE SEGMENTS IN METRIC SPACES

R. C. BUCK, University of Wisconsin

Let M be a metric space, with the distance from x to y denoted by (xy). An ordered set of points $\langle x, p_1, p_2, \cdots, p_n, y\rangle$ is said to be linear if $(xp_1)+\cdots+(p_ny)=(xy)$. Let $[x, y]$, the linear join of x and y, be the set of all p such that $\langle x, p, y\rangle$ is linear. M is said to be *convex* if $[x, y]$ always contains a point other

than x and y. If M is complete and convex, $[x, y]$ is a continuum connecting x and y. M is said to have unique segments if $[x, y]$ is always a single arc and is therefore congruent to a Euclidean line segment of length (xy). Several conditions are known which are equivalent to this condition. The following is typical. In all that follows, M is assumed complete and convex.

THEOREM 1. *Let the linear joins of M obey the following condition. Given x, y, and points p and q in $[x, y]$, either $[x, p]$ intersects $[q, y]$ or $[p, y]$ intersects $[x, q]$. Then, M has unique segments.*

For, if $u \in [x, p] \cap [q, y]$, then $(xu)+(up)=(xp)$, and $(xu)+(up)+(py)=(xy)$. Hence $(up)+(py)=(uy)$ so that $(xy)=(xq)+(qu)+(uy)=(xq)+(qu)+(up)+(py)$ and $\langle x, q, u, p, y\rangle$ is linear.

The purpose of this note is to show that a quadratic metric inequality also implies unique segments. Let us first recall that M is said to have the n-point property if every set of n points of M is congruent with a set of n points in a suitable Euclidean space [1]. Thus, the 3-point property is nothing more than the triangle law. Wilson [2] proved that if M has the 4-point property, M itself is congruent with a subset of a Euclidean space (possibly infinite dimensional), and Blumenthal strengthened this by requiring this hypothesis only for quadruples containing a linear triple [1]. In particular the 4-point property certainly implies that M has unique segments. Our condition will also be a condition on the mutual distances of 4-points which hold for the Euclidean spaces, but which is not as strong as the 4-point property. We first establish a general inequality for the mutual distances of four points in any metric space.

LEMMA. *For any four points x, y, p, q in M,*

$$(xy)^2 + (pq)^2 \leq (xp)^2 + (yq)^2 + 2(yp)[(xy) + (yq)]. \tag{1}$$

We have $|(py)-(xy)| \leq (px)$, and $(pq) \leq (py)+(yq)$ so that $[(py)-(xy)]^2 + (pq)^2 \leq (px)^2 + [(py)+(yq)]^2$ from which (1) immediately comes.

To obtain our 4-point condition on M, we strengthen (1) in a natural way by replacing the term $(xy)+(yq)$ by the smaller number (xq).

DEFINITION. *M has property Q if, for any x, y, p, q,*

$$(xy)^2 + (pq)^2 \leq (xp)^2 + (yq)^2 + 2(xq)(yp). \tag{2}$$

THEOREM 2. *If M has property Q, it has unique segments.*

Let p and q belong to $[x, y]$, with $(xp)=(xq)=a$ and set $(xy)=L$. Then, by (2)

$$L^2 + (pq)^2 \leq a^2 + (L-a)^2 + 2a(L-a)$$

from which $(pq)=0$ and $p=q$.

Property Q is not vacuous, since it obtains in all Euclidean spaces.*

THEOREM 3. *If M is a Euclidean space,* (2) *holds.*

In terms of the inner product in M, we have

$$\begin{aligned}(xy)^2 + (pq)^2 - (xp)^2 - (yp)^2 \\ &= (x - y)\cdot(x - y) + (p - q)(p - q) - (x - p)(x - p) - (y - q)(y - q) \\ &= 2(x\cdot p) + 2(y\cdot q) - 2(x\cdot y) - 2(p\cdot q) \\ &= 2(y - p)(q - x) \\ &\leq 2\|y - p\| \, \|q - x\| = 2(yp)(xq).\end{aligned}$$

It is easy to see that in E^n, equality in (2) forces the four points to be co-planar.

Property Q holds also in some spaces other than the Euclidean spaces. Consider for example the tripod space M formed by the positive Y axis, and the whole X axis, metrized by the geodesic metric in M. For some spaces, the inequality (2) may hold locally, although it fails in the large. For these, we will then have unique segments in the small. This is the case for a 1-sphere or 2-sphere. It would be of interest to know if property Q imposes any further condition on M, such as curvature restrictions, and what other spaces obey it.

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ON THE SKEW QUADRILATERAL†

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Two theorems. In this section we establish two theorems concerning the skew quadrilateral.

THEOREM 1. *A skew quadrilateral has its opposite sides equal in pairs if and only if it has its opposite angles equal in pairs.*

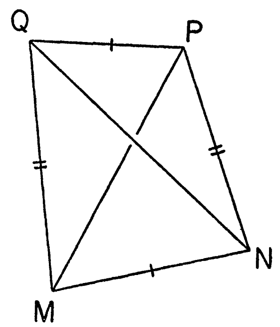
In fact, if the opposite sides MN and PQ of a skew quadrilateral $MNPQ$ are equal, as well as the opposite sides NP and QM, then triangles MNP and PQM are congruent and triangles MNQ and NPQ are congruent, and we have the angular equalities

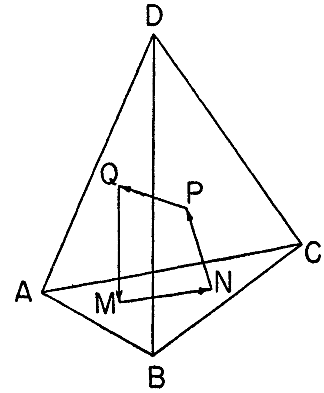
$$(MN, MQ) = (PQ, PN), \qquad (NP, NM) = (QM, QP). \tag{1}$$

* The referee has pointed out that the four-point inequality (2) has been used before in a different connection. (See L. Blumenthal, The norm of a configuration, Bull. Amer. Math Soc., vol. 34, 1928, pp. 726–730.)

† Translated from the French by Howard Eves.

Conversely, let us be given a skew quadrilateral $MNPQ$ for which the angular equalities (1) hold. Draw non-concurrent planes perpendicular to the sides QM, MN, NP, PQ in such a manner that the vectors $\overrightarrow{QM}$, $\overrightarrow{MN}$, $\overrightarrow{NP}$, $\overrightarrow{PQ}$ either all have the sense of the interior normals to the faces ABC, BCD, CDA, DAB





of the tetrahedron T determined by these planes or all have the sense of the exterior normals to these faces. Then the opposite dihedral angles BC and AD of T are equal, as well as the opposite dihedral angles CD and AB. Therefore the trihedral angles A-BCD and C-DAB have the three dihedral angles of one equal to the three dihedral angles of the other, which implies the angular equalities

$$(AB, AC) = (CD, CA), \qquad (AC, AD) = (CA, CB),$$

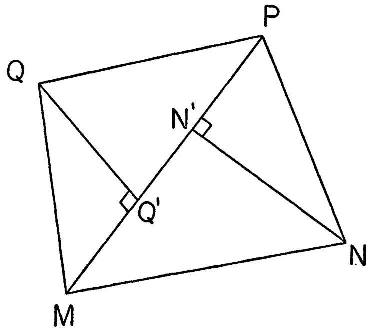
and, consequently, the congruency of triangles ABC and CDA. Now, since by a known theorem the lengths of the sides QM and NP of the quadrilateral are proportional to the areas of the faces ABC and CDA, we have $QM = NP$. Similarly, $MN = PQ$.

THEOREM 2. *A skew quadrilateral has its opposite sides equal in pairs if and only if the perpendiculars erected at the vertices to the planes of the two sides meeting at these vertices constitute a hyperbolic group of lines.*

In fact, for the perpendiculars m, n, p, q at the vertices M, N, P, Q to the planes (MN, MQ), (NP, NM), (PQ, PN), (QM, QP) to be hyperbolic, it is necessary and sufficient that the orthogonal projections on the plane of a face of the tetrahedron $MNPQ$ of the perpendiculars to the other three faces be concurrent lines, and that this property hold for three of the faces. Now the projections of m and p on the plane MNP pass through the point N_1 diametrically opposite to N on the circumference MNP, and the projection of q on the plane MNP coincides with the trace on that plane of the plane drawn perpendicular to MP through the altitude QQ' of triangle MPQ. If

$$MN = PQ, \qquad NP = QM, \tag{2}$$

then the feet N' and Q' of the altitudes NN' and QQ' of triangles MNP and



MPQ are symmetric with respect to the midpoint of side MP, and the projection of q on the plane MNP passes through N_1. We may establish analogous conclusions for the projections of the perpendiculars m, n, p and the perpendiculars m, n, q on the planes MPQ and NPQ.

Conversely, if the projections of the perpendiculars m, p, q on the plane MNP are concurrent in a point N_1, then N_1 is diametrically opposite vertex N on the circumference MNP, and equations (2) follow from

$$(QM)^2 - (QP)^2 = (NP)^2 - (NM)^2,$$
$$(MQ)^2 - (MN)^2 = (PN)^2 - (PQ)^2,$$

by the addition and subtraction of corresponding members.

2. Application to the tetrahedron. Let us designate by M and P, N and Q, the opposite angles of a skew quadrilateral $MNPQ$, and by M' and P', N' and Q', the supplements of these angles. If $M=P$ and $N=Q$ we also have

$$M + P' = P + M' = 180° = N + Q' = Q + N'.$$

On the other hand we can never, for a skew quadrilateral $MNPQ$, have either

$$M + P = 180° = N + Q \quad \text{or} \quad M' + P' = 180° = N' + Q'.$$

To see this latter fact we note that the diagonal NQ of the quadrilateral determines the triangles MNQ and NPQ, the sum of the angles of which is equal to 360°. Since any face angle of a trihedral angle is less than the sum of the other two face angles, we have, first of all, the angular relations

$$Q < (QM, QN) + (QN, QP), \qquad N < (NQ, NM) + (NP, NQ),$$

whence

$$Q + N < 180° - M + 180° - P,$$

and, finally,

$$M + N + P + Q < 360°.$$

We may now state

THEOREM 3. *A tetrahedron cannot have two pairs of opposite dihedral angles supplementary* [1], *nor two pairs of opposite exterior dihedral angles supplementary, but it can have two or three pairs of an interior dihedral angle and an opposite exterior dihedral angle supplementary.*

It is the last configuration (in which case the tetrahedron is isosceles) that should have been considered in the announcement of the author's problem E 617 [1944, 231 and 1945, 160].

Translator's Note. The statement of E 617 is as follows: "From a given tetrahedron we derive another by taking as vertices the points of contact of the insphere with the faces. Show that the dihedral angles at pairs of opposite edges of the first tetrahedron are supplementary if, and only if, the second tetrahedron is trirectangular." The statement of this problem is at fault; to correct it we should replace the words "the insphere" by the words "an escribed sphere" and the word "supplementary" by the word "equal." The error in Kelly's "proof" of the incorrectly stated problem [1945, 160] lies in the fact that the center of the circumsphere of a trirectangular tetrahedron T lies *outside* T, and consequently this circumsphere cannot be the inscribed sphere, but must be an escribed sphere, of the tangential tetrahedron T' of T.

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CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE MATHEMATICAL TRAINING OF ENGINEERS*

RALPH HULL, Purdue University

The Mathematical Association of America has recently introduced into its programs for the annual summer meetings a series of symposia on topics related to the teaching of mathematics. The first symposium was held in September of last year at the University of Minnesota on the question: *What should we teach our undergraduate majors in mathematics, with special reference to their subsequent employment in industry?* The second symposium was held on September 2, 1952, at Michigan State College on the subject: *The mathematical training of engineers*, with Professor Ralph Hull, Head of the Purdue Department of Mathematics, acting as chairman. Three invited speakers presented prepared talks of about twenty-five minutes each on three different aspects of the subject. The talks were followed by an extended and lively discussion.

The first and third speakers were Professor E. B. Allen of the Department of Mathematics, Rensselaer Polytechnic Institute, who spoke on *Mathematics and the engineering curriculum*, and Professor Elizabeth S. Sokolnikoff, of the Department of Mathematics, University of Wisconsin, who spoke on *Problems of mathematicians who teach engineers*. These speakers emphasized the fact that the mathematical training of engineers is a joint responsibility of the faculty of engineering and of the department of mathematics, which may be in some other college of the university. The training can only be effective as a result of close cooperation between the Schools of Engineering and Departments of Mathematics. Such cooperation is generally found in our universities and present conditions are favorable to still closer cooperation. For example, more and more staff members of schools of engineering have had advanced mathematical training; recent textbooks in engineering show a tendency to draw more frequently on mathematical ideas and consequently more of the mathematics which has traditionally been presented in courses on the calculus and differential equations is actually used in engineering courses. On the other hand, many mathematicians have worked directly with engineers in industry or in government research laboratories since the beginning of World War II and consequently are more familiar with the mathematical needs of engineers.

Various ways were suggested of developing still closer relationships between departments of mathematics and the schools of engineering, for example, joint staff appointments, exchange teaching assignments between staff members, the

* Symposium, Mathematical Association of America, September 2, 1952, Michigan State College.

familiarizing of staff members with the engineering curriculum on the one hand and the formal mathematics courses on the other, by the preparation and distribution of extensive course outlines and attendance in courses in other fields as visitors. All of these ways are being utilized but much could be gained by an increase in their use. Since mathematics takes up a large part of the generally crowded engineering curriculum (almost 25% of the first two years' work, measured in semester hours, at some schools), and since more and more engineers are taking more mathematics as undergraduates than in the past, and more and more engineers are prolonging their training to advanced degrees either at schools of engineering or in special programs set up by their employers, it is a fundamental problem of departments of mathematics to see to it that the most effective possible use is made of the time engineers devote to their mathematics courses.

The second speaker, Dr. C. H. Harry, Bureau of Aeronautics, Department of the Navy, spoke on *Mathematics used for engineering designs*. Dr. Harry maintained that the mathematical training of engineers should be based primarily on the fact that whenever an engineer at work employs mathematics he must ultimately arrive at a numerical answer. This is the case whether the numbers he seeks are, for example, merely the lumped constants of a simple electrical circuit which is to have some prescribed property, or the design parameters of a more complicated network or structure, or, at a still higher level, frequently referred to nowadays as *Operational Research*, numbers which represent the relative effectiveness of complicated systems involving machines or weapons, men, and, in military situations, tactics and strategy. It is immaterial to the engineer, and even more so to Management or Command, whether these numbers are arrived at ultimately by the substitution of numerical values of variables in "closed formulas," or otherwise, provided they are as accurate as the data permit, or as reliable as can be hoped for in the more complicated situations. In the simpler cases, the numbers will ultimately be checked by testing the circuit or network. In the more complicated cases of systems evaluation, the testing is much more difficult, as is the type of analysis involved. It is further characteristic of much of the mathematical work of an engineer that it is based largely on relatively elementary mathematical concepts, although more sophisticated mathematical ideas are finding more and more application at the present time.

Dr. Harry stated that in his opinion present mathematics courses for engineers do not give sufficient recognition to the numerical aspect of the engineers' mathematical work. Specifically, for instance, he maintained that in a first course in elementary differential equations, he would consider about two weeks as sufficient for the consideration of differential equations which can be solved in "closed form," and that the rest of the time should be devoted to solution by numerical methods. This point was seriously challenged by several of his listeners. While it was admitted that the engineer seeks always a numerical answer, it was claimed that the proper function of the mathematics department is to

acquaint the engineering student with basic mathematical concepts; that while, admittedly, most of the problems in our elementary mathematics texts used by engineers were not of the type the engineer would be called upon to solve at work, they were essential to the developing of his understanding of the mathematical concepts; that the mathematics departments would not have time in present courses to develop the mathematical concepts and introduce work on numerical methods as proposed by Dr. Harry. The controversy here was summarized as covering the relative emphasis which should be given in mathematics courses for engineers to the three aspects of mathematics itself, the linguistic, the logical, and the technical or computational, or, more briefly, between the conceptual and technical aspects. The resolution of the question should be a major concern of engineering faculties and teachers of mathematics. Probably some experimentation will be required before generally acceptable answers are arrived at. In this connection, a "two-track" system for engineers, one emphasizing the conceptual, the other the technical, aspects of mathematical training, is under consideration for an early trial in at least one school.

A DERIVATION OF THE FORMULAS FOR SIN $(\alpha+\beta)$ AND COS $(\alpha+\beta)$

A. K. Bettinger, The Creighton University

The following procedure for deriving the addition formulas of plane trigonometry has been used for some time by the author. To the best of our knowledge it has not previously appeared in print.

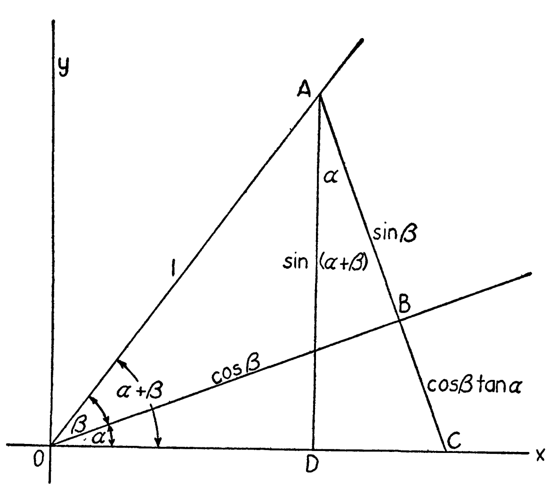

Fig. 1

We shall derive the formulas for the case when both α and β are acute angles and $\alpha+\beta<90°$.

Figure 1 will be used to prove the formula for sin $(\alpha+\beta)$.

In Figure 1, let $OA=1$, draw AC perpendicular to OB, and AD perpendicular to OX. Write the lengths of the sides on the figure as shown.

Now by definition the cosine of angle DAC equals DA/AC, that is,

$$\cos\alpha = \frac{\sin(\alpha+\beta)}{\cos\beta\tan\alpha+\sin\beta}.$$

Solving for $\sin(\alpha+\beta)$, we have

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\alpha(\cos\beta\tan\alpha+\sin\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta.\end{aligned}$$

Figure 2 is found more convenient for deriving the formula for $\cos(\alpha+\beta)$ than the preceding one. Hence we let $OA=1$, draw CA perpendicular to OB, and AD perpendicular to OY.

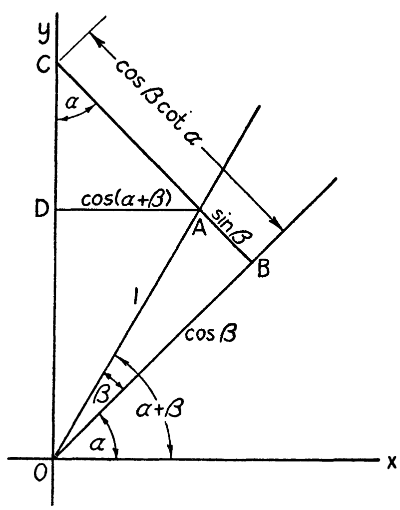

FIG. 2

By definition the cosine of angle DAC equals $\sin\alpha=DA/AC$, hence

$$\sin\alpha = \frac{\cos(\alpha+\beta)}{\cos\beta\cot\alpha-\sin\beta}.$$

Solving for $\cos(\alpha+\beta)$, we have

$$\begin{aligned}\cos(\alpha+\beta) &= \sin\alpha(\cos\beta\cot\alpha-\sin\beta)\\ &= \cos\alpha\cos\beta-\sin\alpha\sin\beta.\end{aligned}$$

The construction is readily extended to other quadrants for demonstrations in different cases. There is no need, however, for showing the construction in the numerous cases that may occur since the generality of the formulas does not

depend on the particular construction used to derive the formulas.

The validity of the formulas in general may be shown by a method similar to that in Todhunter's *Plane Trigonometry*, page 53.

A NOTE ON CAUCHY'S THEOREM

HARRY LASS, University of Illinois

The usual test for analyticity of a complex function $f(z)=u(x, y)+iv(x, y)$ consists in showing that the Cauchy-Riemann equations hold and that the first partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous. It seems desirable, therefore, to obtain a proof of one form of Cauchy's theorem by a direct method involving the above hypotheses on $f(z)$. We proceed to do this using a method due to Bliss* which avoids the use of Green's theorem.

Let S be a simply-connected open region such that the Cauchy-Riemann equations hold and such that the partial derivatives of u and v are continuous at every point of S. We first show that $\int f(z)dz$ vanishes for any rectangle in S.

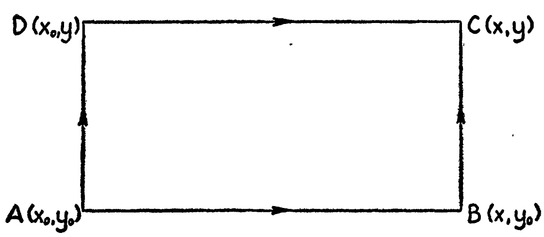

FIG. 1

Consider the rectangle $ABCD$ contained entirely in S, see Fig. 1. We can obtain a complex function of the two real variables x and y by defining

$$F(x, y) = \int_{x_0}^{x} f(t + iy_0)dt + i\int_{y_0}^{y} f(x + it)dt. \tag{1}$$

$F(x, y)$ is simply the sum of the integrals of $f(z)$ along the straight lines AB and BC. To obtain $\partial F/\partial x$ we note that

$$F(x + \Delta x, y) = \int_{x_0}^{x+\Delta x} f(t + iy_0)dt + i\int_{y_0}^{y} f(x + \Delta x + it)dt,$$

and that $\partial F/\partial x=\lim_{\Delta x\to 0}[F(x+\Delta x, y)-F(x, y)]/\Delta x$. We obtain

$$\frac{\partial F}{\partial x} = f(x + iy_0) + i\int_{y_0}^{y} \frac{\partial f(x + it)}{\partial x}\,dt \tag{2}$$

since $f(z)$ is continuous and $\partial f/\partial x$ is uniformly continuous on the closed straight

* American Mathematical Society Colloquium Publications, vol. XVI.

line segment BC. Similarly, $\partial F/\partial y = if(x+iy) = if(z)$.

We now define

$$G(x, y) = i\int_{y_0}^{y} f(x_0 + it)dt + \int_{x_0}^{x} f(t + iy)dt. \tag{3}$$

Note that $G(x, y)$ is the sum of the integrals of $f(z)$ along the straight line paths AD and DC. We easily obtain

$$\begin{aligned} \frac{\partial G}{\partial x} &= f(x + iy) = f(z) \\ \frac{\partial G}{\partial y} &= if(x_0 + iy) + \int_{x_0}^{x} \frac{\partial f(t + iy)}{\partial y} dt. \end{aligned} \tag{4}$$

To show that $\partial F/\partial x = \partial G/\partial x$ we observe that $\partial f/\partial x = (1/i)(\partial f/\partial y)$ from the Cauchy-Riemann equations. Thus (2) becomes

$$\frac{\partial F}{\partial x} = f(x + iy_0) + \int_{y_0}^{y} \frac{\partial f(x + it)}{\partial t} dt = f(x + iy) = f(z).$$

Similarly $\partial F/\partial y = \partial G/\partial y$. Thus $F = G + \textit{constant}$ and since $F = G = 0$ at $A(x_0, y_0)$ we have $F \equiv G$. Hence $\int f(z)dz$ vanishes around the closed rectangular path $ABCD$.

We now consider any closed rectifiable Jordan curve, Γ, inside the rectangle $ABCD$, see Fig. 2.

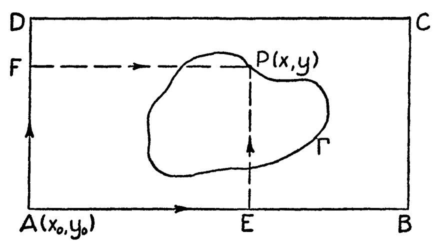

FIG. 2

Since $F \equiv G$ we have (considering the rectangle $AEPF$)

$$\begin{aligned} F(x, y) &= \int_{x_0}^{x} f(t + iy_0)dt + i\int_{y_0}^{y} f(x + it)dt \\ &= i\int_{y_0}^{y} f(x_0 + it)dt + \int_{x_0}^{x} f(t + iy)dt \end{aligned}$$

and $\partial F/\partial x = f(z)$, $\partial F/\partial y = if(z)$. Thus $dF = f(z)(dx + idy) = f(z)dz$ so that $\oint_\Gamma f(z)dz = \oint_\Gamma dF = \oint_\Gamma dU + i\oint_\Gamma dV = 0$, where $F(x, y) = U(x, y) + iV(x, y)$. The Riemann-Stieltjes integrals $\oint_\Gamma dU$, $\oint_\Gamma dV$ certainly vanish.

The final step proceeds as follows. Let C be any closed rectifiable Jordan curve in S. We apply a fine enough rectangular mesh on S so that the rectangles containing C lie inside S. This can be done since C is closed and S is open. The integration of $f(z)$ over all rectangles interior to C plus the integration of $f(z)$ over those boundaries which include C vanish from the preceding results. The sum of these integrations yields $\oint_C f(z)dz=0$.

AB AND BA HAVE THE SAME CHARACTERISTIC ROOTS*

JOEL BRENNER, State College of Washington

1. Introduction. The theory of matrices, and in particular the theory of characteristic roots of matrices, has now progressed far enough so that it should stand on its own feet, and be independent of the theory of determinants. One advantage of establishing an independent theory is this, that many theorems on matrices hold just as well whether the elements are numbers from a non-commutative or from a commutative field or even ring; but the theory of determinants over a non-commutative ring is not effective enough to furnish by itself proofs of those theorems.

A more practical advantage of a proof of a theorem on matrices independent of the theory of determinants is that it is simpler. The proof that the characteristic roots of the matrices AB and BA are the same is an illustration of this fact. One popular proof depends on the fact that the characteristic roots of AB are the solutions of the equation $\det(AB-\lambda I)=0$; if B is non-singular, the matrices $B(AB-\lambda I)B^{-1}$ and $(AB-\lambda I)$ have the same determinant, so the theorem follows. But if B is singular, this proof breaks down, and much more elaborate arguments are needed.

A shorter proof of this theorem is given below. To make this proof independent of any previous knowledge, it would be necessary first to prove that every matrix has a characteristic root, and second that every matrix B has a conjugate $T^{-1}BT$ in triangular form, with the characteristic roots ranged in any convenient order down the main diagonal. The first theorem can be proved simply and without the need for theorems from determinants [1]; the second theorem is very easy to prove; even with the additional restriction that the transforming matrix should be unitary the proof is not very long [2, p. 25].

2. The proofs. The following exposition makes no attempt at generality, but emphasizes method of argument.

Let A, B be $n\times n$ matrices over a field. By a characteristic root of B, we mean a number λ such that a non-null vector x exists with $Bx=x\lambda$.

LEMMA. *If T is non-singular, B and $T^{-1}BT$ have the same characteristic roots.*

* The writing of this article was sponsored by the Office of Ordnance Research.

Proof. If the relation $Bx = x\lambda$ holds, then the relation $T^{-1}BTy = y\lambda$ holds, and conversely. Here y is $T^{-1}x$, x is Ty, and if either x or y is not the null vector, the other is not the null vector.

THEOREM. *The matrices AB and BA have the same characteristic roots.*

Proof. By the lemma, we may suppose that B is triangular, and that the numbers in the main diagonal of B are so arranged that any zeros occur at the end.

If the relation $ABx = x\lambda$ holds, then the relation $BAy = y\lambda$ also holds, where y is the vector Bx. If the vector Bx is not zero, then λ is automatically a characteristic root of BA. On the other hand, if Bx is the null vector, then the equation $ABx = x\lambda$ reads $0 = x\lambda$, so that λ is zero. We have to prove that zero is also a characteristic root of BA. Only in this part of the proof do we need to know that B can be transformed into triangular form. It is clear that B is singular (has no inverse), for otherwise the equation $Bx = 0$ would be impossible with $x \neq 0$. Since B is singular, at least one characteristic root of B must be 0; for the inverse of a triangular matrix, no one of whose diagonal elements is 0, can easily be computed. Thus the last diagonal element of B is 0, so that the last row of B is zero. Hence the last row of BA is zero, and the last row of BAy is zero, whatever y. The matrix equation $BAy = y0$ amounts to a set of $n-1$ equations in n unknowns, so there is always a non-trivial solution [**3**, p. 3].

In the case of matrices over a non-commutative field, the results above hold, although the proof breaks down when $\lambda = 0$. The proof does hold for quaternions, however.

References

1. J. L. Brenner, Matrices of Quaternions, Pacific Journal of Mathematics, vol. 1, 1951, pp. 329–335.
2. F. D. Murnaghan, The Theory of Group Representations, John Hopkins Press, 1938.
3. E. Artin, Galois Theory, Notre Dame, 1948.

PROOF OF THE FIRST MEAN VALUE THEOREM OF THE INTEGRAL CALCULUS

T. PUTNEY, State College of Washington

Professor A. A. Bennett in Volume 31 (1924), page 40, of this MONTHLY called attention to the didactic interest which attaches to a particular form of proof of the first mean value theorem of the differential calculus. A proof of the same character is to be found in the well-known book by Theodore Chaundy *The Differential Calculus* (Oxford University Press, 1935), p. 84. A precisely analogous argument can be employed to obtain a proof of the first mean value theorem of the integral calculus, and may be of interest in classroom instruction.

Define two functions $G(x)$ and $H(x)$ in $a \leqq x \leqq b$ by

$$G(x) \equiv \int_a^x f(t)g(t)dt, \qquad H(x) \equiv \int_a^x g(t)dt$$

where $f(x)$ and $g(x)$ are continuous in $a \leqq x \leqq b$ and $g(x)$ does not vanish in $a < x < b$. In the determinant

$$D(x) \equiv \begin{vmatrix} G(x) & H(x) & 1 \\ G(a) & H(a) & 1 \\ G(b) & H(b) & 1 \end{vmatrix} = \begin{vmatrix} G(x) & H(x) & 1 \\ 0 & 0 & 1 \\ G(b) & H(b) & 1 \end{vmatrix},$$

since $G(a) = H(a) = 0$. Furthermore, $D(a) = D(b) = 0$. Hence, by Rolle's theorem, $D'(\xi) = 0$ for some ξ, $a < \xi < b$ or

$$g(\xi) \begin{vmatrix} f(\xi) & 1 & 0 \\ 0 & 0 & 1 \\ G(b) & H(b) & 1 \end{vmatrix} = 0.$$

Finally, since $g(\xi) \neq 0$, this yields the desired result, namely $G(b) = f(\xi)H(b)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1051. *Proposed by S. W. Golomb, Harvard University*

Given N objects and B boxes, what is a necessary and sufficient condition for at least two boxes to contain the same number of objects?

E 1052. *Proposed by H. H. Berry, U. S. Army Engineering Corps*

Let AOB be a fixed diameter of a given circle (O), and let P be any point on the circle. Denote by Q the foot of the perpendicular from P on AB and by R the foot of the perpendicular from O on PA. Let PQ and RO intersect in N, and let QR and PO intersect NA in L and M, respectively. Find the loci of points L, M, and N as P moves along the given circle.

E 1053. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Evaluate the determinant

$$\begin{vmatrix} x & x-1 & \cdots & x-k \\ x-1 & x & \cdots & x-k+1 \\ \cdot & \cdot & \cdots & \cdot \\ x-k & x-k+1 & \cdots & x \end{vmatrix}.$$

E 1054. *Proposed by F. D. Parker, St. Lawrence University*

Find the probability that the roots of the quadratic equation

$$x^2 + bx + c = 0, \qquad -k < b,\ c < k,$$

are imaginary.

E 1055. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The centers of the four circles passing through triples of vertices of a quadrangle $ABCD$ inscribed in (circumscribed about) a circle (O) are the vertices of a quadrangle $A'B'C'D'$ inscribed in (circumscribed about) a circle (O').

SOLUTIONS

An Invariant Determinant

E 1016 [1952, 328]. *Proposed by Norman Anning, University of Michigan*

Find the element of likeness in: (a) simplifying a fraction, (b) powdering the nose, (c) building new steps on the church, (d) keeping emeritus professors on campus, (e) putting B, C, D in the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ B & a^3 & 1 & a \\ C & D & a^3 & 1 \end{vmatrix}.$$

Solution by C. W. Trigg, Los Angeles City College. The value, $(1-a^4)^3$, of the determinant is independent of the values of B, C, and D. Hence, operation (e) does not change the value of the determinant but merely changes its appearance. Thus the element of likeness in (a), (b), (c), (d), and (e) is only that the appearance of the principal entity is changed. The same element appears also in: (f) changing the name-label of a rose, (g) writing a decimal integer in the scale of 12, (h) gilding the lily, (i) whitewashing a politician, and (j) granting an honorary degree.

Also solved by J. P. Ballantine, Julian Braun, I. A. Dodes, R. P. Eisinger, David Kerr and L. A. Ringenberg (jointly), David Loev, Charles Salkind, Alan Wayne, and the proposer.

A Property of Two Intersecting Circles

E 1017 [1952, 329]. *Proposed by Harry Furstenberg, New York City*

On the common secant AB of two intersecting circles, O and O', are chosen any two points, C and D, outside of either circle. The tangents CQ and CS are drawn to O' and O, respectively, on one side of AB and the tangents DR and DT are drawn to O' and O on the other side of AB. Prove that QR and ST intersect on AB.

Solution by A. Sisk, Maryville, Tenn. (The references in the following solution are to Altshiller-Court, *College Geometry*, 2nd ed., p. 198.)

The common chord AB is the radical axis of circles O' and O. Points Q and S are antihomologous points on O' and O respectively, as also are points R and T (435). Therefore, by definition, RQ and ST are antihomologous chords of O' and O. It follows that RQ meets ST on AB (434).

Also solved by I. A. Dodes, A. L. Epstein, E. I. Gale, L. M. Kelly, W. H. Mead, Jr., Margaret Olmsted, and the proposer.

A Minimum Value

E 1018 [1952, 329]. *Proposed by B. B. Misra, Ravenshaw College, Cuttack, India*

AB is a line segment, which is divided by the point O in the ratio $m:n$. P is a variable point (r, θ) referred to O as pole and OB as initial line. Subject to the conditions

$$0 \leqq r \leqq a < \min(AO, OB), \qquad 0 \leqq \theta \leqq \pi,$$

find the minimum value of $m^2/AP+n^2/BP$, and also find the position of P for which this minimum value is attained.

Solution by I. A. Dodes, Stuyvesant High School, New York City. Let us designate AP by p and BP by q and set

$$z = m^2/p + n^2/q.$$

Since, for fixed θ, p and q both increase when r increases, it follows that $r=a$ at $z_{\min}$ and that $z=z(\theta)$. Hence, for a relative minimum, we must have

$$dz/d\theta = -(m/p)^2 dp/d\theta - (n/q)^2 dq/d\theta = 0.$$

But, setting $AO=km$ and $OB=kn$, we have

$$p^2 = a^2 + k^2m^2 + 2akm \cos\theta,$$
$$q^2 = a^2 + k^2n^2 - 2akn \cos\theta,$$

whence

$$dp/d\theta = -(akm \sin\theta)/p, \qquad dq/d\theta = (akn \sin\theta)/q.$$

Substituting in $dz/d\theta=0$ we find

$$p/q = m/n.$$

It follows that the sought point P is an intersection of the circle of radius a and center O with the circle of Apollonius for the segment AB and the ratio $m:n$. Since PO bisects angle APB we have

$$pq = a^2 + k^2mn.$$

It now follows that, for minimum z,

$$p^2 = m(a^2 + k^2mn)/n, \qquad q^2 = n(a^2 + k^2mn)/m,$$

whence an easy calculation gives

$$z_{\min} = \sqrt{mn}\,(m + n)/\sqrt{a^2 + k^2mn}.$$

Also solved by Julian Braun, L. A. Ringenberg, O. E. Stanaitis, and J. A. Tierney.

A Uniformly Convergent Sequence of Continuous Functions

E 1019 [1952, 329]. *Proposed by A. E. Livingston and L. H. Wegner, University of Oregon*

Let $\{f_n(x)\}$ be a sequence of functions continuous on the closed interval $[a, b]$, and let $\lim_{n\to\infty} f_n(x)=f(x)$ uniformly on the open interval (a, b). Then there exists a function $F(x)$ satisfying: (1) $F(x)$ is continuous on $[a, b]$, (2) $F(x)=f(x)$ for $x\epsilon(a, b)$, (3) $\lim_{n\to\infty} f_n(a)=F(a)$ and $\lim_{n\to\infty} f_n(b)=F(b)$.

Solution by L. A. Ringenberg, Eastern Illinois State College. Let $\epsilon>0$ be given. Since $\lim_{n\to\infty} f_n(x)=f(x)$ uniformly on (a, b), there is an integer $N>0$ such that

$$n \geqq N \text{ and } x\epsilon(a, b) \text{ imply } |f_n(x) - f(x)| < \epsilon. \tag{1}$$

Let $x_0\epsilon(a, b)$ be given. Since $f_N(x)$ is continuous on $[a, b]$, there is a real number $\delta>0$ such that

$$x\epsilon(a, b) \text{ and } |x - x_0| < \delta \text{ imply } |f_N(x) - f_N(x_0)| < \epsilon. \tag{2}$$

Let x be a real number such that $x\epsilon(a, b)$ and $|x-x_0|<\delta$. Then it follows from (1) and (2) that

$$\begin{aligned} |f(x) - f(x_0)| \leqq |f(x) - f_N(x)| &+ |f_N(x) - f_N(x_0)| \\ &+ |f_N(x_0) - f(x_0)| < 3\epsilon. \end{aligned} \tag{3}$$

Let $F(x)=f(x)$ for every $x\epsilon(a, b)$. In view of (3), $F(x)$ is continuous on (a, b).

For $i=1, 2$, let n_i be any integer such that $n_i \geqq N$. Since $f_{n_i}(x)$ is continuous on $[a, b]$, there is a real number $\Delta>0$ such that

$$0 < x - a < \Delta \text{ implies } |f_{n_i}(x) - f_{n_i}(a)| < \epsilon \text{ for } i = 1, 2. \tag{4}$$

Let x be a real number such that $0<x-a<\Delta$. Then, applying (1) and (4), we have

$$\begin{aligned}|f_{n_1}(a) - f_{n_2}(a)| &\leqq |f_{n_1}(a) - f_{n_1}(x)| + |f_{n_1}(x) - f(x)| \\ &\quad + |f(x) - f_{n_2}(x)| + |f_{n_2}(x) - f_{n_2}(a)| < 4\epsilon.\end{aligned}$$

Thus the sequence $\{f_n(a)\}$ satisfies Cauchy's internal criterion for convergence. Let $F(a)=\lim_{n\to\infty} f_n(a)$. Let $M \geqq N$ be any integer such that

$$|f_M(a) - F(a)| < \epsilon. \tag{5}$$

Since $f_M(x)$ is continuous at $x=a$, there is a positive real number ρ such that

$$0 < x - a < \rho \text{ implies } |f_M(x) - f_M(a)| < \epsilon. \tag{6}$$

Let x be a real number such that $0<x-a<\rho$. Then it follows from (5) (6), and (1) that

$$\begin{aligned}|F(a) - F(x)| &= |F(a) - f(x)| \\ &\leqq |F(a) - f_M(a)| + |f_M(a) - f_M(x)| + |f_M(x) - f(x)| < 3\epsilon.\end{aligned}$$

Therefore $F(x)$ is continuous at $x=a$. Similarly $F(x)$ is continuous at $x=b$ if we define $F(b)=\lim_{n\to\infty} f_n(b)$.

Also solved by Harold Johnson, Azriel Rosenfeld, W. R. Smythe, Jr., O. E. Stanaitis, and the proposers.

Johnson pointed out that the desired results are found on pp. 152 and 154 of Carslaw, *Theory of Fourier's Series and Integrals.*

Traveling by the "Ride and Tie" Plan

E 1021 [1952, 407]. *Proposed by W. R. Ransom, Tufts College*

In the "ride and tie" plan (where the first man rides a horse for a stated time and leaves it for a second man, walking, to ride for a like time while the first man walks on) show that the rate of progress is the harmonic mean between the rates of riding and walking, and find for what fraction of the time the horse can rest.

Solution by Julian Braun, Washington, D. C. Let r and w be the riding and walking rates, respectively. At the beginning and end of a "cycle" of time duration t, the two men are together and the horse has rested for a length of time τ. The distance traversed by walking equals the distance traversed by riding, that is, $w(t+\tau)=r(t-\tau)$, whence the fraction of time the horse can rest is

$$\tau/t = (r - w)/(r + w).$$

The rate of progress is thus given by

$$w(t + \tau)/t = w\{1 + (r - w)/(r + w)\} = 2rw/(r + w).$$

Also solved by A. G. Anderson, Leon Bankoff, I. A. Dodes, A. L. Epstein,

Vern Hoggatt, Sr. and Vern Hoggatt, Jr. (jointly), M. S. Klamkin, J. D. E. Konhauser, L. J. Lander, David Mandelbaum, Margaret Olmsted, Azriel Rosenfeld, C. A. Swanson, V. V. Rao, and the proposer.

Klamkin remarked that for m men and one horse the rate of progress is given by

$$mrw/[(m-1)r+w],$$

and the fraction of time the horse can rest is given by

$$[(m-1)r-w]/[(m-1)r+w].$$

The case of m men and h horses, each with a different speed, is a much more involved problem.

A Class of Heronian Triangles

E 1022 [1952, 407]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all triangles for which the three sides and one of the altitudes are, in some order, measured by four integers in arithmetic progression.

Solution by R. W. Huff, College of Wooster. Representing the sides in order of decreasing length and the altitudes to these sides by a, b, c, h_a, h_b, h_c, respectively, and noting that the altitude to any side is smaller than or equal to (the latter prohibited by the conditions of the problem) either of the remaining sides, we arrive at the following four cases for the problem: (I) a, b, c, h_a, (II) a, b, c, h_b, (III) a, b, c, h_c, (IV) a, b, h_c, c, where the segments are given in order of decreasing length. We now express the four segments concerned in terms of a and the common difference d, and substitute the results in the equations

$$(ih_i)^2 = 4s(s-a)(s-b)(s-c), \qquad i = a, b, c,$$

where $2s=a+b+c$. In solving, the only valid solution is found in case (II), where $a=15d$. Hence the only triangles satisfying the problem are: $a=15d$, $b=14d$, $c=13d$, $h_b=12d$, $d=1, 2, 3, \cdots$.

Also solved by Leon Bankoff, N. G. Gunderson, Sam Kravitz, F. D. Parker, Azriel Rosenfeld, E. P. Starke, C. W. Trigg, and the proposer.

The interested reader might like to consult E 695 [1946, 334].

Application of the Maximum-Modulus Theorem

E 1023 [1952, 407]. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Massachusetts Institute of Technology*

(1) Given any set of points in the plane, not all coincident with the origin, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 1.

(2) Given any set of points on the unit circle, not the vertices of a regular polygon, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 2.

Solution by Fritz Herzog, Michigan State College. (1) Let the given points be denoted by $z_1, z_2, \cdots, z_n$. Put

$$P(z) \equiv (z - z_1)(z - z_2) \cdots (z - z_n)$$
$$= z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n.$$

We are to show that, for all values of z with $|z| = 1$, max $|P(z)| \geqq 1$, equality holding if and only if $z_1 = z_2 = \cdots = z_n = 0$, that is, if and only if $P(z) \equiv z^n$. (In the following the maximum is always taken over all values of z with $|z| = 1$.)

Let

$$Q(z) \equiv z^n P(1/z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + 1.$$

Obviously, max $|Q(z)| =$ max $|P(z)|$. But $Q(0) = 1$ and hence, by the maximum-modulus theorem, max $|Q(z)| \geqq 1$, equality holding if and only if $Q(z) \equiv 1$, that is, if and only if $P(z) \equiv z^n$.

(2) We use the above notation and assume that

$$|z_1| = |z_2| = \cdots = |z_n| = 1.$$

As above, we are to show that max $|P(z)| =$ max $|Q(z)| \geqq 2$, equality holding if and only if the z_k are the vertices of a regular n-gon, that is, if and only if $c_1 = c_2 = \cdots = c_{n-1} = 0$.

Assume first that max $|Q(z)| < 2$. Then $|Q(z)| < 2$ for $|z| \leqq 1$, and hence all zeros of

$$Q(z) - 2 = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z - 1$$

lie in $|z| > 1$, which is incompatible with the fact that

$$\text{(I)} \qquad |c_n| = 1.$$

Assume secondly that max $|Q(z)| = 2$. Then $|Q(z)| < 2$ for $|z| < 1$, and hence all zeros of $Q(z) - 2$ must lie in $|z| \geqq 1$ and therefore, by (I), on $|z| = 1$. Let these zeros be denoted by $u_1, u_2, \cdots, u_n$, and let $v_1, v_2, \cdots, v_n$ be the zeros of $Q(z)$ itself; note that the v_m are the reciprocals of the z_m and hence also lie on $|z| = 1$. It is obvious that $u_k \neq v_m$ for all k and m. Let U be the polygon that has the u_k as vertices and V the same for the v_m. Then* the zeros of $Q'(z)$ must lie in the intersection of U and V and hence in $|z| < 1$. It follows that none of the quantities $Q'(u_k)$ and $Q'(v_m)$ vanishes, so that $u_i \neq u_j$ and $v_i \neq v_j$ for $i \neq j$. Let

$$R(z) \equiv c_n z^{n-1} + c_{n-1} z^{n-2} + \cdots + c_2 z + c_1,$$

so that $zR(z) = Q(z) - 1$. We then have $u_k R(u_k) = 1$ and $v_m R(v_m) = -1$, hence $|R(z)| = 1$ at $z = u_k$ and $z = v_m$. Now $|R(e^{i\theta})|^2$ is a trigonometric polynomial in θ, with real coefficients, of degree no higher than $n - 1$, and assuming the value

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* See G. Polya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Section III, Problem 31.

1 at $2n$ distinct points of the interval $0 \leqq \theta \leqq 2\pi$. We conclude† that $|R(e^{i\theta})|^2 \equiv 1$. Considering the constant term of $|R(e^{i\theta})|^2$, we thus obtain

$$|c_n|^2 + |c_{n-1}|^2 + \cdots + |c_2|^2 + |c_1|^2 = 1,$$

and hence, by (I), $c_{n-1} = c_{n-2} = \cdots = c_2 = c_1 = 0$.

Also solved by the proposers. Shapiro further proposed the following similar problem:

(3) Let $P_1, \cdots, P_n$ denote n points on a circle of unit radius, and set τ_k equal to the product of the distances from P_k to the other points. Then

$$\sum_{k=1}^{n} (1/\tau_k) \geqq 1,$$

equality being attained if and only if the n points form a regular n-gon.

Editorial Note. One wonders if purely geometric solutions can be supplied for these problems. The solutions by function theory do not extend themselves even to three dimensions. Problem (1) is not true for one dimension; is it true for any dimension higher than two? Can problems (2) and (3) be extended, in part at least, to higher dimensions?

Concerning the Four Color Theorem

E 1024 [1952, 407]. *Proposed by David Gale, Brown University*

Assuming that the four color theorem is true for planar maps containing a finite number of regions, show that it is also true for planar maps containing an infinite number of regions.

Solution by the Proposer. Let M be the given infinite map. The proof is simplified by using the fact that such a map contains only a countable number of regions. Let the regions be denoted by $R_1, R_2, \cdots$, and let $M_1, M_2, \cdots$ denote the finite sub-maps of M defined by $M_n = R_1 \cup R_2 \cup \cdots \cup R_n$. For each M_n let C_n denote a particular admissible coloring of M (which by assumption exists). Finally, for $i \leqq n$ let $c_{n,i}$ denote the color assigned to the region R_i by the coloring C_n.

We now describe inductively a procedure for coloring M. Let Γ denote the collection of all the colorings, C_n. Consider now the colors $c_{n,i}$. Since there are only a finite number (four) of colors, it follows that one can choose a color c_1 such that $c_1 = c_{n,1}$ for infinitely many n. Now let Γ_1 denote the set of all colorings of Γ for which $c_{n,1} = c_1$. Note that the collection Γ_1 is infinite, so that one can repeat the process, that is, choose a color c_2 such that $c_{n,2} = c_2$ for infinitely many $C_n \epsilon \Gamma_1$, and let Γ_2 be the set of all $C_n \epsilon \Gamma_1$ such that $c_{n,2} = c_2$. By repetition in this way we obtain an infinite sequence of colors, $c_1, c_2, \cdots$, and corresponding infinite subcollections $\Gamma_1, \Gamma_2, \cdots$, of Γ. We now define a coloring C_∞ of M by assigning to R_i the color c_i. To show that C_∞ is admissible, let R_j and R_k be two

† See G. Polya and G. Szegö, *loc. cit.*, vol. 2, Section VI, Problem 14.

contiguous regions of M with $j<k$. By examining the coloring process above it is seen that c_j and c_k are exactly the colors assigned to R_j and R_k by all of the colorings of Γ_k. But since each of these colorings is admissible it follows that $c_k \neq c_j$, which proves the theorem.

The above proof will be recognized as essentially an application of the Cantor diagonal process.

Also solved by S. W. Golomb, who called attention to the article by C. N. Reynolds, *The present status of the four color problem*, Bull. Am. Math. Soc., March, 1952, p. 207, abstract 263.

Harold Kuhn remarked that the problem was stated and solved by D. König, *Über eine Schlussweise aus dem Endlichen ins Undendliche*, Acta Szeged 3 (1927), §3.

Some Finite Series Inequalities

E 1025 [1952, 407]. *Proposed by K. L. Chung, Cornell University*

Show that if

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$$

and

$$\sum_{j=1}^{k} a_j \leqq \sum_{j=1}^{k} b_j$$

for $k=1, \cdots, n$, then

$$\sum_{j=1}^{n} a_j^2 \leqq \sum_{j=1}^{n} b_j^2,$$

equality holding if and only if $a_j=b_j$, $j=1, \cdots, n$.

Solution by G. A. Hunt, Cornell University. We have

$$(a_k - a_{k+1}) \sum_{j=1}^{k} a_j \leqq (a_k - a_{k+1}) \sum_{j=1}^{k} b_j$$

for $k=1, \cdots, n$ and with $a_{n+1}=0$. Summing each side from $k=1$ to $k=n$ and using Abel's partial summation formula we obtain

$$\sum_{j=1}^{n} a_j^2 \leqq \sum_{j=1}^{n} a_j b_j.$$

An application of Schwarz's inequality now gives the result, which can obviously be extended to the case of infinite series.

Also solved by Stephen Chase, W. H. J. Fuchs, Arthur Gregory, Cecil Hastings, Jr., A. E. Livingston, M. Perisastri, J. B. Rosser, O. E. Stanaitis, and R. J. Walker.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4523. *Proposed by H. S. M. Coxeter, University of Toronto*

Given a point P in real projective space, and four skew lines a, b, c, d, find a linear construction for the line through P of the linear congruence determined by a, b, c, d.

4524. *Proposed by E. C. Milner, University of Malaya, Singapore*

Prove that for any positive integers n, N there are blocks of consecutive integers of length greater than N, with the property that each of their totients is divisible by n.

4525. *Proposed by Paul Erdös, National Bureau of Standards, Los Angeles*

Let $u_1 < u_2 < \cdots$ be the sequence of integers all of whose prime factors have exponents exceeding one. Prove that the density of integers of the form $u_i + u_j$ is zero (*i.e.* the number of integers $\leqq x$ of the given form is $< \epsilon x$ for any ϵ if x is large enough).

4526. *Proposed by Peter Ungar, New York University*

Show that for any $k > 1$ there exist k-chromatic graphs which contain no three mutually connected nodes.

Note. A linear graph is the figure obtained by joining certain parts of n points by line segments. If it requires k colors to so color the segments that no two ends of the same color meet at a node, the graph is k-chromatic.

4527. *Proposed by D. J. Newman, Harvard University*

Is there a function $f(x)$ continuous in the closed interval $(0, 1)$ and such that in this interval $f(x) + f(x^2) = x$?

SOLUTIONS

Convergence of an Erdös Sequence

4460 [1951, 636]. *Proposed by Paul Erdös, National Bureau of Standards, Los Angeles, Cal.*

Let $f(1) = 1$, $f(n) = (1 + c_1)f(n-1) - c_2 f(n-2) - \cdots - c_{n-1} f(1)$, $c_i \geqq 0$, $i = 1, 2, \cdots$, $c_1 = c_2 + c_3 + \cdots$. What is the necessary and sufficient condition for the convergence of $f(n)$?

Solution by Michael Golomb, Purdue University. Let $P(z)$ denote the function represented by the power series.

$$1 - (1 + c_1)z + c_2z^2 + c_3z^3 + \cdots = P(z).$$

Since $0 \leqq c_i \leqq c_1$, the series converges for $|z| \leqq 1$, and since $c_1 = c_2 + c_3 + \cdots$ we have $P(z) = (1-z)Q(z)$ where

$$Q(z) = 1 - c_2z - c_3z(1 + z) - c_4z(1 + z + z^2) - \cdots.$$

For sufficiently small z, $1/P(z)$ can be expanded as a power series, $\sum a_kz^k$, where $a_0 = 1$ and

$$a_k - (1 + c_1)a_{k-1} + c_2a_{k-2} + \cdots + c_ka_0 = 0.$$

From the definition of $f(n)$ it is seen that $f(n) = a_{n-1}$. Hence

$$\sum_{n=1}^{\infty} f(n)z^{n-1} = 1/P(z), \qquad \sum_{n=1}^{\infty} f(n)z^n = z/P(z),$$

$$f(1) + \sum_{n=1}^{\infty} \{f(n + 1) - f(n)\}z^n = \frac{1 - z}{P(z)} = \frac{1}{Q(z)}.$$

It follows that

$$\lim_{n\to\infty} f(n) = f(1) + \sum_{n=1}^{\infty} \{f(n + 1) - f(n)\}$$

exists if and only if the power series for $1/Q(z)$ converges for $z = 1$. Hence it is necessary that $Q(z) \neq 0$ for $z = x$ real, $0 \leqq x \leqq 1$, and this condition is also sufficient since $|Q(z)| \geqq 1 - c_2x - c_3x(1+x) - \cdots$ for $|z| \leqq x$. Since $Q(x)$ is monotone non-increasing and $Q(0) = 1$ we have $Q(x) \neq 0$ for $0 \leqq x \leqq 1$ if and only if $Q(1) > 0$, that is, if

$$C = c_2 + 2c_3 + 3c_4 + \cdots < 1.$$

This then is the necessary and sufficient condition for the convergence of $f(n)$. If this condition is satisfied $\lim f(n) = 1/Q(1) = 1/C$.

Also solved by R. P. Agnew, Robert Breusch, Einar Hille, D. J. Newman, H. N. Shapiro, and Gabor Szegö.

Probabilities for a Die of Irregular Shape

4461 [1951, 636]. *Proposed by George Grossman, De Witt Clinton High School, New York City*

A die is made of homogeneous material in the shape of a rectangular parallelepiped and is tossed onto a level surface from which it will not bounce. What is the probability that a particular face will come up on top?

Solution by H. C. Kranzer, Student, New York University. We treat the more general situation where the die may have the shape of any polyhedron and may be loaded in any manner, subject only to the restriction that it be capable of resting upon any of its faces. This last restriction is equivalent to the condition that the polyhedron be convex and that the foot of the perpendicular from the center of mass to the plane of any face lie within that face.

Suppose such a die is tossed onto a level surface in such a way that it will not bounce or roll. It is evident that if, at the moment of first contact between any part of the die and the surface, we drop a plumb line from the center of mass, the face through which this plumb line passes will be on bottom when the die has come to rest. Since the orientation of the die at the moment of first contact is perfectly at random, we may equally well choose a fixed orientation in space for the die and take an arbitrary random direction for the vertical. This shows at once the principal result: *The probability of any given face coming out on bottom is proportional to the solid angle subtended by that face at the center of mass.*

We may find the probability explicitly by the following method: Consider an elementary cone of volume dV, solid angle dw, having its vertex at C the center of mass, and having for its base a region of area dA of the face in question. Let r be the distance of this elementary region from C, and let h be the perpendicular distance from C to the plane of the face. Then $dV=r^3dw/3=hdA/3$; hence $dw=hdA/r^3$ and the required solid angle w is given by

$$w = h\iint dA/r^3,$$

where the integration is performed over the entire area of the face. The probability that the face come out on bottom is then given by

$$p = \frac{w}{4\pi} = \frac{h}{4\pi}\iint \frac{dA}{r^3}\cdot$$

In the special case of a homogeneous rectangular parallelepiped, the probability of a face coming up on top is the same as that of its complementary face coming out on bottom. We will suppose that the required face has sides $2a$ and $2b$, and that the other dimension of the parallelepiped is $2c$. Then C is at the geometrical center, and $h=c$. If we take polar coördinates with the origin at the center of the face and the polar axis parallel to side b, we have $dA=\rho d\rho d\theta$ and $r=(c^2+\rho^2)^{1/2}$. Then

$$p = \frac{1}{\pi}\arcsin\frac{ab}{\sqrt{(a^2+c^2)(b^2+c^2)}}\cdot$$

If $a=b=c$, this reduces to the expected value 1/6.

Also solved by F. G. Fender.

Editorial Note. J. G. Diaz proposes consideration of a die in the shape of a right prism whose bases are regular pentagons. J. H. Braun proposes a cuboctahedron. If the requirement of no bouncing or rolling is removed, the problem becomes much more difficult. Braun notes that in this case the answer cannot be based (solely) on the proportionate solid angle subtended by the faces at the centroid to the whole solid angle owing to the fact that some faces are nearer to the centroid than others, which reduces the tendency of the more remote faces to "show."

A Double Series Summation

4462 [1951, 636]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, New York*

If

$$S_{2m} = \frac{1}{2^{2m}} + \frac{1}{4^{2m}} + \frac{1}{6^{2m}} + \cdots,$$

find the value of

$$\frac{S_2}{2\cdot 3} + \frac{S_4}{4\cdot 5} + \frac{S_6}{6\cdot 7} + \cdots.$$

I. *Solution by V. C. Harris, San Diego State College, California.* If we let

$$T_{2m} = \frac{1}{2\cdot 3}\left(\frac{1}{2m}\right)^2 + \frac{1}{4\cdot 5}\left(\frac{1}{2m}\right)^4 + \frac{1}{6\cdot 7}\left(\frac{1}{2m}\right)^6 + \cdots,$$

then the value of the given expression is the same as that of $\sum_{m=1}^{\infty} T_{2m}$. By integrating $r/(1-r^2) = r + r^3 + r^5 + \cdots$, $|r| < 1$, from 0 to r twice and then setting $r = 1/2m$, we find

$$T_{2m} = 1 - \frac{2m+1}{2}\log\frac{2m+1}{2m} + \frac{2m-1}{2}\log\frac{2m-1}{2m},$$

$m = 1, 2, 3, \cdots$, whence

$$\sum_{m=1}^{n} T_{2m} = n + \log\frac{2^n(n!)}{(2n+1)^{n+1/2}} = \log\frac{2^n e^n(n!)}{(2n+1)^{n+1/2}}.$$

Using Stirling's formula, $\lim e^n n!/n^{n+\frac{1}{2}} = \sqrt{2\pi}$, we obtain

$$\sum_{m=1}^{\infty} T_{2m} = \tfrac{1}{2}\log(\pi/e).$$

II. *Solution by Leonard Carlitz, Duke University.* Starting with the familiar formula

$$\sin x = x \prod_{m=1}^{\infty}\left(1 - \frac{x^2}{m^2\pi^2}\right),$$

we have

$$\log \sin x = \log x - \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k}\,\frac{x^{2k}}{m^{2k}\pi^{2k}},$$

$$\log \sin \frac{x}{2} = \log \frac{x}{2} - \sum_{k=1}^{\infty} \frac{1}{k}\,\frac{x^{2k}}{\pi^{2k}} S_{2k}.$$

Integration yields

$$\int_0^{\pi} \log \sin \frac{x}{2}\, dx = \pi\left(\log \frac{\pi}{2} - 1\right) - \pi \sum_{k=1}^{\infty} \frac{S_{2k}}{k(2k+1)}\,.$$

Since the definite integral is known to have the value $-\pi \log 2$, the desired summation reduces to $\frac{1}{2}(\log \pi - 1)$.

Glaisher (*Messenger of Mathematics*, v. 44, 1914, 1–10) has proved a number of analogous results. In particular the following may be mentioned.

$$\frac{S_2}{3\cdot 4} + \frac{S_4}{5\cdot 6} + \frac{S_6}{7\cdot 8} + \cdots = \frac{1}{4},$$

$$\frac{S_2}{4\cdot 5\cdot 6} + \frac{S_4}{6\cdot 7\cdot 8} + \frac{S_6}{8\cdot 9\cdot 10} + \cdots = \frac{1}{48},$$

$$\frac{S_3}{4\cdot 5\cdot 6} + \frac{2S_5}{6\cdot 7\cdot 8} + \frac{3S_7}{8\cdot 9\cdot 10} + \cdots = \frac{1}{24},$$

$$\frac{S_3}{3\cdot 4} + \frac{S_5}{5\cdot 6} + \frac{S_7}{7\cdot 8} + \cdots = \frac{1}{2} - \frac{1}{2}\gamma,$$

where γ denotes Euler's constant. In connection with the last see also Ramanujan, *Messenger of Mathematics*, v. 46, 1917, 11–21 (or *Collected Papers*, 163–168).

Also solved by T. M. Apostol, Robert Breusch, Calvin Foreman, Samuel Goldberg, Emil Grosswald, Hyman Kaufman, M. S. Klamkin, M. Perisastri, Azriel Rosenfeld, P. C. Sikkema, Michael Skalskyj, O. E. Stanaitis, Chih-yi Wang, R. E. Wild, G. T. Williams, and the Proposer.

A Möbius Involution

4463 [1951, 702]. *Proposed by Roland Deaux, Faculté Polytechnique, Mons, Belgium*

The lines AP, BP, CP drawn through the vertices and a given point P in the plane of triangle ABC meet the opposite sides in A_1, B_1, C_1. Let A', B', C'

be the points which with A, B, C respectively separate harmonically the pairs (B_1, C_1), (C_1, A_1), (A_1, B_1). Prove that the pairs (A, A'), (B, B'), (C, C') belong to a Möbius involution.

Solution by the Proposer. Let F, F' be the foci of the conic section γ which touches in A_1, B_1, C_1 the sides BC, CA, AB. Siebeck's theorem* gives the direct circular transformation

$$\omega = \begin{pmatrix} F & F' & B & C_1 & A \\ F & F' & C & A & B_1 \end{pmatrix}$$

to which the Möbius involution

$$I_a = (F\ F',\ A\ A,\ B_1\ C_1)$$

is harmonic; A' is thus the second fixed point of I_a. Similarly we obtain the Möbius involutions

$$I_b = (F\ F',\ B\ B,\ B'\ B',\ C_1\ A_1), \qquad I_c = (F\ F',\ C\ C,\ C'\ C',\ A_1\ B_1),$$

whence the following Möbius involution

$$I = (F\ F,\ F'\ F',\ A\ A',\ B\ B',\ C\ C').$$

Note. Since γ and I have the same center O, the angles (OA, OA'), (OB, OB'), (OC, OC') have the same bisectors and

$$|OA\cdot OA'| = |OB\cdot OB'| = |OC\cdot OC'| = OF^2.$$

Suggested by the Curvature and Arc-length of a Circle

4464 [1951, 702]. *Proposed by R. Kissling, University of California, Berkeley*

Consider the class of single-valued differentiable functions $f(x)$ on the interval $0 \leqq x < 1$ such that $f(0) = 0$; $f(x) \to \infty$ as $x \to 1^-$. Let

$$K(x) = f'(x)/\{1 + [f(x)]^2\}^{3/2}, \qquad |K(x)| \leqq 1,\ 0 \leqq x < 1.$$

Prove that for any function of this class

$$\int_0^1 \{1 + [f(x)]^2\}^{1/2} dx = \pi/2.$$

I. *Solution by J. A. Tierney, United States Naval Academy.* We have

$$\int_0^1 K(x)dx = \lim_{\epsilon \to 0} \left[\frac{f}{(1+f^2)^{1/2}}\right]_{x=0}^{x=1-\epsilon} = \lim_{f\to\infty} \frac{f}{(1+f^2)^{1/2}} = 1.$$

It follows from $|K| \leqq 1$ on $[0, 1)$ that $K \equiv 1$ almost everywhere on $[0, 1)$. Hence

* See R. Deaux, *Introduction à la Géométrie des nombres complexes*, p. 125 (Brussels, 1947).

$$\int_0^1 (1+f^2)^{1/2}dx = \int_0^1 \frac{f'dx}{K(1+f^2)}$$

has the same value as

$$\int_0^1 \frac{f'dx}{1+f^2} = \tan^{-1} f\Big]_{x=0}^{x=1} = \pi/2.$$

II. *Solution by F. Bagemihl and W. Seidel, University of Rochester.* Let $g(x)=f(x)\{1+[f(x)]^2\}^{-1/2}$, $0\leqq x<1$, $g(1)=1$. Then $g(x)$ is continuous for $0\leqq x<1$; $g'(x)=K(x)$, and therefore $|g'(x)|\leqq 1$ for $0<x<1$; $g(0)=0$, $g(1)=1$. A simple application of the mean value theorem yields $g(x)=x$, and hence

$$f(x) = \pm\, x(1-x^2)^{-1/2}, \qquad 0 \leqq x < 1,$$

from which the relation to be proved follows immediately.

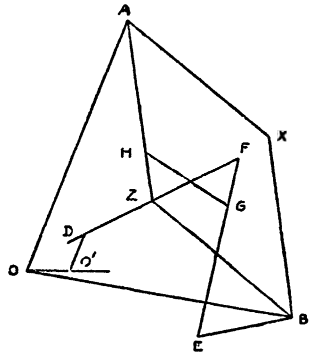
Also solved by J. W. Baldwin, A. B. Farnell, W. Fraser, Emil Grosswald, H. J. Hamilton, P. G. Kirmser, A. M. Peiser and Stanley Katz, J. H. Roberts, O. E. Stanaitis, and the Proposer.

A Conicograph

4465 [1951, 702]. *Proposed by E. I. Gale, University of Saskatchewan*

$AXBZ$ is a jointed rhombus with sides of length $4a$. (See figure. For convenience a is taken considerably greater than half the unit.) AO and BO are bars of equal length. The fixed centers are O and O' with OO' half a unit. Let $O'D=\frac{1}{2}$, $FE=4a$, $FZ=HG=BE=2a$, $HZ=FG=a$. D is an adjustable set screw on bar FZD so that the length ZD can be set at pleasure.

Show that as D moves in a circle about O', X describes the general conic of eccentricity $1/ZD$.



Solution by R. K. Morley, Worcester Polytechnic Institute. Figure $HZBEGF$ constitutes a standard Kempe angle bisector. That is, quadrilaterals $HZFG$ and $FZBE$, having sides proportional and common angle ZFG, and having cor-

responding parts similarly placed, are similar. Hence angles FZB and FZH are equal. Now the Peaucellier cell $OAXBZ$ is symmetrical about OX, so that FD produced passes through O and X.

Also in the Peaucellier cell $OAXBZ$

$$OZ\cdot OX = OA^2 - AX^2 = R^2 \text{ a constant.} \tag{1}$$

Let polar coördinates ρ and θ be OX and angle $O'OX$ respectively. Since $OO' = O'D = \frac{1}{2}$, we have $OD = \cos\theta$ and

$$OZ = DZ + OD = DZ + \cos\theta.$$

By (1)

$$\rho = OX = \frac{R^2}{OZ} = \frac{R^2}{DZ + \cos\theta} = \frac{R^2/DZ}{1 + (1/DZ)\cos\theta}\cdot$$

which is the equation of a conic with eccentricity $1/DZ$. (Note the relation between the different types of conic and the types of limacon traced by point Z.)

Also solved by L. V. Mead, R. C. Yates, and the Proposer.

Function Having Constant Modulus on the Unit Circle

4467 [1951, 703]. *Proposed by D. J. Newman, Harvard University*

Prove that az^n is the only entire function whose modulus is constant for all $|z| = 1$.

Solution by Alex Rosenberg, University of Michigan. Let $f(z)$ be an entire function of constant modulus on $|z| = 1$, and denote the zeros of $f(z)$ within the unit circle by a_i, $i = 1, \cdots, k$. Now the function

$$g(z) = f(z)\prod_i \frac{(1 - \bar{a}_i z)}{(z - a_i)}$$

is an integral function of constant modulus on $|z| = 1$, and having no zeros in the unit circle. By applying the maximum modulus theorem for the unit disk to $g(z)$ and to $1/g(z)$ we see that $g(z) = \alpha$, a constant. Thus

$$f(z) = \alpha\prod_i \frac{(z - a_i)}{(1 - \bar{a}_i z)}\cdot$$

But since $f(z)$ is an entire function, $a_i = 0$, so that $f(z) = \alpha z^n$.

It is clear that the same procedure will show that the most general meromorphic function of constant modulus on $|z| = 1$, is

$$\alpha\prod_i \frac{(z - a_i)}{(1 - \bar{a}_i z)}\prod_j \frac{(1 - \bar{b}_j z)}{(z - b_j)}\cdot$$

Also solved by J. B. Kelly, Jacob Korevaar, M. S. Robertson, W. Seidel, Vidar Wolontis, and the Proposer.

Bounds for the Coefficient a_1 in Certain Series Expansions

4468 [1952, 45]. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be convergent for $|z|<1$, and further let $0<|f(z)|\leqq 1$ for $|z|<1$. Prove $|f'(0)|\leqq 2/e$.

Solution by M. S. Robertson, Rutgers University. Let

$$F(z)=\sum_{n=0}^{\infty} c_n z^n = -\log f(z),$$

$c_0=-\log a_0$, $c_1=-a_1/a_0$. Then $F(z)$ is regular and $RF(z)>0$ for $|z|<1$. Hence a well known inequality* gives $|c_1|\leqq 2Rc_0$. This gives $|a_1|\leqq 2|a_0|\log(1/|a_0|)\leqq 2/e$, since $|a_0|\leqq 1$. Further, $|a_1|=2/e$ when $f(z)=\eta\exp(\epsilon z-1)/(\epsilon z+1)$, for η, ϵ arbitrary complex numbers of absolute value one.

The following generalization can be proved: let $f(z)$ satisfy

1) $f(z)$ regular for $|z|<1$,
2) $|f(z)|\leqq 1$ for $|z|<1$,
3) $f(z)$ has precisely n zeros β_i, $i=1, 2, \cdots, n$, in $0<|z|<1$,
4) $f(0)\neq 0$.

Define μ and ν by

$$\mu=\sum_{i=1}^{n}\frac{1-|\beta_i|^2}{\beta_i}, \qquad \nu=\prod_{i=1}^{n}\beta_i.$$

Then:

a) $|f'(0)|\leqq|\mu\nu|<1$ when $|\mu|\geqq 2$,
b) $|f'(0)|\leqq 2|\nu|e^{|\mu|/2-1}$ when $|\mu|\leqq 2$.

Proof. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and let $g(z)=\sum_{n=0}^{\infty} c_n z^n$ be defined by

$$f(z)=\prod_{i=1}^{n}\left(\frac{\beta_i-z}{1-\bar\beta_i z}\right)g(z).$$

Then $g(z)$ is regular for $|z|<1$, $|g(z)|\leqq 1$ for $|z|<1$, and when $a_0=f(0)\neq 0$ then $g(z)\neq 0$ for $|z|<1$. Thus $|c_1|\leqq 2|c_0|\log(1/|c_0|)$ as in the original problem. However,

$$a_1=\prod_{i=1}^{n}\beta_i\left[c_1-c_0\sum_{i=1}^{n}\frac{1-|\beta_i|^2}{\beta_i}\right],$$

$$|a_1|\leqq|\nu|\left[2|c_0|\log\frac{1}{|c_0|}+|\mu|\cdot|c_0|\right], \qquad |c_0|\leqq 1.$$

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* Carathéodory's inequality. See Nehari, *Conformal Mapping*, 1952, p. 170; also Bieberbach, *Funktiontheorie*, v. 2, p. 140.

For μ and ν fixed, if the right side is maximized over $0<|c_0|\leqq 1$, the stated results follow easily. Equality signs hold for

a) when $f(z) = \prod_{i=1}^{n}\left(\frac{\beta_i - z}{1-\bar{\beta}_i z}\right), \qquad 0<|\beta_i|<1, \ |\mu|\geqq 2;$

b) when $f(z) = \prod_{i=1}^{n}\left(\frac{\beta_i - z}{1-\bar{\beta}_i z}\right)\exp\left(\frac{|\mu|}{2}-1\right)\left(\frac{1+z}{1-z}\right),$

$$0<|\beta_i|<1, \quad |\mu|\leqq 2.$$

A slight modification takes care of the case when $f(0)=0$ instead of 4) above. Then $|f'(0)|\leqq|\nu|<1$, and equality holds when

$$f(z) = z\prod_{i=1}^{n}\left(\frac{\beta_i - z}{1-\bar{\beta}_i z}\right), \qquad 0<|\beta_i|<1.$$

The bounds obtained for $|f'(0)|$ are sharp and in each case less than unity. This is most easily seen from the inequality $|a_0|^2+|a_1|\leqq 1$ which holds for all $f(z)$ for which $|f(z)|\leqq 1$ in $|z|<1$. It would be an interesting problem to prove directly that the bound shown in b) is less than unity.

Also solved by J. B. Kelly, Jacob Korevaar, A. J. Lohwater, W. Seidel, Peter Ungar, Chih-yi Wang, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Theory of Matrices. By Sam Perlis. Cambridge, Mass., Addison-Wesley Press, Inc., 1952. xiv+237 pages. $5.50.

This is a finely written, excellent introduction to matrices. The style is natural and clear. Aimed at intelligent but immature students, a complete but elementary treatise on basic ideas was the author's purpose. In this he has succeeded. In subject material, we find deduction of canonical forms under equivalence, similarity (three different forms), congruence, Hermitian congruence, orthogonal and unitary similarity. Canonical forms for skew and skew-Hermitian matrices occur. These are not always to be found in such books. Matrix methods, exclusively, are employed. The elements are from a field (the only abstraction in the book) but may be taken as real or complex. There is a full treatment of polynomial matrices. The spectral theorem is treated well, including a clarifying preliminary step (9-5, p. 175) of identifying matrices susceptible

to spectral decomposition with those which are diagonable. Then, and earlier, characterizations of such matrices are given, culminating with admission of symmetric and Hermitian matrices, these being treated separately. The superdiagonal form is not given. Elementary divisor theory can be learned (pp. 124–158) with the vital fact (for teachers, and researchers seeking counterexamples) that a matrix exists with preassigned elementary divisors. There is also treatment of polynomials (unique factorization), vector space (as n-tuples), determinants (via elementary matrices). Linear transformations are postponed until the last chapter, and as a result similarity theory, spectral theorem *etc.*, are covered without the fundamental motivation. This, however, may be taken on faith, and the arguments have the advantage (for some) that they are purely arithmetic in nature. All treatments of transformations are based on coordinate systems. Inner products are touched lightly. The symbol (ξ, η) is not used; Schwarz's inequality does not appear; the adjoint of an operation is not defined, and for Hermitian matrices $(\xi, H\eta)=(H\xi, \eta)$ is not featured, but used once (p. 184) in the form $\bar{\xi}'H\eta=(\overline{H}\bar{\xi})'\eta$.

There are appendices scattered with such topics as applications to differential equations and mechanics. Only the barest beginnings are made, however.

The treatment of each facet of a given topic is approached as the student is prepared for it so that there is scattering. For example, linear systems are discussed on pp. 45–7, 54, 83. Thus the value as a text exceeds the utility as a handbook. Also, in reading, I often had the feeling that a subject had been abandoned too soon, only to find it resumed later.

The method of exposition, according to the preface, consists of: " . . . coddling (of students) . . . supplemented by activities leading the student to stand on his own feet." We find instruction on pronunciation of $F[x]$ (p. 108) but not $F(x)$ (p. 122). We are exhorted to write I multiplying the constant term of a polynomial g, in forming $g(A)$, with two tellings (pp. 66–7, 136) and a warning in case of failure. We are warned of strange (p. 77) and unusual (p. 85) fields (those of characteristic 2). The author's solicitude leads him to commit the factorial howler (p. 85) of announcing that in these fields "$1+1=0!$" On the other hand, the language is concise; the statements of many theorems (*e.g.* 3-3, p. 42) will have to be restated with quantifiers before certain students will be able to read them.

Italics distinguish theorems and the first occurrence of words. Reference is aided by occurrence of chapter and section numbers on each page. There are a two-page summary and a bibliography at the end. Theorems and definitions are correctly stated and are self-contained. An important point!

Debits. It is not made clear that a polynomial is not a function with scalar values, *i.e.* that x is an "indeterminate" (p. 17). This is somewhat cleared up on p. 108, but is insufficiently emphasized, and, on p. 132, x is called a variable. The crux is that it is not made clear what equality of two polynomials means.

V_n is a set of n-tuples, V a sub-vector-space of V_n. Can V itself be considered as a V_k? Is a 5-sub-space of V_6 "like" V_5, or a 5-sub-space of V_7, or another

5-sub-space of V_6? These are unanswered and the idea "isomorphism" is missing.

The last four lines of p. 85, and p. 86 give the impression that quadrics with congruent matrices can be obtained from each other by a rigid motion.

On p. 125, we must select a "polynomial of minimal degree in the class of all elements of all matrices equivalent to A." This sounds hard. The author should mention that it is easy in practice.

There are no worked-out examples, the ends of proofs are not clearly marked; I dislike a numbering system that puts Lemma 4-2 after Theorem 4-5.

Index. Not all important mentions of a subject are indexed (*e.g.* diagonal 160, 171, 173, 174, 184), nor are items in problems (*e.g.* non-singular, 232). Omitted are proper matric polynomial, 132; square root, 203; trivial factor, 110. There is a lack of a symbol-index (*e.g.* $N(A)$, 53; N, 161; $O(M)$, 225).

Problems. These are good and cover the topics. They are often quite demanding. No answers are given. Number 3 on page 181 is so trivial ($A=0$) as to be confusing, and number 8 on page 174 is false, *e.g.*

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is not similar to a diagonal matrix.

ALBERT WILANSKY
Lehigh University

Calculus. By Tomlinson Fort. D. C. Heath and Co., 1951. xii+560 pages, \$4.75.

Calculus (revised edition). By J. V. McKelvey. The Macmillan Co., 1951. vii+405 pages, \$4.50.

Calculus. By J. F. Randolph. The Macmillan Co., 1952. x+483 pages.

Calculus. By A. H. Sprague. The Ronald Press Co., 1952. xi+576 pages. \$6.50.

These four textbooks are designed for use in courses in calculus which presuppose a knowledge of plane analytic geometry. Fort and Sprague include a short but adequate discussion of polar coordinates. McKelvey's book also presupposes solid analytic geometry for some of the later chapters. The other three books give an adequate treatment for the purposes of calculus. All these texts give a rather complete treatment of more topics than can usually be covered in a course in differential and integral calculus. Fort and McKelvey include an extensive introduction to differential equations. No attempt will be made to compare these texts, but each has some essential characteristics that are worth pointing out to be interpreted as desired.

(1) Fort has given careful attention to mathematical details, proving most of the basic theorems carefully and clearly. Where proofs are omitted, he has tried to indicate what basic ideas are involved (*e.g.* that the proof of the existence of the limit of a bounded monotonic sequence depends on a systematic

study of the irrational numbers, a fact which is discussed further in the appendix). The first chapter is on infinite series, including the usual elementary tests for convergence and a brief treatment of power series. This is designed to develop the concept of limit and also to introduce power series for use in connection with exponential functions, trigonometric functions, and Taylor series. The power series definitions of sine and cosine are used along with the usual definition in deriving differentiation formulas for the trigonometric functions. He handles this and other seemingly advanced treatments in a way that should be convincing to elementary students and so that if not, the student can fall back on the usual treatment without being confused. The logarithm is introduced as an integral, its properties developed, and the exponential function defined in terms of the logarithm. Length of arc is defined as a definite integral before differentiation of arc length is discussed. This book attempts to handle calculus in a way unusual for American texts, but at the same time covers all the usual material with very clear, interesting, and concise explanations. For example, the concepts of slope, velocity and acceleration are introduced and carefully discussed as preparation for the definition of derivative. The attempt has not been so much to give a rigorous treatment as to make the ideas meaningful. It seems likely that he has succeeded. The appendix contains answers to most odd-numbered problems and tables of integrals, natural and Naperian logarithms, exponential and hyperbolic functions, and trigonometric functions.

(2) McKelvey's text is a revised edition of his calculus published in 1937. The general design of the book has not been changed, though an article on curvilinear motion and a chapter on limits and continuity have been added. The examples have been largely rewritten. They are particularly extensive and excellently chosen, with many examples to show the application of calculus in other fields. The concept of integration is not discussed until all the usual methods and applications of differential calculus have been developed, except for a short section in which arc length, plane areas, and volumes of revolution are computed as antiderivatives. Methods of evaluating indefinite integrals are treated extensively before the definite integral is introduced. Proofs of the fundamental limit and continuity theorems are omitted in the introductory chapter, the attempt being to concentrate on teaching understanding by use of examples. Careful definitions of limit of a sequence and continuity are given in the new chapter (Chapter 21), and a number of elementary theorems on limits are proven. Unfortunately, not so much care is given to the definition of limit of a function. Not only are such undefined concepts as "as x approaches c" used, but $\lim_{x\to c} f(x)$ is defined as $\lim_{n\to\infty} f(x_n)$ for any sequence $x_1, x_2, \cdots$ with $\lim_{n\to\infty} x_n = c$. The appendix contains a table of integrals and answers for all examples.

(3) Randolph has tried to write a text which "will provide some routine knowledge of the subject and its applications for all members of a class, but still challenge the good students." He does this largely in three ways: (1) Some advanced subjects such as line integrals and partial derivative systems are included. (2) Many explanations and proofs that might be omitted in a routine

course are italicised so omission does not seem confusing. Sometimes these are short remarks concerning rigor or lengthy discussions such as forms of the remainder in Taylor's theorem. (3) A number of subjects such as proofs of limit theorems and operations with power series are discussed carefully in the appendix with many interesting exercises given so that these topics may be taught along with the rest of the text. A careful discussion of tangents and velocity precedes the definition of derivative. The definite integral is introduced early. Many problems are given to emphasize the concept of definite integral as the limit of a sum. The definite integral is defined by using equal subdivisions and values at end-points, the usual definition being discussed in italics. This simplifies the definition, but complicates demonstrations of applications. An attempt is made to informally introduce foundations for several advanced topics. *E.g.* differential equations are introduced in connection with anti-derivatives, several simple differential equations are discussed along with methods of integration, and exact differential equations are introduced as "partial derivative systems." Great care is given to the function concept, the idea of ordered pairs being very carefully and clearly discussed. Some of the material in *Analytic Geometry and Calculus* by Randolph and Kac has been used, the books having great similarity in a few places. A table of integrals and "selected answers" are given.

(4) Sprague states that "the purpose of this book is to present the elements of the calculus as simply as possible without sacrificing rigor." In a number of cases (*e.g.* the fundamental limit theorems), proofs are given only after the theorems have been explained and used. An "intuitive" and the usual ϵ-δ definition of limit are given. The latter is used in proofs of limit and continuity theorems. Continuity is handled carefully throughout the text in a way that should be quite teachable. The definition of derivative is given and a number of elementary formulas developed before such usual aids to understanding as tangent lines and velocity are given. The anti-derivative is used briefly in connection with velocity and acceleration, but otherwise differential calculus is thoroughly studied before integration is introduced. Methods of evaluating indefinite integrals are thoroughly studied before the definite integral is defined as the limit of a sum. Some simple differential equations and applications are discussed in connection with indefinite integration. The definition of definite integral is very carefully and clearly given, with a discussion of uniform continuity and upper and lower integrals and a proof of the existence of the definite integral of a continuous function. The chapter on applications to solid analytic geometry includes an introduction to differential geometry. A short table of integrals, a collection of formulas, and answers to all problems are given.

R. C. JAMES
Haverford College

Symmetry. By Hermann Weyl. Princeton University Press, 1952. vii+168 pp. $3.75.

This little book, artistically printed and beautifully illustrated, consists of

four lectures given at Princeton as the author's "swan song" on the occasion of his retirement from the Institute for Advanced Study. It is so full of interesting material that, after beginning to read, one can hardly put it down. Although it is written for laymen, there are very few experts who will not learn something. For, besides mathematics and crystallography, it deals with philosophy, history, poetry, sculpture, painting, architecture, theology, physics, chemistry, physiology, embryology, and botany. The author has resisted the temptation to include an account of his own research on symmetry in space of many dimensions (see, *e.g.*, his contribution to the reviewer's *Regular Polytopes*, New York, 1949, pp. 204–207). Instead, he has collected a multitude of facts and theories, observations and quotations, drawings and photographs, blending them in his own inimitable fashion. The arrangement of text and illustrations is reminiscent of Steinhaus's *Mathematical Snapshots* (New York, 1950), but there is little overlapping of material.

The first lecture, on "Bilateral symmetry," contains an abundance of unusual information. We read, for example, that as many as one person in every five thousand is born with *situs inversus:* the consistent inversion of left and right throughout all the asymmetrical organs of the body. Turning from anatomy to biochemistry, the author gives a dramatic account of Pasteur's separation of laevo- and dextro-tartaric acid, of which only the latter occurs in organic nature. "Nature," he writes, "in giving us the wonderful gift of grapes so much enjoyed by Noah, produced only one of the forms, and it remained for Pasteur to produce the other!" But he soberly adds, "If there is a difference in principle between life and death, it does not lie in the chemistry of the material substratum."

The lecture on "Translatory, rotational, and related symmetries" includes a very readable introduction to the theory of vectors and to groups of transformations. On page 45, he makes the remark (too often overlooked in elementary instruction): "A vector is really the same thing as a translation, although one uses different phraseologies for vectors and translations." He ascribes to Leonardo da Vinci the enumeration of finite groups of congruent transformations in the plane: the cyclic groups $C_1, C_2, \cdots$ and the dihedral groups $D_1, D_2, \cdots$. These groups are illustrated by many examples from nature; *e.g.*, D_6 by snowflakes, D_5 by the geranium, and the rarer C_5 by *vinca herbacea.* At this point he might well have reproduced the 17-gonal section of a strand of equisetum from Hans Meierhofer, *Die Augen auf in unseres Herrgotts Garten!* (Zürich, 1947), page 48.

In connection with the occurrence of dihedral groups in animals, he refers to Sir D'Arcy W. Thompson's *Growth and Form*, "a masterpiece of English literature, which combines profound knowledge in geometry, physics, and biology with humanistic erudition and scientific insight of unusual originality."

One of the photographs reproduced from Thompson's book shows the giant sunflower, *Helianthus maximus*, whose florets form 34 right-handed and 55 left-handed spirals (p. 71). Since 34 and 55 are members of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \cdots,$$

this is an instance of the phenomenon called "phyllotaxis." A simpler instance is provided by any well-formed pineapple. The author cites an article of 1872 by P. G. Tait for the best attempt at an explanation.

He reproduces a page from Haeckel's *Challenger monograph*, which shows the skeletons of several Radiolarians, including an octahedron, an icosahedron and a dodecahedron "in astonishingly regular form." The same page had been used by Thompson (*Growth and Form*, Fig. 340), who made the following comment, in a letter to the reviewer, dated 9th March, 1947: "As to Haeckel, I wouldn't trust him round the corner, and I have the gravest doubt whether his pentagonal dodecahedron and various others ever existed outside his fertile fancy. I believe I may safely say that no type-specimens of these exist in the British Museum, or anywhere else. He was an artist, a pattern-designer, a skilled draughtsman. He had a minute professorial salary in a small University. The *Challenger* paid eight guineas apiece for as many plates as he chose to draw; and he kept on drawing them, and lived on the proceeds (so they used to say) till the end of his life. He represents a thoroughly bad period in Natural Science."

In the lecture on "Ornamental symmetry" we see that the plane tessellation of regular hexagons occurs not only on the floors of bathrooms but in the bees' honeycomb, in the parenchyma of maize, in the retinal pigment of our eyes, and on the surface of certain diatoms. There is an account of the enumeration (by Pólya and Niggli, 1924) of the seventeen two-dimensional crystallographic groups, which were unconsciously used by the Egyptians, Moors and Chinese in their ornaments. In this connection, some illustrations are reproduced from Owen Jones' *The Grammar of Ornament* (London, 1868). In the same lecture we find an introduction to the theory of positive definite quadratic forms and their representation by lattices. There is also a proof that the only possible periods for rotational symmetry of a lattice are 2, 3, 4, and 6. A still neater proof was given long ago by William Barlow, Philosophical Magazine (6 ser., 1, 1901, p. 17).

The final lecture, on "Crystals: the general mathematical idea of symmetry" begins with an outline of the enumeration of the 230 space-groups, which was carried out independently by Fedorov in Russia (1885), Schoenflies in Germany (1891) and Barlow in England (1894). Then the notion of a congruent transformation in Euclidean 3-space is extended to that of a Lorentz transformation in Minkowskian space-time. This "world" is an affine 4-space with a real cone of isotropic lines through each point. It is unfortunate that the author says (on p. 132): "The light cone . . . at a definite world-point O, 'here-now,' . . . divides the world into future and past." He seems to have overlooked the third region, exterior to the cone, which one could reasonably call "the present."

From relativity theory he deftly turns to quantum theory, Galois theory, and cyclotomy, showing the importance of the group of automorphisms in the investigation of any "structure-endowed entity."

The author's command of language is truly amazing. Who else would say (p. 127), "Temperature is the environmental factor *kat'exochen*"? Only twice is he at a loss for the English equivalent of a German word: on page 77 he writes "Umklappung" for *half-turn*, and on page 96 "modul" for *modulus*.

The four lectures are supplemented by two mathematical appendices: "Determination of all finite groups of proper rotations in 3-space" and "Inclusion of improper rotations." The latter is especially neat. Finally, there is a list of acknowledgments for the 72 figures, and a good index (except that the entry "Thompson" confuses Sir D'Arcy with Lord Kelvin, whose space-filling of truncated octahedra is displayed on page 92).

H. S. M. COXETER
University of Toronto

NEW BOOKS RECEIVED

Basic Skills in Mathematics. By H. V. Price and L. A. Knowler. Boston, Ginn and Company, 1952. 8+249 pages. $3.25.

Light (The Student's Physics, Vol. I.). By R. W. Ditchburn. Glasgow, Scotland, Blackie and Son, Ltd., 1952. 22+680 pages.

Science and Method. By Henri Poincaré. New York, Dover Publications, Inc., 1952. 288 pages. Paperbound, $1.25; Clothbound, $2.50.

Science and Hypothesis. By Henri Poincaré. New York, Dover Publications, Inc., 1952. 244 pages. Paperbound $1.25.

Tables of the Bessel Functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, 0-x-1, Washington, D. C., National Bureau of Standards. October, 1952. 60 pages, 40 cents.

Vlakke Meetkunde voor Voortgezette Studie. By P. Wijdenes. Groningen, Djakarta, P. Noordhoff, 1952. F 13.00, geb f14.50. 303 pages.

Statistical Theory in Research. By R. L. Anderson and T. A. Bancroft. New York, McGraw-Hill, 1952. xvi+399 pages, $7.00.

Methods of Statistical Analysis, 2nd Edition. By C. H. Goulden. New York, John Wiley and Sons, Inc., 1952. vi+467 pages. $7.50.

Introduction to the Theory of Games. By J. C. C. McKinsey. New York, McGraw-Hill Book Company, 1952. x+371 pages. $6.50.

Theorie der Geometrischen Konstruktionen (*Lehrbucher und Monographien aus dem Gebiete der Exakten Wissenschaften*). By L. Bieberbach. Verlag Birkhauser, Basel, 1952. 6+162 pages. Bound Frs. 18.70, Unbound Fr. 15.60.

Society of Actuaries' Textbook on *Life Contingencies*. By C. W. Jordan, Jr. 1952. xi+331 pages. $8.00.

The Methods of Statistics. Fourth Edition. By L. H. C. Tippett. New York, John Wiley and Sons, 1952. 395 pages. $6.00.

Introduction to the Foundations of Mathematics. By R. L. Wilder. New York, John Wiley and Sons. xiv+305 pages. $5.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

GENAILLE'S RODS

Always a favorite topic for club discussions is that of "Napier's Rods," a device for reducing multiplication to a series of additions. Napier describes these well-known rods, which became popularly known as Napier's "bones," in his *Rabdologia* published shortly after his death in 1617. The literature contains several accessible articles describing these bones.* Not so well-known are the more ingenious rods invented by Genaille in 1885. Readers familiar with Napier's bones should have no difficulty in understanding the principle underlying Genaille's rods upon studying the figure on the opposite page. Here the rods are placed for the multiplicand 471,963; the partial products are read from right to left, the points of the triangles guiding the eye to the proper figures.

* For example, see *The Pentagon*, Spring 1949, pp. 98–100.

MATHEMATICS AND PHILATELY

An interesting club project would be the collecting of postage stamps which have reference in some way to mathematics or mathematicians. For example, a topical collection could be formed of stamps portraying geometrical figures such as triangles, pentagons, octagons, *etc.* Stamps which carry portraits of mathematicians are of particular interest. A comprehensive description of such stamps was published by C. B. Boyer in 1949.* Since that time, several additional mathematicians have been so honored, and a revised list might be of interest.

The following check-list contains 82 items honoring 34 mathematicians. Catalog numbers and descriptions given are those in *Scott's Standard Postage Stamp Catalogue*, 1952.

Abel, Niels Henrik (1802–1829)
  Norway, 1929:
    145. 10 ö green
    146. 15 ö red
    147. 20 ö chocolate
    148. 30 ö ultramarine
Ampere, Andre Marie (1775–1836)
  France, 1936:
    306. 75 c brown
  France, 1949:
    626. 15 fr sepia
Armero, Julio Garavito (1865–1920)
  Colombia, 1949:
    573. 4 c green on white
Avicenna (979–1037)
  Lebanon, 1948:
    223. 30 pi orange brown and buff
    224. 40 pi Prussian green and buff
  Germany, 1952:
    10N106. 35 pf blue
Bolyai, Farkas (1775–1856)
  Hungary, 1932:
    479. 70 f cerise

* Carl B. Boyer, "Mathematicians and Philately," *Scripta Mathematica*, vol. 15, pp. 105–114, June, 1949.

Boscovich, Roger Joseph (1711–1787)
Croatia, 1943:
59. 3.50 k copper red
60. 12.50 k dark violet brown
Brahe, Tycho (1546–1601)
Denmark, 1946:
300. 20 ö dark red
Chaplygin, Sergei A. (1869–1944)
Russia, 1944:
945. 30 k gray
946. 1 r light brown
Chebychev, Pafnutiy Lvovich (1821–1894)
Russia, 1946:
1050. 30 k brown
1051. 60 k gray-brown
Condamine, Charles-Marie de la (1701–1774)
Ecuador, 1936:
347. 2 c blue
348. 5 c olive-green
349. 10 c orange
350. 20 c purple
351. 50 c salmon
Copernicus, Nicholas (1473–1543)
Poland, 1923:
192. 1000 m indigo
193. 5000 m rose
Poland, 1942–43:
NB23. 1 z+1 z dull myrtle green
NB27. 1 z+1 z rose lake
Descartes, René (1596–1650)
France, 1937:
330. 90 c copper red
331. 90 c copper red
Dürer, Albrecht (1471–1528)
Germany, 1925:
362. 80 pf chocolate
Eötvös, Lorand (1848–1919)
Hungary, 1932:
471. 6 f yellow green
Hungary, 1948:
840. 60 f deep red
Euler, Leonard (1707–1783)
Germany, 1950:
10N58. 1 pf gray
Galilei, Galileo (1564–1642)
Italy, 1933:
D16. 35 c rose red
Italy, 1942:
419. 10 c dark orange and lake
420. 25 c gray green and green
421. 50 c brown violet and violet
422. 1.25 l Prussian blue and ultramarine
Italy, 1945:
D18. 1.40 l blue
Gerbert, i.e., *Pope Sylvester II* (c. 950–1003)
Hungary, 1938:
511. 1 f deep violet
516. 10 f red orange
Gazeta, Matematica
Romania, 1945:
596. 2 l sepia
597. 80 l blue-black
Hamilton, Sir William Rowan (1805–1865)
Irish Free State, 1943:
126. $\frac{1}{2}$ d green
127. $2\frac{1}{2}$ d brown
Huygens, Christiaan (1629–1695)
Netherlands, 1928:
B36. $12\frac{1}{2}$ c+$3\frac{1}{2}$ c ultramarine
Jacobsen, Jacob Christian (1811–1887)
Denmark, 1947:
304. 20 ö dark red
Leibniz, Gottfried Wilhelm (1646–1716)
Germany, 1926:
360. 40 pf deep violet
Lorentz, Hendrik Antoon (1853–1928)
Netherlands, 1928:
B35. $7\frac{1}{2}$c+$3\frac{1}{2}$c vermillion
Mercator, Gerard (1512–1594)
Belgium, 1942:
B324. 1.75 fr+50 c dull blue
Ortelius, Abraham (1527–1598)
Belgium, 1942:
B325. 3.25 fr+3.25 fr lilac rose
Pascal, Blaise (1623–1662)
France, 1944:
B181. 1.20 fr+2.80 fr black
Poincaré, Henri (1854–1912)
France, 1952:
18 fr+5 fr black
Roemer, Ole (1644–1710)
Denmark, 1944:
293. 20 ö henna brown
Stevin, Simon (1548–1620)
Belgium, 1942:
B321. 50 c+10 c fawn
Swedenborg, Emanuel (1688–1772)
Sweden, 1938:
264. 10 ö violet
266. 10 ö violet
267. 100 ö green
Vasilyevich, Mihail (1801–1861)
Russia, 1951:
1604. 40 k black brown on pink

Vinci, Leonardo da (1452–1519)
Italy, 1932:
C29. 1 l violet
C30. 3 l brown red
C31. 5 l deep green
C33. 10 l+2.50 l black brown
C34. 100 l bright blue and greenish black
Italy, 1935:
347. 50 c purple
348. 1.25 l dark blue
Italy, 1937:
404. 50 c light violet
C103. 2 l royal blue
C105. 5 l deep green
Liechenstein, 1948:
C24. 10 rp dark green
France, 1952:
682. 30 fr deep ultramarine
Germany, 1952:
10N104. 20 pf green
Hungary, 1952:
C109. 1.60 fo dark Prussian green
Italy, 1952:
601. 25 l deep orange
Poland, 1952:
B73. 30 g+15 g ultramarine
Romania, 1952:
878. 55 b purple
Witt, Johan de (1625–1672)
Netherland, 1947:
B177. $7\frac{1}{2}$ c+$2\frac{1}{2}$ c dark purple brown
Zhukovsky, Nikolai (1847–1921)
Russia, 1941:
838. 15 k deep blue
839. 30 k carmine rose
840. 50 k brown violet
Russia, 1947:
1098. 30 k sepia
1099. 60 k blue violet

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

EUROPEAN STUDY TRIP FOR MATHEMATICS AND SCIENCE TEACHERS

The School of Education of Boston University is sponsoring a study trip for approximately ten weeks during the summer of 1953. This trip is designed to acquaint mathematics and science teachers with the background and resources of Western Europe which will be applicable to their teaching. England, Scotland, Norway, Sweden, Denmark, the Netherlands, Germany, Switzerland, and France will be visited. Six semester hours of college credit will be credited through Boston University.

The group will be accompanied by Professor H. W. Syer of Boston University who will act as director of the tour and guide to the cities. Plans will be made in advance for the group to be met by mathematics and science teachers from the country being visited, officials from the ministries of education, and experts of the museums and laboratories so that authentic information will be at hand. A specialized guidebook entitled "Resources in Europe for Mathe-

matics and Science Teachers" has been prepared.

For further information write to Professor H. W. Syer, School of Education, 332 Bay State Road, Boston, Massachusetts.

PERSONAL ITEMS

Professor W. H. Fagerstrom of City College of the City of New York represented the Association at the Seventeenth Educational Conference which was held in New York City on October 30–31, 1952.

Professor Einar Hille of Yale University was a representative of the Association at the first Assembly of the International Commission for Mathematical Instruction which was held at Geneva, Switzerland, on October 20–21, 1952.

The honorary degree of Doctor of Science was conferred on Dean Emeritus L. P. Eisenhart by Princeton University on June 17, 1952.

Dr. Kurt Gödel of the Institute for Advanced Study was awarded the honorary degree of Doctor of Science by Harvard University on June 19, 1952.

Dr. A. E. Livingston of the University of Oregon has received a National Science Foundation Postdoctoral Fellowship and is at the Institute for Advanced Study.

Associate Professor V. O. McBrien who has received a Ford Foundation Fellowship is on leave of absence from the College of the Holy Cross and is at Harvard University.

Mr. E. C. Molina of Newark College of Engineering was awarded the Cressan Medal by the Franklin Institute, Philadelphia, Pennsylvania, on October 15, 1952, in recognition of his contributions to telephony.

Professor R. C. Stephens of Knox College has been awarded a Ford Foundation Faculty Fellowship and is spending the year at Princeton University.

Alabama Polytechnic Institute announces: Mr. J. A. Pond has been appointed to an assistant professorship; Miss Frances A. Norton has been appointed to an instructorship.

At Boston College: Dr. Lorenzo Calabi, previously at the University of Rome, has been appointed to an assistant professorship; Reverend J. J. McCarthy, formerly professor of Physics and Mathematics at Weston College, has been appointed to an associate professorship; Mr. P. T. Banks has been promoted to an assistant professorship.

Bradley University reports the following: Mr. J. R. Brown, previously a graduate student at Kansas University, and Mr. H. E. Sandstrom, who has been a graduate student at Northwestern University, have been appointed to assistant professorships; Mr. J. H. Hafferkamp, formerly a graduate student at Bradley University, has been appointed to an instructorship.

Brown University makes the following announcements: Dr. W. G. Lister, who was AEC Postdoctoral Fellow at Yale University during 1951–52, has been appointed to an assistant professorship; Professor R. E. Gilman is on leave of absence until June 30, 1953 and is with the Weapons Systems Evaluation Group, Washington, D. C.; Professor Maurice Heins, who is on sabbatical leave during

1952–53, has a Fulbright award for research and is spending the year in Paris, France; Professor Heins was a delegate of the American Mathematical Society to the International Congress of Mathematicians at Salzburg, Austria, in September, 1952.

At Carnegie Institute of Technology: Associate Professor E. A. Whitman has retired with the title of Associate Professor Emeritus; Mr. C. E. Lemke, Mr. F. B. Smith, and Mr. S. C. Ying have been appointed to instructorships.

Catholic University of America announces: Dr. W. W. Boone has been promoted to an assistant professorship; Dr. R. J. Silverman of the University of Illinois has been appointed to an instructorship.

Johns Hopkins University makes the following announcements: Assistant Professor G. D. Mostow of Syracuse University and Dr. N. C. Ankeny, previously a member of the Institute for Advanced Study, have been appointed to assistant professorships; Dr. T. J. Rivlin, formerly a graduate student at Harvard University, has been appointed to an instructorship; Instructor A. B. J. Novikoff is now with the Institute for Cooperative Research of the University.

Knox College reports the following: Assistant Professor A. O. Lindstrum is Acting Chairman of the Department of Mathematics during the absence of Professor R. C. Stephens; Dr. I. J. Christopher of the University of Oregon has been appointed to an instructorship.

Los Angeles City College announces that the third annual William B. Orange Mathematics Prize Competition for students in the high schools of the Los Angeles School District will be held in May, 1953; one hundred fifty-five students participated in the preceding contest and prizes were awarded to ten of the thirty-two participating teams and to twenty-four individuals.

Montana State College reports: Assistant Professor Bernard Ostle of Iowa State College has been appointed to an associate professorship; Dr. Frederick Young, previously with the United States Naval Ordnance Test Station, China Lake, California, has been appointed to an assistant professorship; Mr. Glen Ingram, formerly a student at the College, has been appointed to an instructorship; Mr. J. E. Whitesitt has been awarded a predoctoral National Science Research Fellowship and is spending the year at the University of Illinois.

North Carolina State College announces the promotions of Miss Anna M. Harris and Mr. C. F. Lewis to assistant professorships.

Reed College announces the following: Dr. Tong Hing of Columbia University has been appointed to an assistant professorship; Mr. Joseph Roberts of the University of Minnesota has been appointed to an instructorship.

At Southern Illinois University Dr. Annette Sinclair of the University of Tennessee has been appointed to an assistant professorship. Professor H. S. M. Coxeter of the University of Toronto was one of the four guest lecturers at the Leonardo da Vinci festival which was held at the University during November 12–25, 1952; he discussed the application of mathematics in the work of da Vinci.

University of Georgia announces: Professor Tomlinson Fort has retired as

Head of the Department of Mathematics but retains the position of Regents' Professor of Mathematics; Professor Fort is on sabbatical leave during 1952–53 and is lecturing at the Royal Naval College, University of Aberdeen, Cambridge University and other British universities; Associate Professor A. C. Cohen, Jr. has been promoted to a professorship; Dr. S. E. Dyer has been appointed to an assistant professorship. Professor W. T. Martin of Massachusetts Institute of Technology gave a series of lectures on Analytic Functions of Several Complex Variables at the Colloquium of the University on November 19–21, 1952.

University of Mississippi reports the following: Mr. J. C. McCall, previously a graduate student at the University, has been appointed to an instructorship; Assistant Professor R. D. Sheffield has been granted a leave of absence for an additional year to continue graduate study.

At the University of Nebraska: Assistant Professor O. C. Collins has been promoted to the position of Associate Professor of Astronomy; Assistant Professor W. G. Leavitt has been promoted to an associate professorship; Dr. T. A. Newton, previously a teaching assistant at the University of Georgia, and Mr. A. J. Tingley, formerly a teaching assistant at the University of Minnesota, have been appointed to instructorships.

Dr. M. A. A. Al-Bassam has been promoted to a professorship at Huston-Tillotson College.

Dr. P. H. Anderson has accepted a position as a price economist with the Office of Price Stabilization, Washington, D. C.

Mr. D. L. Arenson who has been with the Armour Research Foundation has a position as Senior Research Engineer with Cook Research Laboratory, Skokie, Illinois.

Mr. B. J. Ball of the University of Texas has been appointed to an acting assistant professorship at the University of Virginia.

Associate Professor R. H. Bardell of the University of Wisconsin in Milwaukee has been promoted to the position of Professor and Chairman of the Department of Mathematics.

Mr. William Beck, formerly a graduate assistant at the University of Kentucky, has been appointed to a teaching assistantship at the University of Southern California.

Professor H. A. Bernhard of the Evans Signal Corps Laboratory has accepted a position as Senior Engineer with the Curtiss-Wright Corporation.

Mr. R. E. Blasch, previously a student at Hofstra College, has been appointed to an assistantship at the University of Illinois.

Assistant Professor J. H. Blau of Pennsylvania State College has been appointed to an associate professorship at Antioch College.

Mr. Daniel Block of Yeshiva University has been promoted to an assistant professorship.

Professor N. R. Bryan of the University of Maine has retired with the title of Professor Emeritus.

Reverend W. F. Burns, formerly of the College of the Holy Cross, is now at Fairfield University.

Assistant Professor L. J. Burton of Bryn Mawr College has been appointed to an assistant professorship at Lake Forest College.

Dr. R. K. Butz of the University of Georgia has been appointed to an assistant professorship at Colorado Agricultural and Mechanical College.

Miss F. Marion Clarke, formerly at the University of Nebraska, is now at Aberdeen Proving Ground, Maryland.

Mr. D. L. Daly has been appointed to an instructorship at Missouri School of Mines and Metallurgy.

Mr. D. J. Davis, formerly a mathematician in the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland, has accepted a position as a physicist with the Rocket Development Group, Redstone Arsenal, Huntsville, Alabama.

Mr. R. C. Davis of the University of Akron has been promoted to an assistant professorship.

Professor Patrick Du Val of the University of Georgia is now at the University of Bristol, England.

Graduate Assistant E. R. Epperson of Miami University has a position as a graduate fellow at Massachusetts Institute of Technology.

Associate Professor Ky Fan of the University of Notre Dame has been promoted to a professorship.

Mr. G. C. Francis, previously at Carleton College, has accepted a position as a mathematician with Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mrs. Ruth M. Frisch who has been a graduate student at the Polytechnic Institute of Brooklyn has been appointed to a graduate assistantship at Syracuse University.

Mr. H. N. Garber, previously a student at the University of Pennsylvania, has been awarded a National Science Foundation Fellowship and is a graduate student at Massachusetts Institute of Technology.

Assistant Professor W. H. Gottschalk of the University of Pennsylvania has been promoted to an associate professorship.

Dr. Emil Grosswald of the University of Saskatchewan has been appointed to the rank of associate at the University of Pennsylvania.

Dr. Carl Hammer, formerly chairman of the Division of Technical Education, Walter Hervey Junior College, New York City, has accepted a position as Senior Research Engineer at Franklin Institute, Philadelphia, Pennsylvania.

Mr. J. S. Hokanson, previously of Ripon College, has a position with Oscar Mayer and Company, Madison, Wisconsin.

Assistant Professor C. E. Jones of Agricultural and Technical College of North Carolina has been appointed to an associate professorship in the Engineering School of Tennessee Agricultural and Industrial State College.

Dr. S. T. Kao, previously a graduate student at Catholic University of

America, has been appointed to an assistant professorship at the College of St. Joseph on the Rio Grande, Albuquerque, New Mexico.

Associate Professor Leo Katz who is on leave of absence from Michigan State College during 1952–53 is at the Statistical Laboratory, University of California, Berkeley.

Assistant Professor Dora E. Kearney of Iowa State Teachers College, Cedar Falls, has been appointed to an assistant professorship at Westminster College, Salt Lake City, Utah.

Dr. H. W. Kuhn, previously lecturer at Princeton University, has been appointed to an assistant professorship at Bryn Mawr College.

Dr. J. R. Larkin of the University of Kansas has accepted a position as a mathematician with the Bureau of Ordnance, Navy Department, Washington, D. C.

Dr. A. E. Livingston, formerly a graduate assistant at the University of Oregon, is now a member of the Institute for Advanced Study.

Assistant Professor G. R. MacLane of Rice Institute has been promoted to an associate professorship.

Associate Professor Leonard McFadden of Virginia Polytechnic Institute has been promoted to a professorship.

Dr. Paul Meier has accepted a position as a research associate at the School of Hygiene and Public Health, Johns Hopkins University.

Mr. E. J. Musch who has been an instructor at the University of Louisville is now a mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Professor E. N. Oberg, who was on leave of absence during the academic year 1951–52 to serve as an applied mathematician with the North American Aviation, Incorporated, Los Angeles, California, has resumed his duties in the Department of Mathematics and Astronomy at the State University of Iowa.

Mr. T. J. Pignani, previously a graduate student at the University of North Carolina, has been appointed to an instructorship at Loyola University, New Orleans, Louisiana.

Miss V. Elise Qualls, formerly an instructor at Tennessee Polytechnic Institute, has a position as a physicist at the Ballistic Laboratory, E. I. du Pont de Nemours and Company, Wilmington, Delaware.

Mr. A. A. Ritchie who has been teaching at Ponca City Senior High School, Oklahoma, has been appointed to an assistant professorship at Millsaps College.

Mr. P. T. Rotter of the Mutual Benefit Life Insurance Company has been promoted to Associate Mathematician.

Assistant Professor Arthur Saastad of DePaul University has a position as an industrial engineer with United States Steel Company, Chicago, Illinois.

Assistant Professor D. R. Scholz of Southwestern Louisiana Institute has been appointed to an assistant professorship at Louisiana State University.

Dr. R. W. Shephard of the Rand Corporation has accepted a position with the Sandia Corporation, Albuquerque, New Mexico.

Rosenbloom, P. C.; Rosser, J. B.; Seebeck, C. L.; Sheffer, I. M.; Springer, C. E.; Struik, D. J.; Vaughan, H. E.; Wade, T. L.; Wallace, A. D.; Winger, R. M.; Yates, R. C.; Zuckerman, H. S.

Mathematical Notes: Agnew, R. P.; Beckenbach, E. F.; Berkowitz, J.; Blank, A.; Blumenthal, L. M.; Brauer, A. T.; Brenner, J. L.; Boas, R. P.; Busemann, H.; Carlitz, L.; Carver, W. B.; Craig, H. V.; Erdelyi, A.; Feller, W.; Fine, N. J.; Frame, J. S.; Friedman, B.; Grove, V. G.; Gustin, W. S.; Hall, M.; Hanson, E. H.; Harrington, W. J.; Herzog, F.; Hollcroft, T. R.; Isaacson, E.; Kaplansky, I.; Lax, P.; Lee, H. L.; Lehmer, D. H.; Levine, J.; MacDuffee, C. C.; Mann, H. B.; Mann, W. R.; Murnaghan, F. D.; Nirenberg, L.; Parker, W. V.; Pettis, B. J.; Price, G. B.; Rainville, E. D.; Reid, W. T.; Rinehart, R. F.; Robinson, R.; Salzer, H. E.; Shapiro, H.; Sinclair, A.; Snyder, W. S.; Struik, D. J.; Tornheim, L.; Truesdell, C. A.; Wilansky, A.; Wilson, R. L.; Wright, E. M.; Zygmund, A.

Classroom Notes: Apostol, T. M.; Bryan, J. G.; Carver, W. B.; Coddington, E. A.; Cohen, I. S.; Corliss, J. J.; Fort, M. K.; Franklin, Philip; Harrington, W. J.; Hart, J. J.; Hildebrand, F. B.; Huff, G. B.; McCoy, N. H.; Ogilvy, C. S.; Parker, W. V.; Richmond, D. E.; Rudin, Walter; Singer, I. M.; Spear, Joseph; Struik, D. J.

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, Canada, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS, Washburn Municipal University of Topeka, April 11, 1953.

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Teachers College, Columbia University, March 28, 1953.

MICHIGAN, Wayne University, Detroit, April 18, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, April 24, 1953.

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City, October, 1953.

PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, April 17–18, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

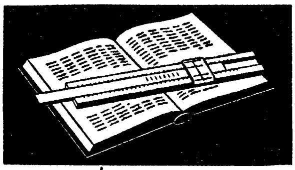
SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN

TEXAS, Fort Worth, April 24–25, 1953.

UPPER NEW YORK STATE, United States Military Academy, West Point, May 9, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.



GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 3

CONTENTS

MARCH 1953

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of College Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

SOME OBSERVATIONS ON UNDERGRADUATE MATHEMATICS IN AMERICAN COLLEGES AND UNIVERSITIES

E. A. CAMERON, University of North Carolina

1. Introduction. During the academic year 1951–1952 it was my privilege, under a grant from the Ford Foundation, to visit the mathematics departments of thirty-three American colleges and universities. These institutions, located in the East, the Midwest, and the Far West, extended from Boston to Los Angeles, from Philadelphia to Seattle. Among them were privately endowed universities, state universities, private colleges, and city colleges. They were selected on the basis of academic excellence, and it is believed that they represent institutions of the highest scholastic standing in the regions mentioned. My primary mission was to study undergraduate mathematics programs. The members of the departments visited were most helpful and generous with their time in supplying information about their own institutions and in participating in discussions on various aspects of undergraduate mathematics instruction. This experience has resulted in certain impressions concerning the present state of undergraduate mathematical education in this country, at least as reflected by the institutions visited, which may be worth passing on to others interested in this matter.

2. Mathematics in general education. Relative to the status of mathematics as a required subject, in only four of the institutions visited is it required of all students for graduation. In three others, mathematics or philosophy is required. The most usual general requirement is one which specifies that a certain number of courses be selected from a group consisting of the natural sciences and mathematics. A question of interest here is the extent to which mathematics is regarded as an essential subject in a liberal education. Concerning this point, I made the following observations. In the institutions which have inaugurated "general education" courses in the humanities, the social sciences, or the natural sciences, with but a single exception mathematics has no place in any of these courses. Indeed, in some institutions which have had the longest experience with such courses, and also in other institutions, the mathematical needs of only those students who follow certain special curricula are seriously considered. This suggests that some mathematics departments are not especially concerned with the contribution their subject might make to a liberal education or else feel that this contribution is automatically provided by the usual introductory courses. On the other hand, one finds individual members of these departments vitally interested in this question, but who, for various reasons, are unable at present to implement their convictions with changes in course content or organization. Another interesting observation is that, generally speaking, in the West one finds fewer instances of mathematics considered as a basic part of a liberal arts program than in the East.

3. Types of freshman courses. The traditional college algebra and trigonometry are still the usual freshman courses in most institutions. However, eighteen of the institutions visited provide courses of a non-traditional character which at least some students can elect or can take to satisfy requirements involving mathematics. These non-traditional courses are usually designed for students not planning to pursue work in which mathematics is needed as a tool. Their content frequently includes elements of analytic geometry and calculus as well as some algebra and trigonometry, and occasionally topics from even more advanced subjects such as number theory and topology. Recently, in some places probability and statistics have come to be regarded as appropriate subjects for inclusion in such a course. With the increasing application of mathematical statistics to more and more areas of human knowledge, there seem to be cogent reasons for teaching as many students as possible something about the nature of statistical inference. In some courses considerable attention is devoted to logic and the character of mathematics as a logical structure. Most of the people with whom I talked agree that much of the trigonometry and some of the algebra in the traditional freshman course can well be replaced by mathematics that is more interesting and of greater significance. Especially is this true for the student who will not take any more mathematics.

The type of course which offers most promise of substantial contribution to a general education is not adequately described by merely listing the topics covered. The spirit in which the subject is treated is of the greatest importance. An understanding of the nature and significance of mathematics is sought through an emphasis on basic concepts, the logical processes used in developing the subject, and the relation of the discipline to other fields through a consideration of its origins and its applications. Techniques, of course, are necessary, but there is plenty of evidence that many students pass their freshman courses by memorizing techniques without obtaining the slightest insight into the true nature of mathematics. Such a procedure could hardly contribute much to a liberal education. These courses, whose nature is so sketchily suggested here, have as their primary objective the fuller realization of the educational values long believed to be inherent in the discipline of mathematics. It is to be emphasized that it is not thought that this can be accomplished by use of descriptive material *about* the subject. Serious mathematics must be the backbone of the course. But exactly what topics constitute the most suitable content for such a course is by no means fully decided. Much experimentation remains to be done. Increasing dissatisfaction with the inadequate contributions of traditional courses to a general education is impelling some institutions to undertake serious investigation in this area. The number of good textbooks suitable for a course of this character is extremely small. This fact has undoubtedly discouraged some institutions from instituting such a course. It seems likely that in time this deterrent will be eliminated.

It would appear that one of the characteristics of many mathematics teachers is their reluctance to try something new. As a result they frequently get

into a rut—it is easy to do in elementary courses in this field—and the listlessness of their students is a reflection of their own boredom with a subject grown stale from endless repetition. The institutions in which the quality of instruction impresses one most favorably are those in which the courses are constantly studied for possible improvement. A stabilized course tends to become a stagnant course.

The freshman course designed for the non-specialist, which we have been discussing, is thought by some also to be the best type of introductory course for students who will take further courses in the field. It is recognized that in the transition from courses of this type to advanced work provision must be made to supply certain knowledge and skills not included in a course planned as terminal.

4. Analytic geometry and calculus. In some institutions with high entrance requirements the usual freshman course consists of analytic geometry and calculus. Whether taught in the freshman year or later, there is an increasing tendency to teach these two subjects together. A goodly number of mathematicians are convinced that from a purely mathematical viewpoint there is much to be gained by teaching them together. Also, from the standpoint of the student's whole program an earlier introduction to the calculus has many advantages; one obvious one is the availability of this tool for use in elementary physics courses. There are topics in analytic geometry which, while interesting mathematics in themselves, are not a necessary part in the mainstream development, and consequently can be and frequently are omitted in these combination courses. Several institutions are experimenting with a more rigorous type of calculus course for their better students. There is considerable difference of opinion as to the degree of rigor feasible in a first course in calculus. Here the importance of variation in ability among students becomes very evident. It appears that only the best ones are able at this stage to assimilate the type of rigor proposed. Another important factor may be the nature of previous work in mathematics, whether a certain maturity in understanding mathematical concepts and proofs has been developed.

5. Upper college courses. In the work of the junior and senior years, a fairly recent innovation is the offering at the majority of the institutions visited of one or more courses in modern algebra. The proper content and level of abstractness for these courses are by no means universally agreed upon. There are those who believe that the first course should be fairly concrete, perhaps devoted mostly to matrices and vector spaces, while others think an introduction to various abstract algebraic systems is the most valuable type of first course. In any event, in the near future instructors will have a considerably larger number of textbooks from which to choose than has been the case previously. In many institutions courses in classical theory of equations are being replaced by some form of modern algebra.

In analysis there is considerable thought being given to courses which make

the transition from elementary calculus to graduate courses in function theory. The nature of these courses varies rather widely among different institutions.

One also finds great variation among institutions in interest and course offering in geometry. Occasionally, mention is made of the need for the reformulation of courses in this field. One possible future trend is the teaching of certain parts of geometry and algebra together—such as introductory projective geometry and linear algebra.

Most departments now offer one or several courses in mathematical statistics. A few institutions provide sequences of special courses designed to give the necessary mathematical training for social scientists. The increasing interest in applications of mathematics in the social sciences may influence courses in the calculus, matrix theory, and other topics in algebra. It is rather surprising to learn that a knowledge of the structure of some of the abstract algebraic systems is proving useful in certain types of investigations in the social science field.

In about half of the institutions visited, something besides formal courses is provided for undergraduates specializing in mathematics. These extra activities take the form of reading courses, honors work, seminars, tutorials, *etc.* The main purpose is to have the student do some independent work under appropriate supervision. The chief deterrent to a more extensive occurrence of these practices is the cost in terms of faculty time. In six of the institutions comprehensive examinations in mathematics are given to all seniors specializing in the subject, and in three others these examinations are given students in the honors program. At several places, theses in the major subject are required of seniors.

6. Teacher training. In the realm of teacher training, courses in algebra and geometry aimed at meeting the special needs of secondary school teachers are frequently offered. It would appear that the character of these courses in some institutions should be scrutinized for their relevance to their purported purpose. Occasionally, courses in fundamental concepts and the history of mathematics are recommended for prospective teachers. It must be admitted that university mathematics departments do not always fully discharge their obligations in the training of teachers. Too often mathematicians are content to complain of the poor preparation students receive in high schools without taking any steps to determine how departments of mathematics might help to remedy the situation. As a result, too large a part of the training of high school teachers is frequently left to less competent agencies.

7. Universities versus colleges. It is generally conceded that undergraduate education in the large universities is by and large not up to the high standards set by the good small colleges. There are, of course, many reasons for this: differences in admission requirements, extensive use in universities of graduate assistants to teach elementary courses, the presence or absence of an atmosphere conducive to good academic work, *etc.* One of the most important factors is that in universities the men with the imagination, the energy, and the enthu-

siasm to raise teaching to truly inspirational levels are frequently so heavily engaged in research, training graduate students, and other activities that they simply do not have the time to devote to elementary teaching. Thus, in many of our great universities the quality of undergraduate instruction has lagged behind that of research and graduate training. There appears to be in some places a genuine need for more mathematicians with the qualities mentioned above, drawing on their learning and inventiveness, to contribute ideas, constructive suggestions, and a part of their time to the essential task of undergraduate education.

State universities have their own peculiar problems. Many of them are due to a heterogeneous student body, representing an incredibly wide range of ability, preparation, and interest. In several states all graduates of accredited high schools must be admitted to the state university. In most of the state institutions some provision for individual differences is made in the freshman year by offering algebra courses at various levels. It is quite usual to find "intermediate algebra"—second-year high school algebra—taught in universities, and frequently college credit is given for the course. In state universities with relatively unselected freshman classes, differentiated programs of study appear to be the only feasible way of providing the type of education appropriate to various levels of ability. The present practice of setting standards and adjusting levels of teaching for the median student frequently results in something which is beyond the grasp of the poorer students and at the same time fails to challenge the better ones. This problem is recognized at many institutions but it is far from being solved.

8. Exchange of information. It became obvious to me that there is a genuine need for a freer exchange of information between institutions, particularly information regarding innovations and experiments. Perhaps the Association, through its meetings and through this MONTHLY, can make a greater contribution to this end. Also, personal correspondence and private conversation, where opportunity permits, would certainly be of great mutual benefit to everyone engaged in this process of trying to improve undergraduate mathematical education.

TYPES OF FUNCTIONS*

H. P. THIELMAN, Iowa State College

1. Peculiar functions. Let f be a function whose domain of definition X is a neighborhood space. Let P be a point property of the function f. The function f is said to be ***peculiar with respect to the property P*** if there exists a partition of X into two subsets X_1 and X_2 each everywhere dense in X and such that the property P holds at every point of X_1 and fails to hold at every point of X_2.

For example, a function whose domain of definition is the interval (a, b), whose points of continuity are everywhere dense in (a, b) and whose points of discontinuity are everywhere dense in (a, b) is peculiar with respect to the property of continuity, and also with respect to the property of discontinuity. The function $f(x)$, which is defined in the open interval $(0, 1)$ as follows:

$$\begin{aligned} f(x) &= 0 \quad \text{if } x \text{ is irrational} \\ &= 1/q \quad \text{if } x = p/q, \text{ where } p \text{ and } q \text{ are relatively prime positive integers} \end{aligned}$$

is such a peculiar function.

We give another example of a function which is peculiar with respect to continuity. In the last example the points of continuity constituted a set of measure one, while the points of discontinuity formed a set of measure zero. In the next example these characteristics are reversed for the corresponding sets, that is, the points of continuity will form a set of measure zero, while the points of discontinuity will constitute a set of measure one.

By the well known method used for the construction of the Cantor ternary set we construct a sequence of sets $N_1, N_2, N_3, \cdots, N_n, \cdots$, such that each of these sets is nowhere dense in the closed interval $[0, 1]$. The complement in $[0, 1]$ of the set N_n consists of the points of a denumerable number of non-overlapping, non-abutting open intervals. We construct the set $N_n (n=1, 2, 3, \cdots)$ so that the sum of the lengths of the complementary intervals is $1/2^n$. Then the set S which is given by

$$S = \sum_{i=1}^{\infty} N_i$$

is a set of Baire's first category whose measure is one. Its complement is everywhere dense in $[0, 1]$ and has measure zero. We now define a sequence $\{f_n(x)\}$ of functions as follows:

$$\begin{aligned} f_n(x) &= 1/2^n \quad \text{if } x \text{ is an element of } N_n, \\ &= 0 \quad \text{if } x \text{ is not an element of } N_n. \end{aligned}$$

* An excerpt from an address presented to the Minnesota Section of the Mathematical Association of America at the invitation of the Executive Committee on May 10, 1952.

The function

$$F(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (0 < x < 1)$$

is the required function.

We first show that $F(x)$ is discontinuous at each point of S. Let x be an element of S. Then x is an element of at least one of the sets $N_n (n=1, 2, 3, \cdots)$. Suppose $x \in N_k$. Then $F(x) \geqq 1/2^k$. But in every neighborhood of x there are points of the everywhere dense complement of S where $F(x)=0$. Hence $F(x)$ is discontinuous at x.

Next we show that $F(x)$ is continuous at every point of the complement of S. Let c be a point of the complement of S, and let a positive number ϵ be given. Since the series $\sum f_n(x)$ converges uniformly in $[0, 1]$ there exists a positive integer m such that the remainder

$$\sum_{n=m+1}^{\infty} f_n(x)$$

is less than ϵ for all x in $[0, 1]$. We can find a subinterval (α, β) of $[0, 1]$ containing c and such that no point of the finite set of nowhere dense sets N_1, $N_2, \cdots, N_m$ lies in (α, β). Then for every x of (α, β), x is either an element of S, or x belongs only to those nowhere dense sets N_n for which $n>m$. In either case $|F(c)-F(x)| = |F(x)| < \epsilon$, and $F(x)$ is continuous at c.

We have thus given two extreme examples of functions which are peculiar with respect to continuity.

We might mention that there can exist no function which is continuous at every rational point and discontinuous at every irrational point. This follows from the fact that the set of irrational points is not of Baire's first category while the points of discontinuity of every pointwise discontinuous function constitute a set of Baire's first category.

It is easy to construct functions which are peculiar with respect to differentiability.

We shall next give examples of functions which are peculiar with respect to a certain property which can be considered as a generalization of the notion of continuity. After that we shall show that there exist point properties with respect to which there exist no peculiar functions.

2. Neighborly functions. Let f be a function whose domain of definition is a neighborhood space X with neighborhoods denoted by N, or by N_x, and let the range of f be a neighborhood space whose neighborhoods will be denoted by M_y or by $M_{f(x)}$. The function f is said to be ***neighborly**** at a point c of its domain of definition if for every neighborhood N_c of c and for every neighborhood

* This concept was defined for the case of functions of a real variable whose ranges are metric spaces by Woodrow W. Bledsoe. Proc. Amer. Math. Soc., vol. 3, pp. 114–115, 1952.

$M_{f(c)}$ of $f(c)$ there exists a neighborhood N contained in N_c (but not necessarily containing c) such that for every element of x of N it is true that $f(x)$ is an element of $M_{f(c)}$. A function which is neighborly at each point of its domain of definition is called a ***neighborly function***. A function is said to be ***non-neighborly*** at a point if it is not neighborly at that point.

It is obvious that if a function is continuous at a point then it is neighborly at that point.

The function $f(x)=\sin 1/x$ if $x\neq 0, f(0)=c$ $(-1\leqq c\leqq 1)$ is neighborly at $x=0$, but it is discontinuous at that point.

A function $f(x)$ is said to be *pointwise discontinuous* in X if the set of points x where $f(x)$ is continuous is everywhere dense in X, but not closed relative to X. A function $f(x)$ is *pointwise continuous* in X if the sets of points x where $f(x)$ is discontinuous is everywhere dense in X but not closed relative to X.

A function $f(x)$ is said to be ***pointwise non-neighborly*** in X if the set of points x where $f(x)$ is neighborly is everywhere dense in X, but not closed relative to X. A function $f(x)$ is said to be ***pointwise neighborly*** in X if the set of points x where $f(x)$ is non-neighborly is everywhere dense in X, but not closed relative to X.

The functions given in the examples of section 1 are pointwise continuous and pointwise discontinuous and also pointwise neighborly and pointwise non-neighborly in their domains of definition. Those functions are thus peculiar with respect to continuity and also with respect to neighborliness.

3. Cliquish functions.† Let the domain of definition of a function f be a neighborhood space whose neighborhoods are denoted by N or N_x, and let the range of the function be a metric space with metric ρ. The function f is said to be *cliquish at a point c* of the closure of the domain of definition if for every positive number ϵ and for every neighborhood N_c of c there exists a neighborhood N contained in N_c (but not necessarily containing c) such that for every two elements x_1 and x_2 of N it is true that

$$\rho[f(x_1), f(x_2)] < \epsilon.$$

It is obvious that if a function is continuous or neighborly at a point it is cliquish at that point. The function $f(x)$ given by the equations

$$\begin{aligned} f(x) &= \sin 1/x && \text{if} \quad x \neq 0, \\ &= 2 && \text{if} \quad x = 0, \end{aligned}$$

is cliquish at $x=0$, but it is not neighborly at that point.

A function which is cliquish at every point of its domain of definition is called a *cliquish function*. The functions of the examples in section 1 are only pointwise continuous and pointwise neighborly but they are cliquish. As a matter of fact, we shall see that every function which is pointwise non-neighborly is cliquish.

† A similar concept, called *neighborly'*, was defined by Bledsoe (loc. cit., p. 115) for functions of a real variable whose ranges are metric spaces.

A function is said to be *non-cliquish at a point* if it is not cliquish at that point. The function which is zero at each rational point and one at each irrational point is an example of a function which is non-cliquish at every point.

A function is said to be *pointwise non-cliquish* in X if the set of points where the function is cliquish is everywhere dense in X but not closed relative to X. A function is *pointwise cliquish* in X if the set of points where the functions is non-cliquish is everywhere dense in X but is not closed relative to X.

A function which was both pointwise cliquish and also pointwise non-cliquish in its domain of definition would be peculiar with respect to cliquishness. We shall now prove that there exist no such functions. Even though cliquishness of a function is similar to continuity and neighborliness of a function we have the interesting result that there exist no functions which are peculiar with respect to cliquishness.

THEOREM I. *Let f be a function with domain of definition X. If f is cliquish at each point of a set dense in X, then f is cliquish on X. In other words, there are no pointwise non-cliquish functions.*

Proof. Let f be the function given in the statement of the theorem. There exists a set C which is everywhere dense in X, and which is such that for each point c of C the function f is cliquish at c. Let x be a given point of X, and let N_x be an arbitrary, given neighborhood of x. In N_x there exists at least one point c of C. Let a positive number ϵ be given, and let N_c be a neighborhood of c such that N_c is contained in N_x. Since f is cliquish at c, there exists a neighborhood N contained in N_c (and hence contained in N_x) such that for every x_1 and x_2 of N it is true that

$$\rho[f(x_1), f(x_2)] < \epsilon.$$

Since N is contained in N_x, and since N_x was an arbitrary neighborhood of x, f is cliquish at x. But x was an arbitrary, given point of X. Therefore f is cliquish at every point of X. In other words f is cliquish.

As a direct consequence we have the result to which we alluded above that every pointwise non-neighborly function is cliquish.

THEOREM II. *The set of points at which a pointwise cliquish function is cliquish is nowhere dense.*

Proof. Suppose the set C where f is cliquish were not nowhere dense in the domain of definition X of f. Then there would exist at least one neighborhood N such that C would be everywhere dense in N. Then f would be cliquish on a dense set in N. Hence by the last theorem f would be cliquish at every point of N. This contradicts the hypothesis that the set of points where f is non-cliquish is everywhere dense in X.

It might be of interest to note that a type of converse of this theorem holds. That is, for every set S which is nowhere dense in an interval (a, b) there exist pointwise cliquish functions which are cliquish on S and non-cliquish at each

point of the everywhere dense complement of the closure of S in (a, b). An example of such functions can be constructed in the following way. The complement of the closure of S in (a, b) is an everywhere dense set which consists of the points of a denumerable number of non-overlapping open intervals. On each of these intervals the function is defined as follows: Let (α, β) be such an interval. Then

$$\begin{aligned} f(x) &= (x-\alpha) \quad \text{if} \quad \alpha < x \leqq \tfrac{1}{2}(\alpha+\beta), \quad \text{and } x \text{ is rational,} \\ &= -(x-\beta) \quad \text{if} \quad \tfrac{1}{2}(\alpha+\beta) < x < \beta, \quad \text{and } x \text{ is rational,} \\ &= 0 \qquad\qquad \text{if} \quad \alpha < x < \beta \quad \text{and } x \text{ is irrational.} \end{aligned}$$

For each point x of the closure of S, $f(x)=0$. This function is easily seen to be continuous, and hence cliquish, at each point of S. On every open interval (α, β) of the everywhere dense complement of S in (a, b), $f(x)$ is totally discontinuous and non-cliquish.

THEOREM III. *The limit $F(x)$ of a sequence of cliquish functions can be non-cliquish at every point of its domain of definition.*

Proof. Let $F(x)=1$ if x is a rational number in the open interval $(0, 1)$, and let $F(x)=0$ if x is an irrational number in $(0, 1)$. This function is non-cliquish at each point of $(0, 1)$. It is, however, the limit of the sequence $\{F_n(x)\}$, where

$$F_n(x) = \sum_{q=2}^{q=n} f_q(x),$$

and where the functions $f_q(x)$ are defined as follows:

$f_q(x)=1$, if $x=p/q$, $p<q$, and p and q are relatively prime

positive integers,

$$f_q(x) = 0, \text{ if } \quad x \neq p/q.$$

Each $F_n(x)$ is obviously cliquish at each point of $(0, 1)$.

The three theorems stated illustrate how different the point property of cliquishness of a function is from the properties of continuity and neighborliness. The corresponding statements for continuous and neighborly functions are false. Thus for example, it is not true that every function, which is continuous on a dense set, is continuous, nor is every function, which is neighborly on a dense set, neighborly. As opposed to Theorem II we have for continuous functions the result that the set of points at which a pointwise continuous function is continuous may be everywhere dense. Instead of Theorem III there is the well known result that the limit of a sequence of continuous (neighborly) functions is at most pointwise discontinuous (non-neighborly).

The next two theorems and their proofs are analogues of similar theorems

and proofs of the theory of pointwise discontinuous and neighborly functions.

THEOREM IV. *The points of discontinuity of every cliquish function constitute a set of Baire's first category.*

THEOREM V. *Every cliquish function is at most pointwise discontinuous.*

COLLEGIATE MATHEMATICS AND THE NATIONAL SCIENCE FOUNDATION

W. L. DUREN, JR., Tulane University

Having recently completed a short term as Acting Program Director for Mathematics in the National Science Foundation, I should like to make some remarks on the relation of collegiate mathematics to the Foundation. By collegiate mathematics I mean the professional activities associated with undergraduate and beginning graduate teaching, whether in a college or university.

The National Science Foundation Act of 1950 (Public Law 507—81st Congress) directed the Foundation in part "to develop and encourage the pursuit of a national policy for the promotion of basic research and education in the sciences; to initiate and support basic scientific research in the mathematical, physical, medical, biological, engineering and other sciences, by making contracts or other arrangements (including grants, loans and other forms of assistance) for the conduct of such basic scientific research and to appraise the impact of research upon industrial development and the general welfare; . . . ; to award scholarships and graduate fellowships in the mathematical, physical, medical, biological and other sciences; . . . " Note that this directive includes the strengthening of education in the sciences. However, we can assume that it does not mean that the Government will enter into the field of education as such.

It is well recognized that collegiate mathematics will be stronger if the teachers continue in active scholarship and research in their fields. If teaching duties interfere too much, the mathematician needs free time to carry out his research. This sort of research is supported by various government agencies, including the National Science Foundation, on a basis of proposals which describe individual research projects. If the proposal is favorably reviewed by mathematical consultants and there are funds, a grant is made to the institution to permit the work of the scientist to be carried out. In mathematics the principal item of the budget of any such proposal is the salary of the investigator for the time to be spent on the work. This is always determined by the investigator's regular academic salary. For example, if the academic salary covers service for nine months,

the full time summer research salary may be as much as one third the academic salary. The payment of a fraction of the salary of a man of professorial rank during the academic year for part-time research, when accompanied by a corresponding reduction in the total service load of the investigator, is still being considered by the National Science Foundation, although there are doubts as to the wisdom of it. These doubts do not apply to the relief of junior research personnel.

Before preparing a proposal for a research grant, the investigator should consult the "Guide for the Submission of Research Proposals" obtainable from the National Science Foundation, Mathematical, Physical and Engineering Sciences Division, Washington 25, D. C. The formal proposal should come from the administrative office of the institution. The budget should include an allowance for the publication which will arise from the research. It may include necessary travel in the United States and may include an item for the indirect costs of the institution not exceeding 15 per cent of the total. It is feasible for the investigator to write directly to the Program Director for Mathematics with a tentative proposal before submitting a formal one. The above describes the individual, short-term research proposal. A continuing group project having a similar effect to a departmental block grant is being described by the author in an article submitted to the Bulletin of the American Mathematical Society.

I have never seen a proposal asking for assistance in conducting a program of undergraduate honors work for gifted young people. Most colleges lack the funds to carry out such a program because the direction of these young people takes more time and is more costly than conventional teaching. But this is something which would obviously contribute much to the welfare of science in the future. Hence a department which has a tradition of some accomplishment in this activity might appropriately submit a proposal for an "undergraduate research" program. The needs might be some fraction of the time of the staff to be devoted to direction of the original investigations of the students selected to work for honors in mathematics. Also, in particular cases, some undergraduate research assistantships (don't call them scholarships) will be needed to permit the honors men to replace part time jobs by earning a comparable amount of money in their scientific work. It would be fine if there were an inexpensively published journal for publishing the better examples of these undergraduate researches, perhaps edited by graduate students.

There are other proper activities of mathematicians in a college or university which would contribute much towards the health of mathematical and scientific research. It will take some imagination to cast these good research-supporting activities in the form of proposals which ask for support in terms which government agencies can consider. One such area which should be explored, in my opinion, is the service of mathematicians as consultants to the scientific research groups in their own institutions. Under present circumstances the institution cannot ordinarily recognize this potentially valuable work by reduction of

normal duties and the mathematical profession does not recognize the work as a substitute for mathematical research. But when there is a mathematician who likes this sort of thing and he is good at it, he can be very valuable. Perhaps in applying for support for the general mathematical consultant, his work might be linked with the teaching of the intermediate level service courses in applied mathematics which are severely limited in a small institution.

Still another domain which needs cultivation in the ranks of mathematicians as a whole is what might be termed developmental mathematical research. It is rather generally recognized that there is an unhealthy tendency, brought about in part by the limitations of publication, to give the exalted title of RESEARCH only to the construction of the very general and abstract theories. The working out of particular problems within these theories is usually relegated to "masters' theses," even though there is no reasonable chance that so immature a student could comprehend what he would need to know. It would be good if there were more mathematicians who would unabashedly commit themselves to this sort of developmental research. It would bring more immature mathematicians into research which would be appropriate to their interests and would contribute to the understanding and use of the general theories laid out by the few real innovators. It would tend to preserve problem-solving ability in American mathematics and in some cases could be expected to lead into new general theory. A cheap journal, associated with an abstracting and evaluation service, perhaps a microcard or microfilm journal, would be a help in recording the results without undue bulging of the libraries or taxing the finances of publications. Even as things are now, there is nothing to exclude this type of proposal.

Some things which are probably not supportable by government agencies are: regional scholarships, sabbaticals, teaching, preparation of new courses, writing of textbooks (though the research which goes into a book may be supported), travel abroad (except under Fulbright Act or travel to a scientific meeting abroad), block grants to an institution without specific determination of the purpose and use and accountability for the results of the work, and anything continuing in perpetuo. Research which derives its scientific interest from a particular geographic location, however meritorious, will usually meet with difficulty. In general, the words "education" and "scholarships" or "fellowships" should be avoided in research proposals, because these are things which are nationalized if supported at all, and come under the Scientific Personnel and Education Division of the Foundation.

Finally, I should like to make a plea to all the members of this Association to submit proposals to the Foundation expressing your real needs in research or support of research. Do not be disturbed if these proposals must be rejected for lack of funds or other reasons, but consider that your time spent was a public service and a service to mathematics. For only from proposals can the real needs of mathematics and mathematicians be determined.

A SUPPLEMENTARY NOTE TO A 1946 ARTICLE ON FERMAT'S LAST THEOREM

H. S. VANDIVER, The University of Texas

Concerning the relation

$$x^l + y^l + z^l = 0, \tag{1}$$

where l is an odd prime >2, the writer (this MONTHLY, vol. 53, pp. 555–578, which article we shall refer to here as (I)) gave a number of references to papers which included results concerning (1) which enabled us to state that Fermat's well known statement is true for certain classes of exponents l in (1) and none of these classes was the null class. In error I omitted several papers which belong to the type I definitely meant to treat. I shall now give reports on these latter papers.

H. J. S. Smith [1] gave a report of some of the principal results which had been found concerning (1) prior to the year 1860, particularly those of Kummer.

K. Hensel [2] commented on Kummer's various results concerning Fermat's last theorem.

Got [3] reproduced with explanations Kummer's [4] memoir of 1857 on Fermat's last theorem.

J. McDonnel [5] proved that if p is an odd prime and

$$x^p + y^p + z^p = 0 \tag{2}$$

with $p>2$ and if

$$(yz + zx + xy, p) = 1, \qquad (x, y, z, p) = 1,$$

and

$$\frac{r^{p-1} - 1}{p} = q(r),$$

$$x^2 - yz \equiv 0 \pmod{r},$$

then it follows that

$$q(r) \equiv 0 \pmod{p}.$$

Also, if (2) holds with

$$(x(y - z)(x^2 + yz), p) = b, \qquad (x, y, z, p) = 1$$

and

$$x^2 + yz \equiv 0 \pmod{r_1},$$

then

$$q(r_1) \equiv 0 \pmod{p}.$$

L. Holzer [6] showed that if (1) holds with l an odd prime, and

$$(x, y, z, l) = 1, \qquad a = xy - z^2 \not\equiv 0 \pmod{l}, \qquad a \equiv 0 \pmod{r},$$

then

$$q(r) \equiv 0 \pmod{l}.$$

Also, if

$$((x - y)(xy + z^2), l) = 1, \qquad xy + z^2 \equiv 0 \pmod{r_1},$$

then

$$q(r_1) \equiv 0 \pmod{l}.$$

Compare this with our preceding reference.

H. Hasse [7] gave an account of some of the principal results for the first case of Fermat's last theorem, including some of those obtained by Kummer, Mirimanoff, and Furtwängler. In particular, he obtained the Kummer criteria, that is, if (1) is satisfied with l an odd prime, and $xyz \not\equiv 0 \pmod{l}$, then

$$B_n f_{l-2n}(t) \equiv 0 \pmod{l}; \qquad n = 1, 2, \cdots, (l-3)/2;$$

where if $m > 1$,

$$f_m(t) = \sum_{i=1}^{l-1} i^{m-1} t^i; \; t = -x/y, \; -y/x, \; -y/z, \; -z/y, \; -x/z, \; -z/x,$$

and B_n is the nth Bernoulli number, $B_1 = 1/6$, $B_2 = 1/30$, *etc.* The method he used to obtain this is different than that which was employed by Kummer, Hasse using explicit expressions for the power characters of units in a cyclotomic field.

M. Krasner [8] using the criteria of Kummer in the form originally given by the latter instead of Mirimanoff's form as given in our last reference, proved the following result:

There exists a number l_0, such that for $l \geqq l_0$, the existence of three integers, x, y, z, prime to l and such that

$$x^l + y^l + z^l = 0,$$

implies that l is a divisor of $[\sqrt{\log l}]$ consecutive Bernoulli numbers, the last of which is $B_{(l-3)/2}$. Here the symbol $[k]$ means the greatest integer in k.

In (I) page 575, paragraph 5, the writer discussed various results which led him to the opinion that Fermat's last theorem is true in Case I, that is, where $xyz \not\equiv 0 \pmod{l}$ in (1). In March of last year I received a letter from Dr. Krasner, which refers to his result quoted above, where l is replaced by p. I quote herewith his letter in part:

"Concerning your discussion of the truth of Fermat's Theorem, I think as

you that in the Case I it is certainly true. Your argument is certainly in order. (This statement refers to the reference given above to the present writer.) Even the numbers p not satisfying my criterion given in Theorem VIII of your article must be very exceptional. But I think that my preceding result given in the *Comptes Rendus* of 1934 furnishes even much stronger arguments in this sense. It seems quite unlikely that all the $[\sqrt[3]{\log p}]$ consecutive Bernoullian numbers

$$B_{(p-1)/2-i}; \qquad 0 < i \leqq \sqrt[3]{\log p}$$

are divisible by p, if only p is not too small. If you admit that the probability for an unknown Bernoullian number to be divisible by p is $1/p$, such a divisibility has only the probability

$$\left(\frac{1}{p}\right)^{[\sqrt[3]{\log p}]} \approx \frac{1}{p^{\sqrt[3]{\log p}}},$$

and a simple calculation shows that the mathematical expectation of the number of primes $p \geqq n$ satisfying this condition does not exceed

$$\frac{2}{n^{\sqrt[3]{\log n}}}.$$

When we use a little different and very likely hypothesis that, if f_p is the frequency of the Bernoullian numbers divisible by p,

$$\frac{\sum_{p \leqq x} p f_p}{\pi(x)} \to 1,$$

the result is not very different, with only, maybe, a greater dispersion."

H. S. Vandiver [9] gave without proof the result that if (1) is satisfied in Case I, then, if $l_1 = (l-3)/2$,

$$B_s \equiv 0 \pmod{l^2}, \qquad (i = 1, 2, 3, 4, 5, 6; s = n_i(l_1 + 1) - i),$$

where the n's each range independently over all positive integral values. He indicated later [10] a proof of this statement as well as discussing a method of examining these criteria in special cases.

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3. Got, Annales de la faculté des sciences de Toulouse, vol. 25, 1911, pp. 21–62.

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5. McDonnel, Bull. Amer. Math. Soc., vol. 36, 1930, pp. 553–558.

6. Holzer, Jour. für Math., vol. 173, 1935, pp. 118–119.

7. Hasse, Bericht über neuere Untersuchungen, etc., Jahresber. Deuts. Math. Verein., II, 1930, pp. 113–122.

8. Krasner, On the first case of Fermat's last theorem, Comptes Rendus, 199, 1934, p. 256. As stated in the first paragraph and second footnote of (I), we were confining ourselves in that paper to citing papers giving results showing that (1) was impossible for certain classes of prime exponents, and it was possible to show that none of these classes was the null-class. No references were listed in the bibliography unless the article cited included results of the type just mentioned, with the exception of M. Krasner's article here reviewed. However, mistakenly, the writer put Krasner's paper in the wrong category.

9. Vandiver, Proc. Nat. Acad. Sci., vol. 16, 1930, p. 298; Bull. Amer. Math. Soc., Feb., 1934, p. 124.

10. Vandiver, Proc. Nat. Acad. Sci., vol. 27, 1941, p. 82; Duke Math. Journal, vol. 5, 1939, pp. 425–427.

ON CEVIANS OF A TRIANGLE*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. The lines, one from each vertex of a triangle, which cut the opposite sides of that triangle in the ratio of the nth powers of the adjacent sides, possess interesting properties. It seems that only the simple cases corresponding to the medians, bisectors, symmedians, that is to $n=0$, 1, 2, have been considered. This note takes up the general case and then arrives at a system of hyperbolic lines associated with a tetrahedron, for which J. Neuberg gave two particular cases in his *Mémoire sur le tétraèdre* (1884).

2. Notation. Let a, b, c $(a+b+c=2p)$, designate the sides BC, CA, AB of a triangle; (O), (O_9), (I), (I_i), $(i=a,\ b,\ c)$, denote the circumcircle, the nine-point circle, the incircle and the excircles of respective radii R, $R/2$, r, r_i; AA', BB', CC', G, H, K, indicate the altitudes, centroid, orthocenter and symmedian point of the triangle $T\equiv ABC$ oriented in the sense ABC.

3. Metric relations. From the theorem of Ceva it follows that the lines AA_1, BB_1, CC_1 which cut the sides BC, CA, AB in the ratios

$$(1)\qquad BA_1/A_1C = c^n/b^n, \qquad CB_1/B_1A = a^n/c^n, \qquad AC_1/C_1B = b^n/a^n$$

are concurrent at a point P_n. The harmonic conjugates A_1', B_1', C_1' of A_1, B_1, C_1 with respect to B and C, C and A, A and B, are collinear as they are on the trilinear polar of P_n. Likewise the sets of points $(B_1,\ C_1,\ A_1')$, $(C_1,\ A_1,\ B_1')$, $(A_1,\ B_1,\ C_1')$ are on the trilinear polars of the associates of P_n with respect to the triangle T. The associates of a point of normal (or barycentric) coordinates $(x,\ y,\ z)$ are the points of coordinates $(-x,\ y,\ z)$, $(x,\ -y,\ z)$, $(x,\ y,\ -z)$.

Furthermore in magnitude and sign

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* Translated from the French by W. E. Byrne.

$$(2)\qquad \begin{aligned} BA_1 &= ac^n/(b^n + c^n), & A_1'B &= ac^n/(b^n - c^n), \\ A_1C &= ab^n/(b^n + c^n), & A_1'C &= ab^n/(b^n - c^n). \end{aligned}$$

Also

$$(3)\qquad AP_n/P_nA_1 = AC_1 \cdot BC/C_1B \cdot AC_1 = (b^n + c^n)/a^n$$

so that

$$(4)\qquad \begin{aligned} AP_n &= bc[(b^{n-2} + c^{n-2})(b^n + c^n) - a^2(bc)^{n-2}]^{1/2}/(a^n + b^n + c^n) \\ &= a^n \cdot P_nA_1/(b^n + c^n). \end{aligned}$$

Similar formulae may be obtained by cyclical permutation. Thus the barycentric coordinates of P_n, and its associates P_a, P_b, P_c are (a^n, b^n, c^n), $(-a^n, b^n, c^n)$, $(a^n, -b^n, c^n)$, $(a^n, b^n, -c^n)$ with respect to T as the reference triangle.

An application of Stewart's theorem to T and to AA_1, AA_1', BB_1, BB_1', CC_1, CC_1' gives

$$(5)\qquad \overline{AA_1}^2 = (bc)^2[(b^{n-2} + c^{n-2})(b^n + c^n) - a^2(bc)^{n-2}]/(b^n + c^n)^2$$

$$(6)\qquad \overline{AA_1'}^2 = (bc)^2[(b^{n-2} - c^{n-2})(b^n - c^n) + a^2(bc)^{n-2}]/(b^n - c^n)^2$$

as well as like formulas obtained by cyclical permutation.

THEOREM. *The square of the distance of any point Q from P_n is given by*

$$(7)\qquad \begin{aligned} \overline{QP_n}^2 &= \sum a^n\overline{QA}^2/(\sum a^n) - (abc/\sum a^n)^2 \cdot \sum (bc)^{n-2} \\ &= \sum a^n\overline{QA}^2/(\sum a^n) - D_n \end{aligned}$$

where D_n is a constant independent of the position of Q [1].

Proof. (7) follows from (5) and an application of Stewart's theorem to triangles BQC, AQA_1.

Likewise

$$(8)\qquad \begin{aligned} \overline{QP_a}^2 &= (-a^n\overline{QA}^2 + b^n\overline{QB}^2 + c^n\overline{QC}^2)/(-a^n + b^n + c^n) \\ &\quad + (abc)^2/(-a^n + b^n + c^n)^2[-(bc)^{n-2} + (ca)^{n-2} + (ba)^{n-2}] \\ &= (-a^n\overline{QA}^2 + b^n\overline{QB}^2 + c^n\overline{QC}^2)/(-a^n + b^n + c^n) + D_a, \end{aligned}$$

where D_i are constants independent of the position of the point Q. For $n=0, 1, 2$ these reduce to

$$\begin{aligned} D_0 &= (a^2 + b^2 + c^2)/9, & D_a &= -a^2 + b^2 + c^2, \cdots \\ D_1 &= 2Rr, & D_a &= 2Rr_a, \cdots \\ D_2 &= 3(abc)^2/(a^2 + b^2 + c^2)^2, & D_a &= (abc)^2/(-a^2 + b^2 + c^2)^2, \cdots . \end{aligned}$$

Thus the measures of the medians, bisectors, symmedians and the squares of the distances of O and H from G, I, K can be obtained. For $n=1$ and $Q=O_9$

formulas (7), (8) reduce to the Feuerbach relations since (7) gives

$$\overline{O_9I}^2 = \sum a\overline{O_9A}^2/2p - D_1.$$

But

$$\overline{O_9A}^2 = (R^2 + b^2 + c^2 - a^2)/4 = (R^2 + 2bc \cos A)/4$$
$$\overline{O_9B}^2 = (R^2 + 2ca \cos B)/4$$
$$\overline{O_9C}^2 = (R^2 + 2ab \cos C)/4$$

so that

$$\sum a\cdot\overline{D_9A}^2/4p = (2pR^2 + 2abc\sum \cos A)/4p$$
$$= (R^2 + 4rR + 4r^2)/4$$

and

$$\overline{O_9I}^2 = (R^2 + 4Rr + 4r^2)/4 - 2Rr = (R - 2r)^2/4.$$

Likewise

$$\overline{O_9I_a}^2 = (R + 2ra)^2/4.$$

Furthermore it follows from (2) that the generalized circles of Apollonius (O_i) described on A_1A_1', B_1B_1', C_1C_1' as diameters are orthogonal to (O).

Since

$$A_1A_1' = 2abc(bc)^{n-1}/(b^{2n} - c^{2n})$$

then

$$A_1A_1'/a^{n-1} + BB_1'/b^{n-1} + C_1C_1'/c^{n-1} = 0.$$

The measures of the diameters of the circles (O_i) are independent of the lengths of the sides of the triangle on which the centers O_i are situated, and the same holds for the powers

$$(b^{2n+2} - c^{2n+2})/(b^{2n} - c^{2n}), \cdots$$

of the vertices A, B, C for the circles symmetric to the (O_i) with respect to the midpoints of BC, CA, AB. If $n=1$, these powers are equal to b^2+c^2, c^2+a^2, a^2+b^2.

4. Associated circles and triangles. THEOREM. *The circle $A_1B_1C_1$ intercepts segments x, y, z on BC, CA, AB which satisfy the equation*

$$\frac{x}{a^{n-1}} + \frac{y}{b^{n-1}} + \frac{z}{c^{n-1}} = 0. \tag{9}$$

Proof. If we designate by A_3, B_3, C_3 the second points of intersection of the circle $A_1B_1C_1$ with BC, CA, AB and if $a \geqq b \geqq c$, then

$$AB_1 \cdot AB_3 = AC_1 \cdot AC_3$$

and like relations for the other vertices of T. These relations with (2) give

$$bc^n[bc^n + (c^n + a^n)y]/(c^n + a^n)^2 = cb^n[cb^n - (a^n + b^n)z]/(a^n + b^n)^2$$
$$ca^n[ca^n + (a^n + b^n)z]/(a^n + b^n)^2 = ac^n[ac^n - (b^n + c^n)x]/(b^n + c^n)^2$$
$$ab^n[ab^n + (b^n + c^n)x]/(b^n + c^n)^2 = ba^n[ba^n - (c^n + a^n)y]/(c^n + a^n)^2.$$

If the above equations are multiplied by a^{2n}, b^{2n}, c^{2n} and added, (9) follows.

If $n=1$, 2 the cevians are the bisectors and the symmedians of T, so

$$x + y + z = 0 \qquad (n = 1) \qquad [2]$$
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 \qquad (n = 2).$$

Relations like (9) hold for the segments intercepted by the circles $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ on BC, CA, AB.

The lines AA_3, BB_3, CC_3 are concurrent in a point P_n' [3]. The points A_1, B_1, C_1, A_1', B_1', C_1' are the vertices of a complete quadrilateral (Q) which has T as its diagonal triangle. The Miquel point M of (Q), which is common to the circles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$, is the focus of the parabola inscribed in (Q). Furthermore the centers of the equilateral hyperbolas circumscribed about the quadrangles $ABCP_n$, and $ABCP_n'$ are distinct points on (O_9) as they are the orthopoles f and f', with respect to T, of the diameters OP_n and OP_n' of the circle ABC. f' coincides with M. The above may be summarized as follows:

THEOREM. *The orthocenters of the triangles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ are on the orthopolar Δ with respect to T of the Miquel point M of (Q). Δ is a diameter of the circle ABC. The Simson line of the point M, with respect to the complementary triangle t of T, is common to the triangles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ and the orthopoles (with respect to t) of the sides of these triangles are on the line Δ.*

When $n=1$, P_1 coincides with the incenter of T and P_1' is on the Kiepert hyperbola circumscribed about T. Hence the circles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ meet at the center of the Kiepert hyperbola. Each of these circles contains one of the Feuerbach points of T [4]. The Simson line with respect to t of the Miquel point M of quadrilateral (Q) is common to the orthic triangle $A'B'C'$ and the triangles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$. Also the orthopoles, with respect to t, of the sides of these triangles are on the Brocard diameter OK of T. See reference [5] for some special cases.

5. Hyperbolic systems. Consider an arbitrary tetrahedron $T \equiv ABCD$ with edges BC, DA, CA, DB, AB, DB of lengths a, a', b, b', c, c'. Let (a_1, b_1, c_1), (a_2, c_1', b_1'), (c_2', a_1', b_2), (a_2', c_2, b_2') designate the feet of the cevians which

divide the sides of the oriented triangles ABC, BCD, CDA, DAB in the ratios

$$\begin{aligned} &Ba_1/a_1C = c^n/b^n, \qquad Cb_1/b_1A = a^n/c^n, \qquad Ac_1/c_1B = b^n/a^n, \\ (10)\qquad &Ba_2/a_2C = b'^n/c'^n, \qquad Cb_2/b_2A = c'^n/a'^n, \qquad Ac_2/c_2B = a'^n/b'^n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned}$$

Let (A_n, B_n, C_n, D_n), (A_n', B_n', C_n', D_n') denote the points of intersection of these sets of cevians in the triangles ABC, BCD, CDA, DAB.

THEOREM. *The squares of the segments* AA_n, BB_n, CC_n, DD_n and AA_n', BB_n', CC_n', DD_n' *are given by*

$$\overline{DD}_n^2 = \left[\sum a^n \cdot \sum (aa')^n - (abc)^2 \sum (bc)^{n-2}\right] / \left(\sum a^n\right) \tag{11}$$

$$\overline{DD}_n'^2 = \left[\sum \frac{1}{a^n} \cdot \sum \frac{1}{a'^{n-1}} - \sum \frac{a^2}{(bc)^n}\right] \Big/ \left(\sum \frac{1}{a^n}\right). \tag{12}$$

Formulas (11), (12) may be obtained by successive applications of Stewart's theorem.

THEOREM. AA_n, BB_n, CC_n, DD_n *are rulings of one system and* AA_n', BB_n', CC_n', DD_n' *are rulings of a second system of an hyperboloid.*

Proof. Lines AB_n, AC_n, AD_n meet the edges DC, DB, BC at points c_2', b_2', a_2 such that

$$Bb_2'/b_2'D = c^n/a'^n, \qquad Dc_2'/c_2'A = a'^n/b^n, \qquad Ca_2/a_2B = b^n/c^n \tag{13}$$

and Bc_2', Cb_2', Da_2 meet at A_n'. Analogous statements hold for the points B_n', C_n', D_n' of the other faces. Hence the line AA_n', for instance, meets AA_n, BB_n, CC_n, DD_n, which are four rulings of one system of an hyperboloid; AA_n', BB_n', CC_n', DD_n' are rulings of the other system.

COROLLARY. *If* AA_n, BB_n, CC_n, DD_n *coincide with* AA_n', BB_n', CC_n', DD_n', *then they are concurrent and the tetrahedron* T *is isodynamic, and conversely.*

Proof. The necessary and sufficient condition that b_1' and b_2' coincide as well as the pairs a_1, a_2 and c_1', c_2' is that

$$(aa')^n = (bb')^n = (cc')^n$$

and conversely. But these equations follow from (10) and (13). So in this case $AA_n = AA_n'$, $\cdots$ and they are concurrent, and T is isodynamic.

COROLLARY. *If* T *is isodynamic the lines* a_1a_1', b_1b_1', c_1c_1' *meet at the point of intersection* L *of* AA_n, BB_n, CC_n, DD_n.

Proof. Line a_1a_1' situated in the planes $Ba_1'C$ and Aa_1D meets the lines AA_n, DD_n and BB_n, CC_n in these planes and hence goes through L.

COROLLARY. *The trilinear polars of the points A_n, B_n, C_n, E_n with respect to the faces BCD, CDA, DAB, ABC of an isodynamic tetrahedron are situated in the polar plane with respect to T of the point L.*

In like manner the following more general theorem may be obtained [6].

THEOREM. *Let λ_1, λ_2, λ_3, λ_1', λ_2', λ_3' be associated with the edges BC, CA, AB, DA, DB, DC of a tetrahedron. Let A_0, B_0, C_0, D_0 be the points of the planes of the faces BCD, CDA, DAB, ABC for which the barycentric coordinates are proportional to the numbers λ associated with the sides of the corresponding face. AA_0, BB_0, CC_0, DD_0 are four rulings of one system of an hyperboloid. They are concurrent if and only if*

$$\lambda_1\lambda_1' = \lambda_2\lambda_2' = \lambda_3\lambda_3'.$$

Particular cases. $n=1$. The points A_n, B_n, C_n, D_n are the incenters of the faces BCD, CDA, DAB, ABC of T. Thus the lines joining the vertices of an arbitrary tetrahedron to the incenters of the opposite faces are four rulings of an hyperboloid [7]. This may be generalized as follows:

THEOREM. *If perpendiculars are erected to the faces of tetrahedron T at the centers A_1, B_1, C_1, D_1 of faces BCD, CDA, DAB, ABC and segments A_1A_2, B_1B_2, C_1C_2, D_1D_2 are laid off toward the exterior (interior) of T proportional to the radii of the inscribed circles, the lines AA_2, BB_2, CC_2, DD_2 are rulings of an hyperboloid.*

Proof. Let us designate by $AA'=h_a$ the altitude of T drawn from A and put $A_1A_2=mr_a$, r_a being the radius of the inscribed circle of face BCD, m arbitrary. The distances of A' from the edges BC, CD, DB are $h_a \cot \alpha$, $h_a \cot \gamma'$, $h_a \cot \beta'$ where α is the measure of the dihedral angle of edge a. AA_2 meets $A'A_1$ at a point A_3 which divides the segment $A'A_1$ in the ratio

$$A'A_3/A'A_1 = h_a/(h_a + mr_a) = k.$$

The distances of A_3 from the edges BC, CD, DB are

$$kr_a(m \cot \alpha + 1), \qquad kr_a(m \cot \gamma' + 1), \qquad kr_a(m \cot \beta' + 1).$$

A_3 has

$$\lambda_1 = a(m \cot \alpha + 1), \qquad \lambda_3' = c'(m \cot \gamma' + 1), \qquad \lambda_1' = b'(m \cot \beta' + 1)$$

as barycentric coordinates with respect to triangle BCD. In like manner the barycentric coordinates of B_3, C_3, D_3, analogous to A_3, with respect to faces CDA, DAB, ABC are

$$(\lambda_3', \lambda_1', \lambda_2), \qquad (\lambda_1', \lambda_3, \lambda_2'), \qquad (\lambda_1, \lambda_2, \lambda_3).$$

The proof is then reduced to the previous theorem.

It may be noted that since

$$m = A_1A_2/r_a = B_1B_2/r_b = C_1C_2/r_c = D_1D_2/r_d,$$

the planes (A_2BC, A_2CD, A_2DB), (B_2CD, B_2DA, B_2AC), (C_2DA, C_2AB, C_2BD), (D_2AB, D_2BC, D_2CA) are equally inclined with respect to the faces BCD, CDA, DAB, ABC, respectively. The configuration of lines AA_2, BB_2, CC_2, DD_2 is then like that of the concurrent lines MM_1, NN_1, PP_1 joining the vertices of a triangle to the vertices of the directly similar isosceles triangles MP_1N, NM_1P, PN_1M. We find thus a particular case of the following theorem: [8], [9].

THEOREM. *Given a tetrahedron $ABCD$ and a surface (S) of the second class, consider a face ABC for instance, draw through each edge a plane which has a given cross-ratio k with the face and the two tangent planes of (S) through that edge. The three planes thus defined meet at a point D'. Likewise points A', B', C' are determined. The lines AA', BB', CC', DD' are hyperbolic. If $k=-1$, D' coincides with the pole of the plane ABC with respect to (S).*

$n=2$. In this case A_n, B_n, C_n, D_n are the symmedian points (Lemoine points) of the faces BCD, CDA, DAB, ABC. The lines joining the vertices of a tetrahedron to the symmedian points of the opposite faces are hyperbolic [10].

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

NOTES ON MATRIX THEORY—II

RICHARD BELLMAN, The Rand Corporation

The purpose of this note is to indicate how a well-known formula of integral calculus may be used to derive some results of interest pertaining to the determinant of a positive definite matrix A.

If $A=(a_{ij})$, $i, j=1, 2, \cdots, n$, is a positive definite matrix we have

$$\frac{C_n}{|A|^{n/2}} = \int_{-\infty}^{\infty} e^{-\sum_{i,j=1}^{n} a_{ij}x_ix_j} \prod_{i=1}^{n} dx_i, \tag{1}$$

where $|A| = |a_{ij}|$ and $C_n = (\sqrt{\pi})^n$. The proof of (1) follows readily if we transform the quadratic form into a sum of squares, $\sum_{k=1}^{n}\lambda_k x_k^2$ by means of an orthogonal transformation, where λ_k are the characteristic roots of A, and use the fact that $|A| = \prod_{k=1}^{n}\lambda_k$.

As a first application, let us prove the known result,

$$|\alpha A + (1-\alpha)B| \geqq |A|^{\alpha}|B|^{1-\alpha}, \tag{2}$$

if A and B are positive definite and $0 \leqq \alpha \leqq 1$. We have

$$\frac{C_n}{|\alpha A + (1-\alpha)B|^{n/2}} = \int_{-\infty}^{\infty} e^{-\alpha\sum_{i,j=1}^{n} a_{ij}x_ix_j - (1-\alpha)\sum_{i,j=1}^{n} b_{ij}x_ix_j} \prod dx_i. \tag{3}$$

Applying Hölder's integral inequality with $0<\alpha<1$, $p=1/\alpha$, $p'=1/(1-\alpha)$, we obtain

$$\begin{aligned} \frac{C_n}{|\alpha A + (1-\alpha)B|^{n/2}} &\leqq \left(\int_{-\infty}^{\infty} e^{-\sum a_{ij}x_ix_j}\prod dx_i\right)^{\alpha}\left(\int_{-\infty}^{\infty} e^{-\sum b_{ij}x_ix_j}\prod dx_i\right)^{1-\alpha} \\ &\leqq \left(\frac{C_n}{|A|^{n/2}}\right)^{\alpha}\left(\frac{C_n}{|B|^{n/2}}\right)^{1-\alpha}, \end{aligned} \tag{4}$$

which is equivalent to (2).

As a second application, let us establish the classical result,

$$\prod_{i=1}^{n} a_{ii} \geqq |A|. \tag{5}$$

To illustrate the method, it is sufficient to consider the first non-trivial case, $n=3$. In (1) set successively $x_1=-x_1'$, $x_2=-x_2'$, $x_3=x_3'$ and add, obtaining

$$\begin{aligned} \frac{C_3}{|A|^{3/2}} = \int_{-\infty}^{\infty} & e^{-a_{11}x_1^2 - a_{22}x_2^2 - a_{33}x_3^2} \\ & \cdot\left[\frac{y_1y_2y_3 + y_1y_2^{-1}y_3^{-1} + y_1^{-1}y_2y_3^{-1} + y_1^{-1}y_2^{-1}y_3}{4}\right]\prod dx_i, \end{aligned} \tag{6}$$

where $y_1 = e^{-a_{23}x_2x_3}$, $y_2 = e^{-a_{13}x_1x_3}$, $y_3 = e^{-a_{12}x_1x_2}$. The arithmetic-geometric mean inequality yields

$$\frac{y_1y_2y_3 + y_1y_2^{-1}y_3^{-1} + y_1^{-1}y_2y_3^{-1} + y_1^{-1}y_2^{-1}y_3}{4} \geqq 1. \tag{7}$$

Therefore

$$\frac{C_3}{|A|^{3/2}} \geqq \int_{-\infty}^{\infty} e^{-a_{11}x_1^2-a_{22}x_2^2-a_{33}x_3^2} \prod dx_i = \frac{C_3}{(a_{11}a_{22}a_{33})^{3/2}}, \tag{8}$$

which is equivalent to (5) for $n=3$.

A METHOD FOR FINDING PRIMES

JOHN THOMPSON, Yale University

The simplest way to show that there exists an infinite number of primes is probably by Euclid's method of taking the n first primes

$$p_1 = 2, \cdots, p_n \tag{1}$$

and observe that the number

$$N = 1 + p_1p_2 \cdots p_n$$

must be divisible by primes different from those in (1).

A slight variation of this method goes further and actually yields a method to determine new primes more explicitly from those given in (1). This depends on the observation that if one splits the primes in (1) up into two different groups

$$q_1, \cdots, q_a; \qquad r_1, \cdots, r_b$$

then the difference between the two products

$$D = (q_1 \cdots q_a) - (r_1 \cdots r_b) \tag{2}$$

is also a number which cannot be divisible by any of the primes (1). Furthermore if the difference (2) should turn out to be less than $(p_n+2)^2$ the difference (2) is evidently a prime.

Following are some simple examples:

$$\begin{aligned}
(11\cdot 5) - (2\cdot 3\cdot 7) &= 13\\
(7\cdot 11) - (2\cdot 3\cdot 5) &= 47\\
(2\cdot 3\cdot 11) - (5\cdot 7) &= 31\\
(3\cdot 5\cdot 7) - (2\cdot 11) &= 83\\
(3\cdot 13\cdot 17\cdot 23) - (2\cdot 5\cdot 7\cdot 11\cdot 19) &= 619\\
(3\cdot 5\cdot 7\cdot 11\cdot 13) - (2\cdot 17\cdot 19\cdot 23) &= 157
\end{aligned}$$

One may ask whether a suitable choice of the products in (2) will always produce a prime. A wider range of possibilities is opened up by using exponents for the primes in (2).

AN INEQUALITY FOR REARRANGEMENTS

G. G. LORENTZ, University of Toronto

Let $f_1(x)$, $f_2(x)$, $\cdots$ denote positive measurable functions on $(0, 1)$ and $f_1^*(x)$, $f_2^*(x)$, $\cdots$ their equimeasurable decreasing rearrangements (see [1], [3]). For the work dealing with rearrangements, the following simple inequality is basic:

$$(1)\qquad \int_0^1 f_1(x)f_2(x)dx \leqq \int_0^1 f_1^*(x)f_2^*(x)dx.$$

There are, however, also other combinations of $f_1, f_2, \cdots$ for which relations similar to (1) hold. One of these was given by Ruderman [2, Theorem II]. In this note we propose to determine, quite generally, necessary and sufficient conditions on a continuous function $\Phi(x, u_1, \cdots, u_n)$ defined for $0<x<1$, $u_k \geqq 0$, $k=1, 2, \cdots, n$, under which

$$(2)\qquad \int_0^1 \Phi(x, f_1(x), \cdots, f_n(x))dx \leqq \int_0^1 \Phi(x, f_1^*(x), \cdots, f_n^*(x))dx$$

is satisfied for each set $f_k(x)$, $k=1, \cdots, n$, of positive bounded measurable functions on $(0, 1)$. (We assume the $f_k(x)$ bounded in order to insure the existence of both integrals in (2).)

In inequalities containing values of the function Φ at different points, we shall omit those of the arguments $x, u_1, \cdots, u_n$ which take the same but arbitrary values. For a set I of indices i, $1 \leqq i \leqq n$, we put $U_I = \{u_i\}_{i \in I}$. We also put $U_I + U_I' = \{u_i + u_i'\}_{i \in I}$, if $U_I' = \{u_i'\}$.

THEOREM. *In order that Φ satisfy* (2) *it is necessary and sufficient that Φ have the properties*

$$(3)\qquad \Phi(u_i + h, u_j + h) - \Phi(u_i + h, u_j) - \Phi(u_i, u_j + h) + \Phi(u_i, u_j) \geqq 0,$$

$$(4)\qquad \int_0^\delta \{\Phi(x - t, u_i + h) + \Phi(x + t, u_i) - \Phi(x + t, u_i + h) - \Phi(x - t, u_i)\}dt \geqq 0$$

for all $0<x<1$, $u_k \geqq 0$, $k=1, \cdots, n$, $h>0$, $0<\delta<x$, $\delta<1-x$, and $i \neq j$. If Φ has continuous second partial derivatives with respect to all variables, conditions (3), (4) *are equivalent to*

$$(3a)\qquad \frac{\partial^2 \Phi}{\partial u_i \partial u_j} \geqq 0,$$

$$(4a)\qquad \frac{\partial^2 \Phi}{\partial x \partial u_i} \leqq 0.$$

Proof. Suppose $0<a<1$, $0<\delta<a$, $\delta<1-a$, $i \neq j$. Define $f_i(x) = u_i + h_i$ for

$x \leqq a-\delta$ and $a<x\leqq a+\delta$ and $f_i(x)=u_i$ for other x, $f_j(x)=u_j+h_j$ for $x\leqq a$, $f_j(x)=u_j$ for $x>a$, further $f_k(x)=u_k$, $0<x<1$ for k different from i and j. Then the inequality (2) reduces to

$$\int_0^\delta \{\Phi(a-t, u_i+h_i, u_j+h_j) - \Phi(a+t, u_i+h_i, u_j) - \Phi(a-t, u_j, u_j+h_j) + \Phi(a+t, u_i, u_j)\}\,dt \geqq 0.$$

Putting here $h_j=0$, we obtain (4). Dividing through by δ and making $\delta\to 0$, we obtain (3).

To prove that the conditions are sufficient, we first deduce from (3) that for any two disjoint groups of indices I, J and h_i, $h_j\geqq 0$,

$$\Phi(U_I+H_I, U_J+H_J) - \Phi(U_I+H_I, U_J) - \Phi(U_I, U_J+H_J) + \Phi(U_I, U_J) \geqq 0. \tag{5}$$

From (3) we have

$$\Phi(u_i+sh, u_j+h) - \Phi(u_i+sh, u_j) - \Phi(u_i+(s-1)h, u_j+h) + \Phi(u_i+(s-1)h, u_j) \geqq 0.$$

Adding these relations for $s=1, 2, \cdots, p$ we deduce

$$\Phi(u_i+ph, u_j+h) - \Phi(u_i+ph, u_j) - \Phi(u_i, u_j+h) + \Phi(u_i, u_j) \geqq 0. \tag{6}$$

Treating now the second argument in (6) in the same way we obtain, for positive integers p, q and $h_i=ph$, $h_j=qh$,

$$\Phi(u_i+h_i, u_j+h_j) - \Phi(u_i+h_i, u_j) - \Phi(u_i, u_j+h_j) + \Phi(u_i, u_j) \geqq 0. \tag{7}$$

An appeal to the continuity of Φ establishes (7) for arbitrary h_i, $h_j\geqq 0$.

To prove (5), let I' be the group consisting of I and the index k, which belongs neither to I nor to J. Then

$$\begin{aligned}
&\Phi(U_{I'}+H_{I'}, U_J+H_J) - \Phi(U_{I'}+H_{I'}, U_J) - \Phi(U_{I'}, U_J+H_J) + \Phi(U_{I'}, U_J)\\
&\quad = \{\Phi(U_I+H_I, u_k+h_k, U_J+H_J) - \Phi(U_I+H_I, u_k+h_k, U_J)\\
&\qquad - \Phi(U_I, u_k+h_k, U_J+H_J) + \Phi(U_I, u_k+h_k, U_J)\}\\
&\qquad + \{\Phi(U_I, u_k+h_k, U_J+H_J) - \Phi(U_I, u_k+h_k, U_J)\\
&\qquad - \Phi(U_I, u_k, U_J+H_J) + \Phi(U_I, u_k, U_J)\}.
\end{aligned} \tag{8}$$

Applying this relation we can, beginning with (7), prove (5) by induction with respect to the number of elements of I and J.

In the same way, we can generalize (4) to

$$\int_0^\delta \{\Phi(x-t, U_I+H_I) + \Phi(x+t, U_I) - \Phi(x+t, U_I+H_I) - \Phi(x-t, U_I)\}\,dt \geqq 0. \tag{9}$$

Replacing in identity (8) u_k by $x-t$, u_k+h_k by $x+t$, and combining (5) and (9),

we obtain finally

$$(10)\quad \int_0^\delta \{\Phi(x-t, U_I+H_I, U_J+H_J) - \Phi(x-t, U_I, U_J+H_J) \\ - \Phi(x+t, U_I+H_I, U_J) + \Phi(x+t, U_I, U_J)\}dt \geqq 0.$$

We can now prove (2) under the assumption that each of the functions $f_k(x)$ is a step-function, constant on each of the intervals $((s-1)/p, s/p)$, $s=1, \cdots, p$. For $1 \leqq s < p$ we consider the following *elementary operation* which gives a new set of functions $\bar{f}_k(x)$. We put $\bar{f}_k(x) = f_k(x)$ outside of $((s-1)/p, (s+1)/p)$; on $((s-1)/p, (s+1)/p)$, $\bar{f}_k(x)$ is the decreasing rearrangement of $f_k(x)$ on this interval. If I consists of the indices k for which $f_k(x)$ increases on $((s-1)/p, (s+1)/p)$, J of the indices for which $f_k(x)$ decreases, u_k is the smaller, u_k+h_k the larger of the two values of $f_k(x)$, then (10) with $x=s/p$, $\delta=1/p$ is exactly the inequality

$$\int_0^1 \Phi(x, f_1, \cdots, f_n)dx \leqq \int_0^1 \Phi(x, \bar{f}_1, \cdots, \bar{f}_n)dx.$$

By a finite number of elementary operations we can transform $f_1, \cdots, f_n$ into $f_1^*, \cdots, f_n^*$. This proves (2) in our particular case. In the general case we consider sequences $f_1^{(p)}, \cdots, f_1^{(p)}$, $p=1, 2, \cdots$ of uniformly bounded step-functions of our type such that $f_k^{(p)}(x) \to f_k(x)$ almost everywhere and pass to the limit $p \to \infty$ in the relation (2) for the $f_k^{(p)}$. This gives (2) in full generality.

It remains to show that (3) is equivalent to (3a) and (4) to (4a), if Φ has continuous second derivatives. If (4) holds, then for any i, $0<x<1$, $u_k \geqq 0$, there are arbitrary small $t>0$ with

$$\Phi(x+t, u_i+t) - \Phi(x-t, u_i+t) - \Phi(x+t, u_i) + \Phi(x-t, u_i) \leqq 0.$$

Dividing by $2t^2$ and making $t \to 0$, we obtain (4a). Conversely, from (4a) we deduce a relation stronger than (4), namely

$$(4b)\quad \Delta^2\Phi = \Phi(x+t, u_i+h) - \Phi(x+t, u_i) - \Phi(x, u_i+h) + \Phi(x, u_i) \leqq 0.$$

For if (4b) does not hold, there is a $c>0$ and a rectangle $R=(x, x+t; u_i, u_i+h)$ with side lengths t, h for which $\Delta^2\Phi \geqq cht$. Subdividing R, we obtain a sequence of rectangles with the same property which converge to a point (x^0, u_i^0). Then

$$\frac{\partial^2\Phi}{\partial x^0 \partial u_i^0} = \lim \frac{\Delta^2\Phi}{ht} \geqq c > 0,$$

which contradicts (4a). In the same way we treat the pair of relations (3), (3a).

Examples. The inequality (2) holds if $\Phi(u_1, \cdots, u_n) = u_1 \cdots u_n$. It holds for $\Phi = F(u_1+u_2+\cdots u_n)$ if and only if $F(u)$ is convex, that is $F(u+2h) - 2F(u+h) + F(u) \geqq 0$. For example, $F(u) = -\log u$ has this property. Writing (2) in this

case for sums instead of integrals, we obtain Ruderman's inequality [2, Theorem II]

$$\prod_{s=1}^{p}\sum_{k=1}^{n} a_{sk} \geqq \prod_{s=1}^{p}\sum_{k=1}^{n} a_{sk}^{*}, \tag{11}$$

where $a_{sk}\geqq 0$ and the a_{sk}^*, $s=1, \cdots, p$ are the a_{sk}, $s=1, \cdots, p$ arranged in order of decreasing magnitude.

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ON SUMS INVOLVING BINOMIAL COEFFICIENTS*

EMIL GROSSWALD, Institute for Advanced Study

In some problems of algebra† we are led to consider sums of the form $\sum_{\nu\geqq 0} k_\nu A(n,r,\nu)\, B(n,r,\nu)\, C(n,r,\nu,)$ where $A, B, C, \cdots$, are binomial coefficients, depending on ν and also on one or two other integral parameters, and where the summation proceeds up to the first value of ν, for which one of the factors vanishes. A certain number of such sums can be found in [1] and [3]. However, the sums computed in (4), do not seem to appear in the literature. They do not follow readily by the methods of [3], and a direct proof, or a proof by induction, seems rather difficult. In what follows, we give a simple proof of (4), using well-known properties of Legendre's polynomials $P_n(x)$ and of the hypergeometric function $F(a, b; c; x)$.

Let $P_n(x)=\sum_{r=0}^{n} a_r^{(n)}x^r$ be the nth Legendre polynomial, and let $P_n^{(r)}(x)$ be its rth derivative. Then, by Maclaurin's formula, $P_n(x)=\sum_{r=0}^{n}\{x^rP_n^{(r)}(0)/r!\}$ so that

$$P_n^{(r)}(0)/r! = a_r^{(n)}. \tag{1}$$

Here the values of $a_r^{(n)}$ are (see [2], p. 11)

$$\begin{aligned} a_r^{(n)} &= (-1)^{(n-r)/2}2^{-n}\binom{n+r}{n}\binom{n}{(n+r)/2} && \text{if} \quad n\equiv r \pmod 2 \\ &= 0 && \text{otherwise.}\end{aligned} \tag{2}$$

It also is known (see [4], pp. 61–62) that‡ the hypergeometric function

* This paper was prepared under contract with the Office of Naval Research.

† *E.g.* the study of the algebraic irreducibility of Legendre's polynomials in the field of rational numbers.

‡ The idea of this proof is due to Professor E. D. Rainville, who kindly suggested it to me in a letter.

$$F(a, b; c; x) = 1 + \sum_{s=1}^{\infty} \frac{(a)_s(b)_s}{(c)_s} x^s \text{ satisfies } P_n(x) = F(-n, n+1; 1; (1-x)/2).$$

Here $(m)_s$ stands for $m(m+1) \cdots (m+s-1)$.

From the formula for the rth derivative of the hypergeometric function,

$$\frac{d^r}{dx^r} F(a, b; c; x) = \frac{(a)_r(b)_r}{(c)_r} F(a+r, b+r; c+r; x),$$

it follows that

$$P_n^{(r)}(x) = \frac{(n+r)!}{(n-r)!r!2^r} F(-n+r, n+1+r; 1+r; (1-x)/2),$$

and, in particular,

$$P_n^{(r)}(0) = \frac{(n+r)!}{(n-r)!r!\,2^r} \sum_{\nu=0}^{n-r} (-2)^{-\nu} \frac{(n-r)!}{\nu!(n-r-\nu)!} \frac{(n+r+\nu)!r!}{(n+r)!(r+\nu)!}.$$

This can be written in any of the following ways:

$$\begin{aligned} 2^r P_n^{(r)}(0)/r! &= \binom{n}{r} \sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n-r}{\nu} \binom{n+r+\nu}{n} \\ &= \binom{n+r}{r} \sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{\nu} \qquad (3) \\ &= \sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{n} \binom{r+\nu}{r}. \end{aligned}$$

If we substitute now in (1) the values of $a_r^{(n)}$ from (2) and also, successively, the three values of $P_n^{(r)}(0)$ from (3), after some obvious simplifications we obtain, respectively, for $r \equiv n \pmod 2$,

$$\sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n-r}{\nu} \binom{n+r+\nu}{n} = (-1)^{(n-r)/2} 2^{r-n} \binom{2n}{n} \binom{n}{(n+r)/2} \binom{2n}{n+r}^{-1}$$

$$(4) \qquad \sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{\nu} = (-1)^{(n-r)/2} 2^{r-n} \binom{n}{(n+r)/2}$$

$$\sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{n} \binom{r+\nu}{r} = (-1)^{(n-r)/2} 2^{r-n} \binom{n+r}{n} \binom{n}{(n+r)/2}.$$

If $r \not\equiv n \pmod 2$ the sums vanish.

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NOTE ON A FORMULA OF GROSSWALD

L. CARLITZ, Duke University

Using properties of the Legendre polynomials and the hypergeometric function, Grosswald has proved* the formula

$$\sum_{\nu=0}^{n-r} (-2)^{-\nu}\binom{n}{r+\nu}\binom{n+r+\nu}{\nu} = (-1)^k 2^{-2k}\binom{n}{k}, \tag{1}$$

where $n-r=2k$. We wish to point out that (1) is easily obtained by means of the known formula (W. N. Bailey, Generalized hypergeometric series, Cambridge, 1935, p. 11, formula (2)):

$$F(a, b; \tfrac{1}{2}(a+b+1); \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)}. \tag{2}$$

Indeed the left member of (1) evidently

$$\begin{aligned}
&= \binom{n}{n-r}\sum_{\nu=0}^{n-r}(-2)^{-\nu}\frac{(n-r)!r!}{(n-r-\nu)!(r+\nu)!}\binom{n+r+\nu}{\nu} \\
&= \binom{n}{n-r}\sum_{\nu=0}^{n-r}\frac{(r-n)_\nu(n+r+1)_\nu}{\nu!(r+1)_\nu}(\tfrac{1}{2})^\nu \\
&= \binom{n}{n-r}F(r-n, n+r+1; r+1; \tfrac{1}{2}),
\end{aligned}$$

and using (2) this becomes

$$\begin{aligned}
\binom{n}{n-r}&\frac{\Gamma(\frac{1}{2})\Gamma(r+1)}{\Gamma(\frac{1}{2}-k)\Gamma(r+k+1)} \\
&= \frac{n!}{(n-r)!(r+k)!}(\tfrac{1}{2}-k)_k \\
&= \frac{n!}{(2k)!(n-k)!}(-1)^k 2^{-k}(2k-1)(2k-3)\cdots 1 \\
&= (-1)^k 2^{-2k}\binom{n}{k}.
\end{aligned}$$

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* This MONTHLY, vol. 60, p. 179.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

POPULATIONS WITH MEANS AND STANDARD DEVIATIONS OF SAMPLES INDEPENDENTLY DISTRIBUTED

G. A. BAKER, University of California, Davis

1. Introduction. Deming in his recent book [1] speaking of the joint distribution of the mean, $\bar{x}$, and standard deviation, s, of samples of n says, "These estimates $(\bar{x}, s)$ are independent in normal theory and nearly independent in some nonnormal theory, although no proof can be given by the author." Geary [2] was the first to prove that a necessary and sufficient condition for the normality of the parent distribution is that the sampling distributions of the mean and of the variance be independent. Of course suitable restrictions must be imposed on the parent distributions with regard to range, continuity, and differentiability. Geary used R. A. Fisher's general formulae for seminvariants. Another proof was given by Lukacs [3] using characteristic functions. The purpose of this note is to present a proof that makes use of only a few simple notions of function theory without appeal to the more sophisticated mathematical statistical concepts of seminvariants and characteristic functions.

2. A functional equation. A. T. Craig [4] has shown that if the parent population is $f(x)$, $-\infty < x < \infty$, then the joint distribution of $\bar{x}$ and s for samples of two is

$$F(\bar{x}, s) = 4f(\bar{x} - s)f(\bar{x} + s). \tag{2.1}$$

It is assumed that our functions are sufficiently well behaved as far as continuity and differentiability are concerned. If the mean and standard deviation are independent in general, they must be independent for samples of two. Thus a necessary condition for independence can be stated as

$$4f(\bar{x} - s)f(\bar{x} + s) = g(\bar{x})h(s). \tag{2.2}$$

Put $s=0$ and we obtain

$$4[f(\bar{x})]^2 = h(0)g(\bar{x}). \tag{2.3}$$

Put $\bar{x}=0$ and we obtain

$$4f(-s)f(s) = g(0)h(s). \tag{2.4}$$

Hence

$$4[(f(\bar{x})]^2f(-s)f(s) = g(0)h(0)f(\bar{x} - s)f(\bar{x} + s) \tag{2.5}$$

which gives a condition on $f(x)$, the sampled population, so that the means and standard deviations of samples of two shall be independent.

We can now state the following theorem.

THEOREM 2.1. *Let a population be represented by $f(x)$, $-\infty < x < \infty$, $f(x) > 0$, then a necessary condition that the means $\bar{x}$, and standard deviations s, of samples are distributed independently is*

$$[f(\bar{x})]^2 f(-s)f(s) = kf(\bar{x} - s)f(\bar{x} + s) \tag{2.6}$$

where k is a constant.

3. Solution of the functional equation (2.6). It will now be shown that the only analytical function which satisfies the functional equation (2.6) is the normal curve. This proof was supplied by Professor C. A. Hayes, Jr. of the Mathematics Department, University of California at Davis. Take the logarithm of (2.6) and we have

$$2u(\bar{x}) + u(s) + u(-s) = \log k + u(\bar{x} + s) + u(\bar{x} - s) \tag{3.1}$$

where $u(\bar{x}) = \log f(\bar{x})$ and similarly for the other terms. Assume that derivatives exist, differentiate (3.1) with respect to $\bar{x}$, and obtain

$$2u'(\bar{x}) = u'(\bar{x} + s) + u'(\bar{x} - s). \tag{3.2}$$

Put $\bar{x} = s$ and get

$$2u'(\bar{x}) = u'(2\bar{x}) + u'(0). \tag{3.3}$$

Assume that $u(x)$ has a power series representation, substitute in (3.3), equate coefficients on both sides, and we find that, necessarily,

$$u(x) = u_0 + u_1 x + u_2 x^2 \tag{3.4}$$

or

$$f(x) = \exp(u_0 + u_1 x + u_2 x^2) \tag{3.5}$$

where the additional requirement that $f(x)$ be a distribution function requires that u_2 be negative. The linearity of $u'(x)$ can be established more generally from (3.2).

Thus it follows that the only population that is similar to the normal population which has means and standard deviations of samples independently distributed is the normal distribution.

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ON THE QUADRATIC FACTORS OF A POLYNOMIAL OF THE FOURTH DEGREE

J. G. CAMPBELL, University of Kentucky

Let $f(x)=x^4+ax^3+bx^2+cx+d$, where a, b, c and d are integers and suppose that $f(x)$ is the product of two factors of the form x^2+Ax+B, where A and B are integers. The following theorem may be used to isolate the coefficients A and B.

THEOREM. *If m is any positive or negative integer or zero, and m_i represent the j divisors of $f(m)$, $i=1, 2, \cdots, j$, then the set of numbers* $(m_1-m^2, m_2-m^2, \cdots, m_j-m^2)$ includes $Am+B$.

Proof. Let $x^4+ax^3+bx^2+cx+d=(x^2+Ax+B)\ (x^2+A_1x+B_1)$. Then $f(m)=(m^2+Am+B)(m^2+A_1m+B_1)$. Thus (m^2+Am+B) is one of the divisors of $f(m)$ and $(m^2+Am+B)-m^2=Am+B$. Hence $Am+B$ is in the given set of numbers.

If n is any other integer, and n_h represent the k divisors of $f(n)$, $h=1, 2, \cdots, k$, then the set $(n_1-n^2, n_2-n^2, \cdots, n_k-n^2)$ includes $An+B$.

Solving the system

$$mA+B=m_i-m^2$$
$$nA+B=n_h-n^2,$$

we get

$$A=\frac{m_i-n_h}{m-n}-(m+n) \qquad i=1, 2, \cdots, j,$$

$$B=\frac{mn_h-nm_i}{m-n}+mn \qquad h=1, 2, \cdots, k.$$

Direct substitution of the divisors m_i and n_h yield the desired integers. Note that A_1 and B_1 are also determined by the above equations. By equating coefficients of the product of the quadratic factors to the coefficients of $f(x)$, we have

$$A+A_1=a, \qquad BB_1=d, \qquad B+B_1+AA_1=b, \qquad AB_1+A_1B=c.$$

These relations may be used to further restrict the integers A and B. We select only pairs of values of the A's whose sum is equal to a and only pairs of values of the B's whose product is equal to d. These pairs of values are further restricted by testing them in $B+B_1+AA_1=b$ and $AB_1+A_1B=c$.

A numerical example is used to illustrate the method.* Find the roots of $x^4-2x^3-5x^2+10x-3=0$. Here $a=-2$, $b=-5$, $c=10$, $d=-3$. Set $m=0$ and $n=1$, and note that $f(0)=-3, f(1)=1$. The divisors m_i are, $(1, -1, 3, -3)$. The divisors n_h are, $(1, -1)$. The values of A and B can now be found from

* N. B. Conkwright, Introduction to the Theory of Equations, problem no. 5, p. 80.

$A=-(m_i-n_h)-1$ and $B=m_i$. Computing the A's we have $(-1, -3, 1, -5, 3)$. Selecting pairs of values whose sum is -2, we get $(1, -3)$ and $(-5, 3)$. Computing the B's, we get, $(1, -1, 3, -3)$. Selecting pairs of the B's whose product is -3, we have $(1, -3)$ and $(-1, 3)$. Substituting the B's in $B+B_1+AA_1=b$, we have $AA_1=-3$ or -7. The pair of A's satisfying these relations is $(1, -3)$; the pair $(-5, 3)$ is discarded. Let $A=1$, $A_1=-3$, then $B_1-3B=10$. The only pair of values satisfying this equation is $B_1=1$, $B=-3$. Thus, our problem reduces to solving

$$(x^2+x-3)(x^2-3x+1)=0.$$

The roots are, $(-1+\sqrt{13})/2$, $(-1-\sqrt{13})/2$, $(3+\sqrt{5})/2$ and $(3-\sqrt{5})/2$.

If $f(m)$ and $f(n)$ present few divisors, the labor incident to computing a root by this method is not greater than that of finding a rational root of the resolvent cubic equation.

SQUARE ROOTS IN GROUPS

W. R. UTZ, University of Missouri

In beginning the study of finite groups a student soon observes that if a group is commutative, then the elements off the principal diagonal of the group table are symmetric with respect to the diagonal and any given element occurs in the table a number of times equal to the order of the group, from which it easily follows that the elements of the group occur exactly once in the diagonal when, and only when, the order of the group is odd. An equivalent way of stating this is that each element of a finite commutative group has a unique square root (*i.e.*, the equation $x^2=b$ has a unique solution for each element b of the group) when, and only when, the group is of odd order. That this is a property of all finite groups is not as easily established but it is possible to give a proof sufficiently elementary to be understood early in the study of groups and, in most classes, there will be some students who can provide the proof themselves when the theorem is suggested.

We first observe that the "perfect squares" in the group appear on the diagonal of the group table. Thus if any element appears twice on the diagonal, then some element of the group is omitted from the diagonal and such an element does not have a square root. This proves the following lemma.

LEMMA. *If G is a finite group and each $a\epsilon G$ has a square root, then the root is unique.*

THEOREM 1. *Each element of a finite group has a square root if, and only if, the order of the group is odd.*

Proof. Suppose that G is a finite group of even order. Some element of G has period 2. To see this, write $G=S\cup R$ where S is the set of all elements of G of period greater than 2. If $x\epsilon S$, then $x^{-1}\epsilon S$ and $x\neq x^{-1}$. Thus S contains an even

number of elements of G. Since S has an even number of elements of the even ordered group G, R contains the identity and an odd number of elements of period 2 hence at least one element of period 2.

Now suppose that G is of even order and that each element of G has a square root. From the previous paragraph there exists $a \epsilon G$, $a \neq i$, where i denotes G's identity, such that $a^2 = i$. This is contrary to the Lemma since i has at least two square roots i and a.

To prove the other half of the theorem, let G be a group of odd order $n = 2k + 1$. Then $1 = n - 2k$ and since $b^n = i$ for each b in G,

$$b = b^{n-2k} = b^{-2k} = (b^{-k})^2 = c^2,$$

where $c = b^{-k} = (b^{-1})^k$. For each b, the element c is uniquely determined and qualifies as a square root of b. (The author is indebted to the referee for shortening this proof.)

The theorem given above is a special case of the following theorem.

THEOREM 2. *If G is a finite group and r is a positive integer, then each element of G has an r-th root when, and only when, r is prime to the order of G.*

The "only when" half of this theorem is a consequence of the corollary to Sylow's Theorem which states: If a prime number p is a factor of the order of a finite group G, then G contains an element of period p. The other half is readily established as in Theorem 1, for if r is prime to n, the order of G, then there exist integers, h and k, such that $1 = hr + kn$.

The theorems given above, in one form or another, may be known to many algebraists. However, the author does not know a comparable theorem for infinite groups. It seems that even a sufficient condition that in a group, at least as general as the group of all non-singular $n \times n$ matrices over the complex field, the equation $x^2 = b$, for each b of the group, have a solution would be of interest since for the matrix group this is commonly established by special methods that are not group-theoretic (see, for example, Bôcher, Higher Algebra, p. 299).

ON AN ELEMENTARY DERIVATION OF CRAMER'S RULE

D. E. WHITFORD and M. S. KLAMKIN, Polytechnic Institute of Brooklyn

The purpose of this note is to point out an elementary derivation of Cramer's rule which should be easily understood by freshman students.

Consider the simultaneous set of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{1}$$

Now

$$x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

by the elementary transformations of a determinant. Hence if x is to satisfy equations (1) it is necessary that

$$x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

or

$$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \Delta, \quad \text{provided} \quad \Delta \neq 0,$$

where Δ is the determinant of the coefficient matrix of the system (1). Similarly

$$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \Delta, \qquad z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \Delta.$$

That these conditions are sufficient, when $\Delta \neq 0$, can be established by substituting back into (1), which gives

$$a_r\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} + b_r\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} + c_r\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = d_r\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

That this is true follows from

$$\begin{vmatrix} a_r & b_r & c_r & d_r \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0$$

since the top row is equivalent to one of the other rows.

The method can be extended immediately to n linear equations in n unknowns.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1056. *Proposed by W. R. Ransom, Tufts College*

A rectangular room has a spider one foot down from the ceiling at the middle of one end, and a fly one foot up from the floor at the middle of the other end. There are three paths by which the spider can crawl to the fly, and which become straight lines when the sides of the room are properly developed. What dimensions of the room will make these three paths equal in length?

E 1057. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find the sum of the first n terms of the series

$$\sec\theta + (\sec\theta\sec 2\theta)/2 + (\sec\theta\sec 2\theta\sec 4\theta)/4 + \cdots .$$

E 1058. *Proposed by M. Perisastri, M. R. College, Vizianagram, India*

If $d_1, d_2, \cdots, d_k$ are the divisors of n, show that

$$(1)\qquad (d_1 d_2 \cdots d_k)^2 = n^k,$$

$$(2)\qquad d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k} = n^{d_1+d_2+\cdots+d_k} e^{-nrk}, \qquad 0 \leqq r \leqq (\log 3)/3.$$

E 1059. *Proposed by Chih-yi Wang, Hampton Institute*

Let a circle and an inscribed closed polygon of n sides be given. Show that the product of the distances of a point on the circumference of the circle from the sides of the polygon is equal to the product of the distances of the same point from the sides of the tangential polygon (*i.e.*, the polygon formed by the tangents to the circle at the vertices) of the given polygon.

E 1060. *Proposed by H. K. Crowder, Case Institute of Technology*

Let D be a determinant of order n whose ith row, $i=1, \cdots, n$, is

$$a_i, a_{i-1}, \cdots, a_1, 1, 0, \cdots, 0.$$

Show that

$$D = (-1)^n \Sigma \frac{(-1)(-2)\cdots(-\Sigma\alpha_i)}{\alpha_1!\alpha_2!\cdots\alpha_n!} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n},$$

where the sum is taken over all non-negative integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n.$$

SOLUTIONS

Multiplications Involving Only Two Digits

E 1012 [1952, 249]. *Proposed by Oystein Ore, Yale University*

In a multiplication like

$$\begin{array}{r} 11 \\ \times 11 \\ \hline 11 \\ 11 \\ \hline 121 \end{array}$$

there occur only the two digits 1 and 2. Find all multiplications involving exactly two digits.

Solution by the Proposer. The two numbers to be multiplied shall be denoted by A and B. Each contains at most two different digits a and b. We shall divide the problem into simpler cases.

(I) A and B have only a single digit each.

By looking at an ordinary multiplication table one finds that there are only the cases

$$0 \times a = a \times 0 = 0, \qquad 1 \times a = a \times 1 = a,$$
$$2 \times 2 = 4, \qquad 3 \times 3 = 9, \qquad 5 \times 5 = 25, \qquad 6 \times 6 = 36,$$

involving only two digits.

(II) $A = a \neq 0$ has a single digit, B has more than one digit.

Through the use of (I) it is verified that there are only the solutions

$$2 \times (22 \cdots 2), \qquad 3 \times (33 \cdots 3), \qquad 1 \times (bb \cdots b), \qquad a \times (11 \cdots 1).$$

(III) A has more than one digit, $B = b$ is a single digit.

This is the most involved case and it seems necessary to examine the various possibilities for b separately. The results are, except for $b = 0$ and $b = 1$,

$$(22 \cdots 2) \times 2, \qquad (33 \cdots 3) \times 3,$$
$$(11 \cdots 1) \times b \text{ arbitrary}, \qquad 86 \times 8.$$

(IV) A and B contain at least two digits each.

Each line in a multiplication is then of the type given in (III) and it is readily verified that the only permissible digits can be 0 and 1. If all digits are unity the only solution is found to be

$$(11 \cdots 1) \times 11.$$

Thus this is the only solution to our problem for at least two digits in A and B, with zero not permitted as a digit. If zero is permitted the result is not so simple to formulate.

A Code Problem

E 1026 [1952, 465]. *Proposed by Michael Golomb, Purdue University*

Orders have been given in a certain country that from now on all scientific writing shall be done in a secret transcription, consisting of a permutation of the alphabet of the country. A foreigner, unacquainted with the language and its alphabet is given the privilege of seeing the transcribed form of any five letters he may designate. He wishes to learn the common form of the word *atom*, which he is told has five letters. How can he manage this, if reference to the inverse transcription is ruled out?

Solution by the Proposer. Let x be the word for atom, $c_1(x)$ the transcribed form of x, and $c_k(x)=c_1(c_{k-1}(x))$ for $k=2, 3, \cdots$. The foreigner asks: "What is the transcribed form of $c_{p-1}(x)$ where $p=n!$ (or $p=$ l.c.m. of numbers $2, 3, \cdots, n$) and n is the number of letters in your alphabet." It is clear that $c_p(x)=x$.

Also solved by C. S. Ogilvy.

Total Area of a Set of Circles

E 1027 [1952, 465]. *Proposed by Ruth Clark and Robert Oeder, Los Alamos Scientific Laboratory*

Find the total area enclosed by the set of circles formed as follows: Construct the inscribed circle of a triangle ABC and draw the tangent to this circle which is parallel to a selected side of the triangle; construct the inscribed circle of the new triangle so cut off and draw the tangent to this circle which is parallel to the selected side of triangle ABC; repeat the process indefinitely. Also show that the set of maximum area is obtained if the lines are drawn parallel to the shortest side of triangle ABC.

Solution by Julian Braun, Washington, D. C. Let K, s, r_0, r_n denote the area, semiperimeter, radius of the excircle on the selected side a, and radius of the nth constructed incircle, of the given triangle. Then

$$r_0 = K/(s-a), \qquad r_1 = K/s,$$

whence

$$r_1/r_0 = r_{n+1}/r_n = 1 - a/s.$$

This ratio is largest, and thus the required area a maximum, if a is the shortest side of the triangle. The total area of the set of incircles is

$$\sum_{i=1}^{\infty} \pi r_i^2 = \pi(K/s)^2 \sum_{i=0}^{\infty} (1-a/s)^{2i}$$
$$= \pi K^2/(2as - a^2) = \pi K^2/a(b+c).$$

Also solved by Leon Bankoff (in two ways), D. H. Browne, C. V. Fronabarger, Harry Furstenberg, Arthur Gregory, Vern Hoggatt, H. I. James, M. S.

Klamkin, A. E. Livingston, B. Martin and J. C. Wu (jointly), C. S. Ogilvy, F. D. Parker, W. O. Pennell, R. R. Phelps, L. A. Ringenberg, W. J. Robinson, Azriel Rosenfeld, K. Subba Rao, C. W. Trigg, J. V. Whittaker, and the proposers. Late solutions by C. L. Gape and D. E. Konhauser.

Pythagorean Triangles and Fibonacci Numbers

E 1028 [1952, 465]. *Proposed by Vern Hoggatt, Oregon State College*

Do there exist Pythagorean triangles whose sides are Fibonacci numbers?

Note by Norman Miller, Queen's University. This problem was assigned as an exercise to my freshman class in algebra and a number of answers were received which made use of one or other of the following arguments:

(1) No three numbers of a Fibonacci sequence can represent the sides of *any* triangle, since of any three chosen the sum of the two smaller is not greater than the third;

(2) If three numbers from a Fibonacci sequence, in order of magnitude, are a, b, and c, then $c \geqq a+b$ and $c^2 > a^2+b^2$.

Remarks by W. F. Cheney, Jr., University of Connecticut. It is quite likely that (3, 4, 5) and (5, 12, 13) are the only Pythagorean triangles *two* of whose sides are numbers in the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \cdots .$$

If we refer to these Fibonacci numbers by their position numbers in the sequence, those numbered $6n$ will always be an even leg of a Pythagorean triangle; those numbered $6n\pm 1$, $6n\pm 2$ will always be an odd leg of some Pythagorean triangle; those numbered $6n\pm 1$ will always be the hypotenuse of some Pythagorean triangle; some, but not all, of those numbered $6n+2$ may be a hypotenuse; none numbered $6n+3$ can constitute any side of a Pythagorean triangle; those numbered $6n+4$ can never be a hypotenuse.

Also solved by A. P. Boblétt, Arthur Gregory, P. G. Kirmser, M. S. Klamkin, Sam Kravitz, R. J. Mercer, Leo Moser, J. V. Pennington, L. L. Pennisi, L. A. Ringenberg, Azriel Rosenfeld, K. Subba Rao, J. V. Whittaker, Margaret Willerding, and the proposer.

E 854 Again

E 1029 [1952, 465]. *Proposed by A. E. Currier, United States Naval Academy*

Sum the series $1+1/3+2!/(3)(5)+3!/(3)(5)(7)+\cdots$.

Solution by K. Subba Rao, M. R. College, Vizianagram, India. We have, in the interval $-1<x<1$,

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \cdots$$

$$= x\left[1 + \frac{1}{3}2x^2 + \frac{2!}{3\cdot 5}2^2x^4 + \frac{3!}{3\cdot 5\cdot 7}2^3x^6 + \cdots\right].$$

Putting $x=1/\sqrt{2}$ in each member, and simplifying, we have

$$1 + 1/3 + 2!/(3)(5) + 3!/(3)(5)(7) + \cdots = \pi/2.$$

Also solved by Julian Braun, L. Carlitz, Richard Courter, H. E. Fettis, Harry Furstenberg, H. W. Gould, N. G. Gunderson, Vern Hoggatt, A. S. Howard, P. G. Kirmser, M. S. Klamkin, W. D. Lambert, A. E. Livingston, Norman Miller, F. R. Olson, W. O. Pennell, L. L. Pennisi, M. R. Spiegel, O. E. Stanaitis, C. A. Swanson, Chih-yi Wang, J. V. Whittaker, R. E. Wild, and the proposer.

Editorial Note. It has been pointed out that this problem is the same as E 854 [1949, 633–35]. There five different solutions were published, and three references given.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4528. *Proposed by Paul Erdös, National Bureau of Standards, Los Angeles*

Consider sequences of consecutive positive integers, $n, n+1, \cdots, n+k$, which have the property that one of the integers is relatively prime to all the others. (1) Prove that, if one of the integers is a prime, the property holds. (2) Show that sequences exist which do not have the property.

4529. *Proposed by C. D. Olds, San Jose State College, California*

Can one find a convergent series $a_0+a_1x+a_2x^2+\cdots$, where the a's are real and positive and such that all the roots of each of the equations

$$a_0 + a_1x + a_2x^2 = 0$$

$$a_0 + a_1x + a_2x^2 + a_3x^3 = 0$$

$$\cdots\cdots\cdots\cdots$$

are real?

4530. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$, let A', B', C', D' be the feet of the altitudes AA', BB', CC', DD'. The planes drawn through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ perpendicular to BC, CA, AB, DA, DB, DC respectively, are concurrent at a point P, which is the radical center of the spheres described with the vertices A, B, C, D as centers and with the altitudes AA', BB', CC', DD' as radii.

4531. *Proposed by R. M. Redheffer, University of California, Los Angeles*

Let $f(z)=\sum a_n z^n$ have simple poles with positive residues at the points $d_m \neq 0$, and similarly for $g(z)=\sum b_n z^n$ at the points $e_m \neq 0$. Both functions are supposed regular elsewhere. Then the function $\sum a_n b_n z^n$ has simple poles with negative residues at the points $d_m e_n$ and no other finite singularities.

4532. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime $\geqq 3$ and let g denote a primitive root (mod p). Prove

1. $$1+\sum_{r=1}^{(p-3)/2} \frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)} \equiv \prod_{r=1}^{(p-3)/2} (1+g^r) \pmod{p}.$$

2. If $p \equiv 3$ (mod 4), then

$$1+\sum_{r=1}^{(p-3)/2} g^{r(r+1)/2} \frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)} \equiv \prod_{r=1}^{(p-3)/4} (1+g^{2r}) \pmod{p};$$

If $p \equiv 1$ (mod 4), the sum vanishes.

SOLUTIONS

A Definite Integral

4469 [1952, 45]. *Proposed by A. J. Coleman, University of Toronto*

Evaluate the integral

$$\int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{x^2(1+x^2)}\,dx.$$

The dominant term when a is very much smaller than b is also desired. (The integral was met in a discussion of the diffusion of gamma rays.)

Solution by the Proposer. Since $1/x^2(1+x^2)=x^{-2}-(1+x^2)^{-1}$, we have

$$I(a,b) = J(a,b) - K(a,b),$$

where $I(a, b)$ is the given integral and

(1) $$J(a,b) = \int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{x^2}\,dx,$$

$$K(a, b) = \int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{1+x^2}\,dx. \tag{2}$$

The substitution $x=z^{-1}$ reduces (1) to a known integral*

$$J(a, b) = \int_0^\infty e^{-b^2z^2-a^2/z^2}dz = \frac{\sqrt{\pi}}{2b}\,e^{-2ab}. \tag{3}$$

To evaluate K, we note that the substitution $x=y^{-1}$ gives

$$K(a, b) = K(b, a) = K. \tag{4}$$

Further

$$K - \frac{1}{2a}\frac{\partial K}{\partial a} = \int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{1+x^2}\,dx + \int_0^\infty \frac{x^2e^{-a^2x^2-b^2/x}}{1+x^2}\,dx$$

$$= \int_0^\infty e^{-a^2x^2-b^2/x^2}dx = J(b, a) = \frac{\sqrt{\pi}}{2a}\,e^{-2ab}$$

by (3). Thus we obtain

$$\frac{\partial}{\partial a}(e^{-a^2}K) = -\sqrt{\pi}\,e^{-a^2-2ab},$$

$$e^{-a^2}K(a, b) = f(b) - \sqrt{\pi}\int_0^a e^{-x^2-2bx}dx,$$

where $f(b)$ does not involve a. It follows that

$$\begin{aligned} e^{-a^2-b^2}K(a, b) &= F(b) - \sqrt{\pi}\int_0^a e^{-(x+b)^2}dx \\ &= F(a) - \sqrt{\pi}\int_0^b e^{-(x+a)^2}dx, \end{aligned} \tag{5}$$

where we have made use of (4), and $F(b)=e^{-b^2}f(b)$. Putting $a=0$ in (5) and then $b=0$ in the result, we have

$$F(b) = e^{-b^2}K(0, b) = F(0) - \sqrt{\pi}\int_0^b e^{-x^2}dx,$$

$$F(0) = K(0, 0) = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

We continue as follows:

* Goursat-Hedrick, *Mathematical Analysis*, v. I, p. 373. In exercise 3, replace x by bz and a by a/b.

$$e^{-a^2-b^2}K(a,b) = \frac{\pi}{2} - \sqrt{\pi}\int_0^b e^{-x^2}dx - \sqrt{\pi}\int_0^a e^{-(x+b)^2}dx$$

$$= \frac{\pi}{2} - \sqrt{\pi}\int_0^{a+b} e^{-x^2}dx = \sqrt{\pi}\left[\int_0^\infty e^{-x^2}dx - \int_0^{a+b} e^{-x^2}dx\right]$$

$$= \sqrt{\pi}\int_{a+b}^\infty e^{-x^2}dx.$$

$$K(a,b) = \sqrt{\pi}\, e^{a^2+b^2}\int_{a+b}^\infty e^{-x^2}dx.$$

Thus we have finally

$$I(a,b) = \sqrt{\pi}\, e^{-2ab}\left[\frac{1}{2b} - e^{(a+b)^2}\int_{a+b}^\infty e^{-x^2}dx\right]. \tag{6}$$

The integral involved is the well known, and much tabulated, remainder of the Error Function. Since†

$$e^{y^2}\int_y^\infty e^{-x^2}dx = \frac{1}{2y}\left[1 - \frac{1}{2y^2} + \frac{1\cdot 3}{2^2y^4} - \frac{1\cdot 3\cdot 5}{2^3y^6} + \cdots\right],$$

we have, for large values of $a+b$,

$$I(a,b) = \sqrt{\pi}\, e^{-2ab}\left[\frac{1}{2b} - \frac{1}{2(a+b)} + \frac{1}{4(a+b)^3} - \cdots\right]. \tag{7}$$

For small a/b, the dominant term of this will be

$$I(a,b) \sim \frac{\sqrt{\pi}}{2}\frac{a}{b^2}e^{-2ab}. \tag{8}$$

Throughout the above proof the differentiation under the integral sign is readily justified by the powerful convergence of e^{-x^2}.

An Integral Inequality

4471 [1952, 46]. *Proposed by Ky Fan, University of Notre Dame*

Let $K(x, y)$ be a non-negative Lebesgue integrable function over the square $a\leq x\leq b$, $a\leq y\leq b$. Suppose that B is a positive constant such that $\int_a^b K(x, y)dy \leq B$ for almost all x in $[a, b]$, and also $\int_a^b K(x, y)dx\leq B$ for almost all y in $[a, b]$. If two finite-valued functions $f(x)$, $g(x)$ are both non-negative and non-increasing in $[a, b]$, prove that

$$\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy \leq B\int_a^b f(x)g(x)dx. \tag{1}$$

† Jahnke and Emde, *Tables of Functions*, Dover, 1943, p. 24

[For the particular case when $K(x, y)$ is constant, (1) is known as Tchebychef's inequality. See Hardy, Littlewood, and Polya, *Inequalities*, Cambridge 1934, p. 168, Theorem 236.]

Solution by the Proposer. Obviously we may assume $B=1$. Let

$$h(x) = \int_a^b K(x, y)g(y)dy.$$

Then (1) can be written

$$\int_a^b f(x)[h(x) - g(x)]dx \leqq 0. \tag{2}$$

Consider an arbitrary point c in $[a, b]$. Let

$$k(y) = \int_a^c K(x, y)dx.$$

Then

$$k(y) \geqq 0 \text{ in } [a, b]; \qquad k(y) \leqq 1 \text{ for almost all } y \text{ in } [a, b]; \tag{3}$$

$$\int_a^b k(y)dy \leqq c - a; \tag{4}$$

and

$$\int_a^c h(x)dx = \int_a^b k(x)g(y)dy = \int_a^c k(y)g(y)dy + \int_c^b k(y)g(y)dy. \tag{5}$$

As $g(x)$ is non-increasing and non-negative, (3) and (4) imply

$$\int_a^b k(y)g(y)dy \leqq g(c)\int_c^b k(y)dy \leqq g(c)\left[c - a - \int_a^c k(y)dy\right].$$

Consequently we have, by (5):

$$\int_a^c h(x)dx \leqq \int_a^c k(y)[g(y) - g(c)]dy + (c - a)g(c)$$

and then, by the second part of (3):

$$\int_a^c h(x)dx \leqq \int_a^c [g(y) - g(c)]dy + (c - a)g(c) = \int_a^c g(y)dy.$$

Hence,

$$\int_a^c [h(x) - g(x)]dx \leqq 0 \qquad \text{for all } c \text{ in } [a, b]. \tag{6}$$

As $f(x)$ is finite-valued and non-increasing, we can apply the Second Mean Value Theorem. There exists a point c in $[a, b]$ such that

$$\int_a^b f(x)[h(x) - g(x)]dx = f(a)\int_a^c [h(x) - g(x)]dx + f(b)\int_c^b [h(x) - g(x)]dx.$$

Since $f(a) \geqq f(b)$, we have by (6):

$$f(a)\int_a^c [h(x) - g(x)]dx \leqq f(b)\int_a^c [h(x) - g(x)]dx$$

and therefore, since $f(b) \geqq 0$:

$$\int_a^b f(x)[h(x) - g(x)]dx \leqq f(b)\int_a^b [h(x) - g(x)]dx \leqq 0,$$

which is the desired inequality (2).

Editorial Note. H. J. Zimmerberg points out the similarity of the above inequality to Theorem 2.1 in W. T. Reid, Symmetrizable completely continuous linear transformations in Hilbert space, *Duke Mathematical Journal*, v. 18 (1951) pp. 41–56. The inequality is shown to hold also under different hypotheses. It does not appear to be a simple problem to derive one from the other.

A Problem in Plane Areas

4472 [1952, 46]. *Proposed by G. T. Williams, Elmont, New York*

If n is an odd integer >1, the curve $x^n + y^n = 1$ has the line $x + y = 0$ as asymptote, and the area in the second (or fourth) quadrant between the curve and its asymptote exists and is equal to the area in the first quadrant divided by $2\cos(\pi/n)$.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. The asymptote may be easily obtained by familiar methods.

Let us denote by A_1 the area in the first quadrant and by A_2 the area between the curve and its asymptote in the fourth quadrant. Then

$$A_1 = \int_0^1 (1 - x^n)^{1/n}dx, \qquad A_2 = \frac{1}{2} + \int_1^\infty [x + (1 - x^n)^{1/n}]dx.$$

Since $n \geqq 3$ and $x + (1 - x^n)^{1/n} = O(1/x^{n-1})$, the second integral exists. Making the substitutions $x = (\sin t)^{2/n}$ and $x = (\sin t)^{-2/n}$ in A_1 and A_2 respectively, we obtain

$$A_1 = \frac{2}{n}\int_0^{\pi/2} (\sin t)^{2/n-1}(\cos t)^{2/n+1}dt,$$

$$A_2 = \frac{1}{2} + \frac{2}{n}\int_0^{\pi/2} [1 - (\cos t)^{2/n}](\sin t)^{-4/n-1}\cos t dt.$$

Integration by parts yields

$$A_2 = \frac{1}{n}\int_0^{\pi/2} (\sin t)^{1-4/n}(\cos t)^{2/n-1}dt.$$

These integrals are well known in terms of the Beta-function. Hence their ratio is found to be

$$\frac{A_1}{A_2} = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)} = \frac{\pi}{\sin\frac{\pi}{n}}\,\frac{\sin\frac{2\pi}{n}}{\pi} = 2\cos\frac{\pi}{n}$$

which gives the desired result immediately.

Also solved by G. P. Henderson, M. Perisastri, M. R. Spiegel, F. Underwood, R. E. Wild, and the Proposer.

A Theorem on Differences (mod p)

4473 [1952, 109]. *Proposed by J. B. Kelly, Institute for Advanced Study*

Show that the differences of order $(p+1)/2$ of the sequence

$$\left(\frac{1}{p}\right), \left(\frac{2}{p}\right), \cdots, \left(\frac{x}{p}\right), \cdots, \left(\frac{p-1}{p}\right)$$

are divisible by the prime p. Here (x/p) is the Legendre symbol. Show, conversely, that the only sequences, $\{f(x)\}$, of length $p-1$ consisting only of 1's and -1's, and having this property are $f(x)\equiv 1$, $f(x)\equiv -1$, $f(x)=(x/p)$, $f(x)=-(x/p)$, $(x=1, 2, \cdots, p-1)$.

Solution by Leonard Carlitz, Duke University. Since the rth differences of a polynomial of degree $<r$ vanish and since

$$\left(\frac{x}{p}\right) \equiv x^{(p-1)/2} \pmod{p},$$

it follows at once that

$$\Delta^r\left(\frac{x}{p}\right) \equiv 0 \pmod{p} \qquad \text{for } r \leqq (p+1)/2.$$

To prove the converse, we remark first that by the Lagrange interpolation a function defined on the set $1, 2, \cdots, p-1$ and with values (mod p) is given by

$$f(x) = -\sum_{a=1}^{p-1} af(a)\,\frac{x^{p-1}-1}{x-a}. \tag{1}$$

In other words, $f(x)$ can be represented by a polynomial. Now suppose the numbers 1, 2, $\cdots$, $p-1$ separated into two sets A, B such that $f(a)=1$ for $a\in A$, $f(b)=-1$ for $b\in B$. Then (1) becomes

$$f(x) = -\sum_a a\frac{x^{p-1}-1}{x-a} + \sum_b b\frac{x^{p-1}-1}{x-b}. \tag{2}$$

In the next place the hypothesis $\Delta^{(p+1)/2}f(x)\equiv 0$ implies deg $f(x)\leqq (p-1)/2$. Accordingly (2) implies

$$\sum_a a^r = \sum_b b^r, \qquad (1\leqq r<(p-3)/2). \tag{3}$$

Put $g(x)=\prod_a(x-a)$, $h(x)=\prod_b(x-b)$, so that $x^{p-1}-1\equiv g(x)h(x)$. Now if deg $g(x)>$deg $h(x)$, it follows from (3) and the familiar congruence

$$\sum_{c=1}^{p-1} c^r \equiv 0 \pmod p \qquad (1\leqq r\leqq p-2)$$

that $h(x)$ reduces to a power of x; consequently $h(x)=1$ and $f(x)=1$. Similarly the supposition deg $g(x)<$deg $h(x)$ implies $g(x)=1$ and $f(x)=-1$. If then we take deg $g(x)=$deg $h(x)=(p-1)/2$, we see that (3) implies

$$g(x) - h(x) = 2C \text{ (constant)},$$
$$x^{p-1} - 1 = g^2(x) - 2Cg(x) = \{g(x) - C\}^2 - C^2,$$

so that $C=\pm 1$. It follows that

$$g(x) = x^{(p-1)/2} \pm 1, \qquad h(x) = x^{(p-1)/2} \mp 1. \tag{4}$$

Taking the lower signs in (4) we find that (2) reduces to $f(x)=x^{(p-1)/2}$, while the upper signs lead to $f(x)=-x^{(p-1)/2}$

Also solved by the Proposer.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

ON INTEGRAL COORDINATES

NORMAN ANNING, University of Michigan

In the classroom integral coordinates have some obvious uses. Obvious is a teacher's word for "as plain as the truck you overtake on the road (*ob viam*)."

The teacher who wants a circle or a parabola with integral points has no trouble in making the arrangements. Here are some handy ellipses.

Obvious solutions of the equation $x^2+3y^2=4(a^2+ab+b^2)$ are

$$\begin{array}{llll} x^2 = (a-b)^2, & (2b+a)^2, & (2a+b)^2, \\ y^2 = (a+b)^2, & a^2, & b^2. \end{array}$$

Choose for a and b any positive integers and the stage is set. If at any time such a conic with its dozen integral points begins to seem too pat and regular, variety can be introduced by the transformations $x=Y+X$, $y=Y-X$. Thus $x^2+3y^2=52$ becomes $X^2-XY+Y^2=13$ and it will be found that the twelve integral points have gone over unharmed. Anyone who feels the need of more than a dozen integral points on one ellipse should try replacing 52 by 196, or 13 by 91, or 13 by 1729.

Only one application will be suggested: the verification of the Pascal Theorem can be brought within the range of the freshman.

Editorial Note to Students. Professor Anning's note suggests avenues of further research. Can you discover other "handy" curves, in particular conic sections? We shall be glad to receive any that you find.

NUMBER MAGIC

To guess a person's age: 1) Have him add 1953 to his age A, and then determine the digit-sum S of the total. Since $A\equiv S$ mod 9, the person's age should be within the range of guesswork. For example, if $A\equiv 5$ mod 9, then $A=5$, 14, 23, 32, 41, or 50 and one should be able to match the proper value of A and the individual. 2) Or, have him multiply his age by 1953, add his age to the product, and then determine the digit-sum of the total. Again, $A\equiv S$ mod 9, and we proceed as in (1).

ON SHORT METHODS OF MULTIPLICATION

Many short methods of multiplication have been suggested, but most of them are applicable only to special situations. Here is a brief list, without explanations, of some of the more useful of these short methods.

1. To square numbers ending in 5.
   Rule. $(10a+5)^2=100a(a+1)+25$.
2. To square numbers near 25.
   Rule. $(25+x)^2=100x+(25-x)^2$.
3. To square numbers near 50.
   Rule. $(50+x)^2=100(25+x)+x^2$.
4. To square numbers near 100.
   Rule. $(100+x)^2=100(100+2x)+x^2$.
5. To find the product of two "teen" numbers.
   Rule. $(10+a)(10+b)=10(10+a+b)+ab$.

6. To find the product of numbers having small complements.

   Rule. $(10-a)(10-b)=10(\overline{10-a}-b)+ab,$

   $(100-a)(100-b)=100(\overline{100-a}-b)+ab,$ etc.

7. To find the product of numbers having small supplements.

   Rule. $(100+a)(100+b)=100(\overline{100+a}+b)+ab,$ etc.

8. Generalization of (6).

   Rule. $ab=(x-a)(x-b)+x(\overline{a+b}-x).$

We add to this list another method which does not seem to be as well known. The method is quite general in its application; indeed, it includes some of the other methods as special cases.

9. Rule. $ab=(a-c)(b+c)+(a-\overline{a-c})(b-\overline{a-c})$

   $=(a+c)(b-c)+(\overline{a+c}-a)(\overline{a+c}-b).$

(Note that the second term on the right is the product of the differences between *two* factors of one set and *one* factor of the other set.)

Examples. $(27)(19)=(30)(16)+(3)(11)=480+33=513.$ $(c=3)$

$(52)(33)=(50)(35)+(2)(-17)=1750-34=1716.$ $(c=2)$

$(146)(86)=(150)(82)+(4)(64)=12300+256=12556.$ $(c=4).$

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

An Introduction to the Theory of Differential Equations. By Walter Leighton. McGraw-Hill Book Company, 1952. viii+174 pages. \$3.50.

This is a textbook for the mathematics major and other students interested in the theory of elementary differential equations. Emphasis is placed on a careful statement of the existence theorems for solutions of the various types of differential equations studied. The book is limited to the study of ordinary differential equations of the first degree.

The topics discussed in the order in which they appear are: the nature of solutions of differential equations with a statement of the fundamental existence theorems for equations of the first and second orders; solutions of first order equations with simple applications; the linear equation with variable coefficients including linear dependence and existence theorems with most of the work limited to second order equations; the method of variation of parameters; the linear equation with constant coefficients with applications chiefly from classical

mechanics; solutions by means of power series with a discussion of the existence of solutions and, for the second order equation, a discussion of singular points; systems of linear equations; the method of successive approximations; oscillation theory with Sturm's separation and comparison theorems as they apply to self-adjoint linear equations of the second order; characteristic functions and the problem of expanding a function in a series of functions orthogonal with respect to a weight function. Appendix 1 deals with a proof of the fundamental existence theorem for two simultaneous first order equations, Appendix 2 gives a proof of the implicit function theorem, Appendix 3 proves the existence of an integrating factor for the first order equation, and Appendix 4 is devoted to regular integrals.

Some of the topics are treated so briefly that one wonders why they were included. The author states in the preface that he has given more than the usual amount of attention to the method of successive approximations but the reviewer finds that only two pages of the text are devoted to this method as it applies to differential equations. This method is used in the appendix to prove the fundamental existence theorem. The work on functions orthogonal with respect to a weight function and the problem of the expansion of a function in a series of such functions is covered in four pages. On the other hand, in the chapter on oscillation theory, the self-adjoint differential equation of the second order is studied in some detail.

The author seems to have succeeded very well in putting before the student statements of the theorems relating to solutions of equations of the types studied. Since the emphasis is on theory, the number of problems to be solved is not great and it does not seem probable that the student will gain much facility in solving differential equations. Most of the techniques usually appearing in an introductory text are missing. In general, the exposition is clear but brief.

The book is pleasing in appearance and the number of typographical errors seems to have been kept to a minimum. The reviewer believes that most teachers of introductory courses in the subject would benefit by a careful reading of this text.

D. S. MORSE
Union College

Differential Equations. By R. C. Yates. New York, McGraw-Hill Book Company, 1952. vii+215 pages. $3.75.

Most of the older text-books on Differential Equations approach the subject as follows: a general "type" differential equation is given, and then a "step one, step two, etc." process is exhibited which "solves" the equation. A course taught from such a book gives the student a great deal of drill in the manipulation of certain mathematical formulae, but teaches him very little in the way of understanding this subject which is of great importance in both pure and applied mathematics.

The author of the text under review proposes to remove one of the chief objections to such an approach by introducing each type through practical examples chosen from the fields of science. The preface states, "The structure of the text is built upon the realistic principle of having the student face the physical or geometrical situation first. He will thus recognize the need for a study of differential equations. He initially analyzes the problem and establishes the mathematical "set-up." Then his attention is directed to the selection and use of appropriate methods of solution. Finally, he must transcribe his mathematical results into the language of words. It is in this way that meaning and understanding are most firmly implanted."

The author has in general followed this procedure throughout the text, and the result is a text-book which is a definite improvement over the older method. The book contains many interesting and pertinent examples. Another interesting feature is the insertion of chapters on "Summary and Review Problems." It is believed that these will be helpful to both student and teacher.

The contents cover, in general, the material included in the usual introductory text-book on differential equations, the chapter titles being as follows: 1. Definitions; Formation of Equations; Physical and Geometrical Interpretations. 2. Separation of Variables and Homogeneity. 3. Equations of First Order, First Degree; The Bernoulli Equation; Integrable Forms. 4. Summary and Review Problems. 5. The Linear Equation with Constant Coefficients, Right Member Zero. 6. The Linear Equation of Second and Higher Order with Constant Coefficients, Right Member Not Zero; Equations with Variable Coefficients. 7. The Derivative Operator. 8. Summary and Review Problems. 9. Some Special Forms and Their Applications. 10. Approximate Numerical Solutions. 11. Summary and Review Problems. 12. Solutions in Series. 13. The Legendre and Bessel Equations; The Gamma Function. 14. Expansions. 15. Summary and Review Problems. 16. Engineering Problems Leading to Partial Differential Equations. 17. The Wave Equation and Separation of Variables. 18. Fourier Series. 19. Review and General Summary. Each chapter contains many well chosen problems of varying degree of difficulty.

For those who prefer a rather rigorous approach to the subject, it should be stated that the present text is written mainly for students of "applied mathematics." No existence theorems are stated—in fact no theorem is stated as such. The approach throughout is intuitive, and one who is an exponent of rigor would object to phrases such as, for example, those on page 22 which refer to dS as a small change in S, and dt as a small change in time. In most cases, no restrictions are placed on the functions discussed. For instance, in the chapter on Fourier Series, no conditions for convergence are stated.

In spite of the above criticism, the reviewer believes that this book will be welcomed by teachers of differential equations.

J. S. LEECH
Colorado College

Calculus of Variations. By Robert Weinstock. McGraw-Hill Book Company. 1950. x+326 pages. $6.50.

In the author's words, the two-fold purpose of this book is "(i) To provide for the senior or first-year graduate student in mathematics, science, or engineering an introduction to the ideas and techniques of the calculus of variations . . . (ii) To illustrate the application of the calculus of variations in several fields outside the realm of pure mathematics. (By far the greater emphasis is placed upon this second aspect of the book's purpose.)" Thus this book is primarily an introduction to calculus of variations for physicists and engineers; and it may well prove to be a popular one, for it appears to be generally well planned and carefully and clearly written. The only other book in English which is at all comparable in aim is C. Lanczos's *The Variational Principles of Mechanics*, University of Toronto Press (1949), a more traditional treatment concerned with presenting the role of calculus of variations in mechanics from the viewpoint of the history of ideas.

The desire to survey a variety of applications in a book of moderate size has forced upon the author some rather severe pruning of the material on calculus of variations proper. Thus not only are all questions of existence and sufficiency finessed, but there is no discussion whatever of second-order conditions or even of first-order corner conditions; problems in parametric form are barely mentioned; and for single integrals only the simplest variable-end point situations in the plane are treated. Problems with side conditions, being particularly important for applications, are considered somewhat more fully. Happily, sound one-parameter-embedding arguments replace use of the slippery "variational operator" δ. Less happily, the term "extremum" is used throughout as synonymous with "stationary value."

A list of the later chapter headings suggests the range of applications: geometrical optics: Fermat's principle; dynamics of particles; two independent variables: the vibrating string; the Sturm-Liouville eigenvalue-eigenfunction problem; several independent variables: the vibrating membrane; theory of elasticity; quantum mechanics; electrostatics. The treatment of several of these topics owes much to Courant-Hilbert. The Ritz method is illustrated early for the vibrating string and discussed in connection with the various other vibration problems. The chapter on quantum mechanics clearly sets forth the important role of variational methods in the early development of the subject; and one can well afford to be reminded that the variational approach to the hydrogen atom has definite virtues of elegance in comparison with the differential-equations method usual in current physics texts. There are numerous well-selected exercises some of which extend the theory developed in the text.

T. A. BOTTS
University of Virginia

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INSTITUTES FOR TEACHERS OF MATHEMATICS

The Association of Teachers of Mathematics in New England announces that its fifth annual Institute for Teachers of Mathematics will be held during August 20–27, 1953 at Colby College, Waterville, Maine.

The University of Houston announces its third Mathematics Institute to be held on its campus in Houston, Texas, June 23–26, 1953. For information write to C. B. Rader, Box 234, University of Houston, Houston 4, Texas.

The University of Washington will sponsor an Institute for Teachers of Mathematics to be held in Seattle, June 22–26, 1953. Further information may be obtained from the Department of Mathematics, University of Washington, Seattle 5, Washington.

A SUMMER CONFERENCE IN COLLEGIATE MATHEMATICS

There will be held at the University of Colorado in 1953 an eight-week Summer Conference in Collegiate Mathematics. The National Science Foundation first made the suggestion that such a conference be held, the Committee on the Regional Development of the National Research Council made the preliminary plans including designation of location and objectives, and the National Science Foundation has granted financial support.

The primary object of the conference will be to assist teachers in colleges and universities not closely associated with the big research centers to improve the quality of the undergraduate majors in mathematics; this to be accomplished by giving such teachers an opportunity to think and discuss mathematics under the guidance of men who are not only nationally known but who can give to non-specialists clear and inspiring pictures of the sweep and depths of present-day mathematics. Throughout the conference Professor Emil Artin of Princeton University will lecture on Modern Developments in Algebra, and Professor Wilder of the University of Michigan on Foundations of Analysis and Geometry. A number of additional lectures will be arranged, including a series by Professor G. Pólya on problem solving. Ample opportunity will be given for formal and informal discussion.

The members of the conference will be housed chiefly in one of the newest dormitories of the University where provision will be made for single persons, couples, and families at a cost of about $160 per person for room and board for the period of the conference—approximately June 15 to August 8. Some accommodations of other types will be available and the Director will be glad to

assist in securing other types of housing for those who desire it.

The plan includes stipends of $300 each for a limited number of participants and it is hoped that many can come without such support, in some cases with assistance from their own universities. Those who wish to participate are asked to apply for membership and agree to stay for the entire period of the conference. It is expected that such membership will be freely granted up to the limit of size imposed by accommodations available. Applications for housing must be made well in advance of the opening date. There will be no cost to participants in the conference except a one-dollar registration fee.

Recreational week-end opportunities in and about Boulder are many. It is about thirty-five miles from Rocky Mountain National Park and thirty miles from Denver. Red Rocks Amphitheater and the Central City summer opera are within an hour's drive. The University of Colorado has an extensive recreational program in the summer, including mountain hiking, bus trips, horseback riding, and other sports. One or two special excursions will probably be arranged for those attending the conference.

Notice of this conference has been sent to all heads of departments of mathematics and Presidents of colleges and universities listed in the directory of the Mathematical Association of America. For further information and application blanks for membership, stipends, and housing, write Burton W. Jones, Director, 119E Hellems Building, The University of Colorado, Boulder, Colorado. Applications for stipends must be in the hands of the director by April 1.

MARIE J. WEISS MEMORIAL SCHOLARSHIP FUND

A fund has been established to endow a scholarship at Newcomb College, Tulane University, in memory of Marie J. Weiss. Those desiring to give to this fund should mark their contributions "for the Marie J. Weiss Memorial Scholarship Fund," and mail them to the Tulane Alumni Fund, Tulane University, New Orleans, Louisiana.

PERSONAL ITEMS

Professor P. O. Bell of the University of Kansas has been awarded a National Science Foundation Postdoctoral Fellowship and is spending the year at the University of California, Berkeley.

Professor Claude Chevalley of Columbia University was elected to the National Academy of Sciences during 1952.

Mr. H. C. Griffith of the University of Tennessee has been awarded a National Science Foundation Fellowship for the academic year 1952–53.

Dr. John Wermer of Yale University has been awarded a National Science Foundation Fellowship for the year 1952–53.

Agricultural and Mechanical College of Texas announces: Associate Professors R. E. Basye and J. A. Daum have been promoted to professorships; Assistant Professor J. T. Kent has been promoted to an associate professorship; Dr. E. R. Keown has been appointed to an assistant professorship; Mr. D. B. Alexander has been appointed to an instructorship.

At Florida State University: Dr. J. W. Ellis, formerly a graduate student at Tulane University, Dr. M. J. Walsh, previously a graduate student at the University of Illinois, and Dr. R. E. Wheeler of the University of Kentucky, have been appointed to assistant professorships; Mr. C. M. Callahan, who has been an instructor at Lincoln Memorial University, has been appointed to an instructorship; Associate Professor Olga Larson has retired.

Lehigh University announces the following: Mr. Samuel Goldberg has been promoted to an assistant professorship; Dr. C. C. Hsiung, previously research fellow at Harvard University, has been appointed to an assistant professorship; Dr. S. I. Goldberg, who has been a research scientist with the Defense Research Board, Valcartier, Quebec, Canada, Dr. Felix Haas of Massachusetts Institute of Technology, Dr. J. A. Schatz, previously editorial assistant for Mathematical Reviews, and Dr. H. H. Wicke, formerly part-time instructor at State University of Iowa, have been appointed to instructorships; Mrs. Marjorie Halpern, Mr. R. R. Hohl, and Mr. C. B. Sensenig have been appointed to graduate assistantships; Associate Professor K. W. Lamson has retired with the title of Associate Professor Emeritus.

McMaster University makes the following announcements: Associate Professor J. D. Bankier has been appointed Chairman of the Department of Mathematics; Assistant Professor N. D. Lane of Carlton College, Ottawa, Canada, has been appointed to an assistant professorship.

Miami University reports the following: Associate Professor G. W. Spenceley has been promoted to a professorship; Mr. R. W. Emmert of Earlham College has been appointed to an instructorship.

Mississippi State College reports the following: Associate Professor S. B. Murray has been promoted to a professorship; Acting Instructor R. H. Hopkins has been promoted to an instructorship.

Washington Square College, New York University announces the following: Associate Professor Morris Kline has been promoted to a professorship; Dr. S. C. Lowell has been appointed Research Assistant Professor; Associate Professor W. M. Maiden and Professor R. G. Putnam have retired.

At Ohio University: Dr. Robert Butner of the University of Iowa has been appointed to an assistant professorship; Miss Mary Colberg, formerly associated with a research project of the University of Michigan, Mr. Neal Newby, Jr. of Harvard University, and Mr. Algray Verssen, who has been teaching in American schools in Germany, have been appointed to instructorships; Mrs. Neva Johnson has retired.

Pomona College makes the following announcements: Professor C. G. Jaeger, chairman of the Department of Mathematics, is on sabbatical leave during the second semester of 1952–53 and is in Europe; Mr. John Walsh has been appointed to a part-time instructorship.

Syracuse University announces: Associate Professor D. E. Kibbey has been promoted to a professorship; Assistant Professors Nancy Cole, H. W. Farnham, and R. D. Whitney have been promoted to associate professorships; Dr. O. O.

Pardee has been promoted to an assistant professorship; Dr. G. F. Leger, previously a mathematician at Bell Aircraft Company, has been appointed to an instructorship.

At Texas Southern University: Professor J. A. Pierce, acting head of the Department of Mathematics, has been appointed Chairman of Graduate School Administration; Mr. Calvin Culbreath, previously with the School of Vocational and Industrial Education of the University, has been appointed to an instructorship; Instructors A. H. Wardlaw and J. E. Westberry have been awarded General Education Board Fellowships and are on leave of absence.

University of Arizona announces the following appointments: Dr. G. M. Petersen of Brandon College to an instructorship; Mr. J. E. Householder and Mr. Gilbert Puente to assistantships.

University of Cincinnati makes the following announcements: Assistant Professor E. F. White has been promoted to an associate professorship; Professor C. A. Ludeke has been awarded a grant by the Research Corporation, New York City.

University of Delaware announces the following: Assistant Professor E. V. Lewis has been promoted to an associate professorship; Mr. G. O. Peters, previously research engineer at Franklin Institute, has been appointed to an assistant professorship; Assistant Instructor J. L. Howell of Yale University and Instructor W. L. Marshall of Hofstra College have been appointed to instructorships; Instructor A. C. Nelson is now a graduate student at the University of North Carolina.

At the University of Houston: Professor Albert Newhouse has been appointed Chairman of the Department of Mathematics; Mr. E. B. Adams, Mrs. Flora H. Bosworth, Mr. H. C. Lefkovits, and Mr. L. S. Lockingen have been serving as part-time instructors.

University of Kansas reports the following: Professor Robert Schatten is spending the year 1952–53 at the Institute for Advanced Study; Associate Professor J. L. Kelley of Tulane University has been appointed Visiting Associate Professor; Dr. W. F. Donoghue, Jr., formerly at the Applied Physics Laboratory, Silver Spring, Maryland, and Dr. K. T. Smith, who spent the year 1951–52 in Nancy, France, on a Fulbright Fellowship, have been appointed to assistant professorships; Dr. Arne Magnus, previously a graduate student at Washington University, has been appointed to an instructorship.

University of Kentucky announces: Associate Professor W. H. Pell of Brown University has been appointed Professor and Head of the Department of Mathematics; Professor H. H. Downing, formerly head of the Department of Mathematics, retains the position of Professor of Mathematics; Mr. D. C. Rose has been promoted to an instructorship; Graduate Assistants R. H. Sprague and W. C. Swift have been promoted to part-time instructorships; Mr. Ceslovas Masaitis of the University of Kaunas, Lithuania, has been appointed to an instructorship; Mr. Gene Adkins, Mr. Joseph Cornelison, Mr. T. F. Droege, Mr. M. I. Rose, and Mr. R. E. Shely have been appointed to graduate assistantships.

University of Maine announces the appointment of Dean Emeritus Paul Cloke of the College of Technology as Lecturer in the Department of Mathematics and the appointment of Mr. Phillip Hamm to a temporary instructorship.

University of Michigan announces the following: Associate Professor C. J. Nesbitt has been promoted to a professorship; Assistant Professor M. O. Reade has been promoted to an associate professorship; Instructors Raoul Bott, K. B. Leisenring, and A. J. Lohwater have been promoted to assistant professorships; Professor J. A. Dieudonné of the University of Nancy, France, has been appointed Visiting Professor for the year 1952–53; Dr. B. J. Tepping of the Bureau of Census has been appointed Lecturer; Dr. E. L. Griffin, formerly a graduate student at the University of Chicago, has been appointed to an instructorship; Assistant Professor N. H. Anning is on retirement furlough during 1952–53.

University of North Carolina at Chapel Hill reports the following: Dr. J. S. MacNerney of Northwestern University has been appointed to an assistant professorship; Professor T. F. Hickerson has retired with the title of Kenan Professor Emeritus.

At Woman's College, University of North Carolina: Assistant Professor Lila P. Walker has returned to the College after a year's leave of absence; Miss Frances Wolfe has been appointed Administrative Assistant to the Chancellor of the College.

University of Richmond announces the following: Assistant Professor D. F. Atkins of Bowling Green State University has been appointed to an assistant professorship; Dr. E. R. Sleight has retired from his position as Visiting Lecturer.

University of Saskatchewan makes the following announcements: Dr. G. H. M. Thomas, previously a graduate student at the University of Wisconsin, and Mr. Victor Linis, formerly a graduate student at McGill University, have been appointed to instructorships; Associate Professor Peter Scherk is on leave of absence and is Visiting Professor at University of California, Los Angeles.

University of South Carolina reports the following: Mr. Herbert Wolf, previously a graduate student at the University of North Carolina, has been appointed to an assistant professorship; Mrs. Elizabeth Wolf, who has been a graduate student at the University of North Carolina, has been appointed to an instructorship; Assistant Professor R. A. Lytle is on leave of absence and is studying at the University of Georgia.

University of Tennessee announces the following appointments to instructorships: Dr. Haskell Cohen, previously a graduate student at Tulane University; Mr. LaVerne Flatt, formerly a graduate student at George Peabody College for Teachers; Mrs. Ruth E. Hofstra of Syracuse University; Miss Thelma Peacock of the University of Maine.

University of Utah reports the following: Associate Professor C. J. Thorne has been promoted to a professorship; Professor R. N. Thomas is on leave of

absence for the academic year 1952–53 and is serving as a visiting professor at the Harvard College Observatory.

West Virginia University announces: Associate Professor J. K. Stewart has been promoted to a professorship; Assistant Professor A. B. Cunningham has been promoted to an associate professorship; Mr. R. A. Roberts, previously a teaching fellow at the University of Michigan, has been appointed to an assistant professorship; Mrs. Margaret S. Marshall has been appointed to an instructorship.

Yale University announces the following: Instructors R. R. Bernard and W. H. Mills have been promoted to assistant professorships; Dr. W. G. Bade of the University of California at Berkeley, Dr. R. G. Bartle, previously AEC Postdoctoral Fellow at Yale University, and Dr. Lawrence Markus, formerly research fellow and instructor at Harvard University, have been appointed to instructorships; also Dr. F. P. Pedersen, previously lecturer at the University of Southern California, Dr. F. D. Quigley, formerly teaching assistant at the University of Chicago, and Dr. J. T. Schwartz, who was an AEC Predoctoral Fellow at Princeton University, have been appointed to instructorships; Mr. P. C. Curtis, Jr., Mr. P. E. Klebe, Jr., Mr. T. A. Paley, and Mr. G. F. Simmons have been appointed to assistantships; Associate Professor J. I. Tracey has retired with the title of Associate Professor Emeritus.

Dr. J. W. Armstrong of Purdue University has accepted a position with the Aerophysics Section, Convair, Ft. Worth, Texas.

Associate Professor A. L. Blakers of Lehigh University has been appointed to a professorship at the University of Western Australia.

Mr. M. J. Bratt, formerly of the University of Cincinnati, is employed now by the General Electric Company, Cincinnati, Ohio.

Mr. H. W. F. Bruns, previously an instructor at Bowling Green State University, is teaching at Anthony Wayne High School, Waterville, Ohio.

Miss Helen Butcher, formerly an instructor at the University of Miami, is teaching at Miami Beach High School.

Dr. Peter Chiarulli of Carnegie Institute of Technology has been appointed to an assistant professorship in the Graduate Division of Applied Mathematics, Brown University.

Dr. G. F. Cramer of the Radio Corporation of America has a position as Staff Scientist with the Engineering Research Associates, Arlington, Virginia.

Dr. R. Y. Dean of California Institute of Technology has accepted a position as a mathematician with the General Electric Company, Richland, Washington.

Professor Chester Feldman of Antioch College has a position as a mathematical consultant with the Aircraft Marine Products, Harrisburg, Pennsylvania.

Mr. B. G. Fetterman, previously an instructor at Bowling Green State University, is teaching at Cleveland Heights High School, Ohio.

Dr. Mary E. Hamstrom of the University of Texas has been appointed to

an assistant professorship at Goucher College.

Assistant Professor L. A. Henkin of the University of Southern California has been promoted to an associate professorship.

Professor D. H. Hyers of the University of Southern California is on sabbatical leave during the current academic year and is at New York University.

Dr. R. V. Kadison, formerly at the Institute for Advanced Study, has been appointed to an assistant professorship at Columbia University.

Assistant Professor William Karush of the University of Chicago has been promoted to an associate professorship.

Professor Jacob Korevaar of the Institute of Technology, Delft, Netherlands, has been appointed to an assistant professorship at the University of Wisconsin.

Mr. R. E. Krucklin of the American Viscose Corporation has accepted a position as Senior Engineer with A. H. Johnson and Company, New York City.

Professor Hans Lewy has been appointed to a visiting professorship at Harvard University.

Mr. A. H. Moore of Pratt Institute has accepted a position as an operations analyst with the Operations Research Office, Johns Hopkins University.

Professor E. E. Moots of Cornell College has retired.

Dr. S. F. Neustadter of Harvard University is now a staff member with the Project Lincoln, Massachusetts Institute of Technology.

Mr. E. M. Olson has been appointed Lecturer at Columbia University.

Mr. R. E. Ozimkoski of Fordham University has been promoted to an assistant professorship.

Mr. P. E. Pfeiffer of Rice Institute has been promoted to the position of Assistant Professor of Electrical Engineering.

Dr. J. D. Riley, previously a graduate student at the University of Kansas, has a position as a mathematician with the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland.

Assistant Professor H. J. Ryser of Ohio State University has been promoted to an associate professorship.

Dr. H. S. Shapiro, formerly of Massachusetts Institute of Technology, is employed as a member of the technical staff of the Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. T. M. Simpson, Dean Emeritus of the Graduate School and Head Professor Emeritus of the Department of Mathematics, University of Florida, has accepted a position as Consultant in General Education for the Arkansas Experiment in Teacher Education sponsored by the Ford Foundation; he is located at Henderson State Teachers College.

Assistant Professor Wayman Strother of the University of Alabama has been appointed to an assistant professorship at the University of Miami.

Dr. J. A. Sullivan of the University of Notre Dame has been promoted to an assistant professorship.

Miss Helen J. Terry, previously an instructor at the University of Idaho, has a position with Sandia Corporation, Albuquerque, New Mexico.

Associate Professor F. E. Ulrich of Rice Institute has been promoted to a professorship.

Assistant Professor J. S. Vigder of the University of Saskatchewan has accepted a position with the Defence Research Board, Ottawa, Canada.

Dr. S. S. Walters has a position as an operations analyst with the Operations Research Office, Chevy Chase, Maryland.

Associate Professor L. E. Wear of California Institute of Technology has retired.

Mr. W. D. Wood, previously a graduate student at Purdue University, is now a staff member with the Sandia Corporation.

Mr. C. J. Kaufman died on November 10, 1952.

Dr. F. R. Moulton died on December 7, 1952. He was a charter member of the Association.

Professor Emeritus H. E. Van Buskirk of California Institute of Technology died on November 21, 1952. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 60 persons have been elected to membership by the Board of Governors on applications duly certified.

F. S. Badger, M.S. (Western Reserve) Dean of Students, Alliance College.

B. J. Ball, Ph.D. (Texas) Acting Asst. Professor, University of Virginia.

R. D. Beckey, B.A. (Wittenberg) Teacher and Director of Audio-Visual Education, Milton-Union High School, West Milton, Ohio.

C. J. Biggerstaff, Electronics Technician, United States Navy.

A. J. Bosgang, M.A. (C.C.N.Y.) Mathematician, Ballistics Research Laboratory, Aberdeen Proving Ground, Md.

J. R. Brown, Ph.D. (Kansas) Asst. Professor, Bradley University.

H. S. Clair, M.S. (Chicago) Lecturer, Illinois Institute of Technology.

A. H. Cockshott, M.A. (N.Y.U.) Asst. Professor, Manhattan College.

W. J. A. Culmer, M.A. (Minnesota) Instr., Departments of Mathematics and Mechanics, University of Minnesota.

E. K. Dorff, Student, University of Minnesota, Duluth, Minn.

M. S. Dorgan, M.S. (Iowa) Asst. Professor, Western State College, Gunnison, Colo.

E. I. Duggan, M.S. (Fordham) Instr., Iona College.

Erna Eanes, Student, Lenoir-Rhyne College.

J. R. Elliott, B.S. (Alabama Poly.) 2nd Lieutenant, United States Marines.

T. J. ENGLERT, Student, Kent State University.
P. J. FINN, Student, St. John's College.
I. E. GLOVER, M.A. (Michigan) Asst. Professor, Langston University.
Mr. JEAN GREGOIRE, Student, Laval University.
J. B. HERRESHOFF, Student, University of California at Berkeley.
RUTH A. HUFFMAN, B.S. (Oklahoma) Statistician, Tinker Air Force Base, Oklahoma City, Okla.
REV. BROTHER CALIXTUS JAMES, M.A. (Manhattan) Head of Department, LaSalle Military Academy, Oakdale, N. Y.
W. D. JAMES, Student, Northern Illinois State Teachers College.
D. H. JONES, B.A. (Oberlin) Teaching Fellow, University of Washington.
R. P. KELISKY, B.S. (Texas Technological) Teaching Fellow, University of Texas.
H. W. KELLEY, Student, Alabama Polytechnic Institute.
R. R. KISSLING, 2611 Piedmont Avenue, Berkeley, Calif.
M. J. KNIEDLER, JR., M.S. (West Virginia) Engineer, Pratt and Whitney Aircraft, East Hartford, Conn.
L. H. LANGE, M.S. (Stanford) Instr., Valparaiso University.
J. S. LEW, Student, Yale University.
A. H. MCCABE, Student, St. John's College.
D. W. MCLEAN, B.S. (Wayne) Grad. Student, University of Chicago.
R. T. MCLEAN, M.A. (Bowling Green State) Instr., College of Steubenville.
E. B. MCLEOD, JR., M.S. (Stanford) Research Assistant, Stanford University.
R. E. MITCHELL, B.S. (Detroit) Gear Mathematician, Vinco Corporation, Detroit, Mich.
R. L. MOENTER, M.A. (Ohio State) Asso. Professor, Midland College.
MORRIS MONSKY, B.A. (Columbia) Associate Actuary, Mutual Life Insurance Company of New York, New York City.
J. E. MUELLER, Student, Butler University.
P. A. NURNBERGER, B.A. (Wabash) Head, Departments of Mathematics and Science, Tri-State College.
R. E. OZIMKOSKI, M.S. (Fordham) Asst. Professor, Fordham University.
E. C. PAXHIA, Student, University of Rochester.
K. C. PENG, M.S. (Michigan) Statistical Quality Control Engineer, Parke, Davis and Company, Detroit, Mich.
D. A. QUARLES, JR., M.A. (Yale) Senior Mathematician, Applied Science Department, International Business Machines Corporation, New York City.
D. M. ROSS, M.S. (Northwestern) Substitute Teacher, City of Chicago High Schools.
A. A. SARDINAS, M.A. (Harvard) Associate Research Engineer, Burroughs Adding Machine Company, Philadelphia, Pa.
J. W. SEHESTEDT, B.D. (Southwestern Baptist Theological Seminary) Minister, Hoyt, Okla.
C. B. SENSENIG, B.S. (Franklin & Marshall) Grad. Assistant, Lehigh University.
Mrs. DOROTHY B. SHAFFER, M.A. (Radcliffe) Asst. Mathematician, Cornell Aeronautical Laboratory, Buffalo, N. Y.
E. S. SHAPIRO, M.A. (California) Mathematician, United States Naval Radiological Defense Laboratory, San Francisco, Calif.
M. G. SHULTS, M.S. (Math., Ft. Hays Kansas State; Meteor., C.I.T.) Instr., Northern Oklahoma Junior College, Tonkawa, Okla.
SISTER MARY STEPHANIE, Ph.D. (Catholic) Teacher, Georgian Court College.
PAUL SLEPIAN, B.S.(M.I.T.) Grad. Assistant, Brown University.
W. J. STEWART, Electrical Engineering Draftsman, San Antonio General Depot, Tex.
F. H. STILLINGER, JR., Student, University of Rochester.
MORRIS TENENBAUM, M.A. (Columbia) Grad. Student, Cornell University.
N. O. TIFFANY, M.A. (Southern California) Instr., Alfred University.
S. B. TOWNES, Ph.D. (Chicago) Asso. Professor, University of Hawaii.
MARION D. WETZEL, Ph.D. (Northwestern) Asso. Professor, Denison University.
H. H. WICKE, Ph.D. (Iowa) Instr., Lehigh University.
MRS. ELIZABETH S. WOLF, M.A. (North Carolina) Instr., University of South Carolina.
F. G. WOLFORT, JR., Student, St. John's College.

THE THIRTY-SIXTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-sixth annual meeting of the Mathematical Association of America was held at Washington University, Saint Louis, Missouri, on Tuesday, December 30, 1952, in conjunction with the annual meetings of the American Mathematical Society, the American Association for the Advancement of Science, and the Association for Symbolic Logic. About five hundred persons were registered, including the following two hundred and eighty-one members of the Association:

V. W. Adkisson, L. W. Akers, A. A. Albert, H. W. Alexander, C. B. Allendoerfer, R. D. Anderson, P. M. Anselone, Beulah Armstrong, Miriam C. Ayer, W. G. Bade, B. J. Ball, I. A. Barnett, M. A. Basoco, P. T. Bateman, J. D. Baum, P. R. Beesack, E. G. Begle, R. H. Bing, A. H. Black, H. D. Block, Brother J. A. Boose, R. D. Boswell, Jr., C. A. Bridger, R. W. Brink, R. H. Bruck, H. D. Brunk, P. B. Burcham, R. S. Burington, Jewell H. Bushey, R. K. Butz, S. S. Cairns, H. H. Campaigne, K. H. Carlson, J. O. Chellevold, Y. W. Chen, Elsie T. Church, R. V. Churchill, D. E. Coffey, L. A. Colquitt, Brother Damian Connelly, Byron Cosby, W. A. Couch, N. A. Court, Mary L. Cummings, J. C. Currie, Robert Davies, A. H. Diamond, Jim Douglas, Jr., T. L. Downs, W. C. Doyle, Nelson Dunford, W. L. Duren, William H. Durfee, J. M. Earl, J. C. Eaves, Samuel Eilenberg, R. E. Ekstrom, H. Margaret Elliott, J. G. Elliott, D. O. Ellis, Bernard Epstein, D. H. Erkiletian, Jr., H. J. Ettlinger, G. M. Ewing, W. H. Fagerstrom, F. D. Faulkner, H. M. Feldman, William Feller, W. T. Fishback, Harley Flanders, A. J. Flynn, W. C. Foreman, G. E. Forsythe, J. S. Frame, C. V. Fronabarger, H. M. Gehman, J. J. Gergen, Reverend F. J. Gerst, A. M. Gleason, H. E. Goheen, V. D. Gokhale, Michael Goldberg, W. A. Golomski, Reverend H. H. Gottbrath, S. H. Gould, R. F. Graesser, W. W. Graham, L. M. Graves, F. L. Griffin, J. S. Griffin, Jr., H. C. Griffith, H. T. Guard, D. L. Guy, Franklin Haimo, Edwin Halfar, A. E. Hallerberg, P. C. Hammer, H. W. Handsfield, J. R. Hanna, Frank Harary, H. G. Harp, Nola A. Haynes, P. W. Healy, I. L. Hebel, G. A. Hedlund, C. E. Heilman, Melvin Henriksen, I. N. Herstein, Fritz Herzog, Edwin Hewitt, T. H. Hildebrandt, J. J. L. Hinrichsen, F. E. Hohn, D. L. Holl, L. Aileen Hostinsky, Aughtum S. Howard, Reverend J. A. Hratz, S. T. Hu, G. B. Huff, Ralph Hull, M. Gweneth Humphreys, M. H. Ingraham, Fritz John, H. J. Johnson, L. W. Johnson, B. W. Jones, F. B. Jones, P. S. Jones, W. C. Kalinowski, Irving Kaplansky, J. L. Kelley, M. R. Kenner, J. R. F. Kent, D. E. Kibbey, V. L. Klee, Jr., S. C. Kleene, George Klein, L. A. Knowler, J. C. Koken, R. L. Lambert, C. E. Langenhop, R. E. Langer, Paolo Lanzano, Leo Lapidus, H. D. Larsen, W. G. Leavitt, A. S. Lee, Joseph Lehner, R. A. Leibler, Walter Leighton, B. L. Lercher, W. J. LeVeque, D. R. Lintvedt, H. D. Lipsich, Lee Lorch, T. A. Love, L. L. Lowenstein, C. I. Lubin, G. R. MacLane, Saunders MacLane, H. M. MacNeille, J. F. Manogue, C. G. Maple, Morris Marden, R. H. Marquis, J. M. Marr, Ella Marth, W. S. Massey, W. C. McDaniel, E. J. McShane, G. M. Merriman, R. J. Michel, R. R. Middlemiss, J. S. Minas, Nellie P. Miser, W. L. Miser, T. W. Moore, W. L. Moore, Thirza A. Mossman, H. T. Muhly, Zeev Nehari, Albert Newhouse, C. V. Newsom, T. A. Newton, C. O. Oakley, J. H. Oppenheim, Morris Ostrofsky, T. P. Palmer, S. T. Parker, G. W. Patterson, H. P. Pettit, C. F. Pinzka, C. G. Pitner, J. C. Polley, G. B. Price, J. E. Pryor, O. J. Ramler, J. F. Randolph, L. T. Ratner, L. M. Reagan, Mina Rees, R. F. Reeves, Francis Regan, Haim Reingold, J. G. Renno, P. R. Rider, J. K. Riess, J. D. Riley, R. F. Rinehart, L. A. Ringenberg, L. V. Robinson, L. D. Rodabaugh, I. H. Rose, Alex Rosenberg, P. C. Rosenbloom, J. B. Rosser, W. C. Royster, Charles Saltzer, Hans Samelson, R. G. Sanger, R. D. Schafer, Edith R. Schneckenburger, J. A. Schumaker, Nathan Schwid, L. L. Scott, W. R. Scott, R. R. Seeber, Jr., D. M. Seward, M. Anice Seybold, Harold Shniad, Marlow Sholander, Edward Silverman, Annette Sinclair, Sister M. Leonarda, Sister M. Pachomia, Sister Mary Teresine, Sister Presentation, Sister Rose Margaret, Sister Seraphim, M. F. Smiley, James C. Smith, R. L. Snider, W. S. Snyder, C. P. Sousley, E. J. Specht, R. L. Sternberg, A. G. Swanson, Gabor Szego, C. T. Taam, T. T. Tanimoto, Mildred E. Taylor, H. P. Thielman, R. H. Thompson, A. W. Tucker, J. L. Ullman, W. R. Utz,

B. O. Van Hook, G. B. Van Schaack, H. E. Vaughan, Reverend Dunstan Velesz, T. L. Wade, G. L. Walker, R. J. Walker, H. S. Wall, C. Y. Wang, J. A. Ward, L. E. Ward, Jr., A. M. Wedel, J. V. Wehausen, L. M. Weiner, Brother Bernard Alfred Welch, F. J. Weyl, P. M. Whitman, G. T. Whyburn, R. L. Wilder, W. L. Williams, J. L. Wilson, R. H. Wilson, Jr., G. N. Wollan, Alice K. Wright, J. W. T. Youngs.

Sessions of the Association were held on Tuesday morning and afternoon in Louderman Auditorium of Washington University. Vice-President Jewell H. Bushey presided at the morning session and President Saunders MacLane at the afternoon session. The Program Committee for the meeting consisted of Walter Leighton, Chairman, V. W. Adkisson, and M. K. Fort, Jr.

FIRST SESSION OF THE ASSOCIATION

"The Mathematical Work of Jacques Hadamard," by Professor Szolem Mandelbrojt, Rice Institute and College de France.

"Recent Applications of Convex Functions," by Professor J. W. Green, University of California at Los Angeles. (Presented by title due to illness of Professor Green.)

"The Scientific Research Program of the Office of Ordnance Research," by Dr. A. H. Diamond, Office of Ordnance Research, U. S. Army.

"Mathematics and the Educational Octopus," by Professor S. S. Cairns, University of Illinois.

SECOND SESSION OF THE ASSOCIATION

Symposium: The Teaching of Service Courses in Mathematics.

"Engineering Mathematics at Mid-Century," by Professor S. H. Caldwell, Electrical Engineering, Massachusetts Institute of Technology.

"What Shall We Teach in Service Courses?", by Professor F. E. Hohn, University of Illinois.

"Mathematics for Physicists, Pure or Applied?", by Professor J. W. Buchta, Physics, University of Minnesota.

"Abstract Mathematics for Scientists," by Professor W. L. Duren, Jr., Tulane University and National Science Foundation.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Monday morning in the Lounge of Brown Hall, with sixteen members present. Among the more important items of business transacted were the following:

Professor W. L. Duren, Jr., of Tulane University was elected Second Vice-President for 1953–1954.

Approval was given to the appointment by President MacLane of the following Nominating Committee for 1953: W. T. Martin, Chairman, J. C. Brixey, and G. K. Kalisch.

The Board voted to establish a standing committee on the Earle Raymond Hedrick Lectures and to discharge with thanks the Committee on Expository

Lectures (G. B. Price, Chairman, G. C. Evans, and J. C. Oxtoby) which had arranged for the first two series of Hedrick Lectures. It was also voted to approve the invitation to Professor P. R. Halmos of the University of Chicago to deliver the second series of Hedrick Lectures at the 1953 Summer Meeting on the subject of Axiomatic Set Theory.

The Thirty-fourth Summer Meeting of the Association will be held on August 31 and September 1, 1953, at Queen's University and the Royal Military College, Kingston, Ontario, Canada.

The Board voted to approve the appointment by the President of a committee to study the possible establishment of an employment bureau, of a joint committee (with the National Council of Teachers of Mathematics) on teacher education in mathematics, of a committee on the Undergraduate Mathematical Program, and of a joint committee (with the National Council of Teachers of Mathematics) to explore the possibility of publishing a mathematical journal for high school students. The President was also authorized to appoint two representatives on the United States sub-committee of the International Mathematical Instruction Committee.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Tuesday, December 30, 1952 at 2:00 P.M. in Louderman Auditorium of Washington University, Saint Louis, Missouri. President Saunders MacLane presided.

The Secretary announced the results of the balloting for officers, in which 1222 votes were cast. E. J. McShane of the University of Virginia was elected President for the two-year term 1953–1954. S. S. Cairns of the University of Illinois and A. W. Tucker of Princeton University were elected Governors for the three-year term 1953–1955.

MEETING OF SECTION OFFICERS

A meeting of Section Officers of the Association was held on Sunday evening in Room 107 of Brown Hall. President MacLane presided. Thirty-five persons were present representing twenty-one of the twenty-five sections of the Association. The following matters were discussed: problems pertaining to programs of section meetings, special activities of sections, such as contests for high school students, traveling lectureships, study of curriculum and teaching of secondary school mathematics, and collection of problem material from industry. Several suggestions for future activities were submitted. The method of electing sectional governors was described.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on Saturday, December 27, and continued through Monday, December 29. The Josiah Willard Gibbs Lecture was delivered by Professor Marston Morse of the Institute for Advanced Study on "Topology and Geometrical Analysis." The invited ad-

dress by Professor A. M. Gleason of Harvard University on "Natural Co-ordinate Systems" was awarded the Newcomb Cleveland Prize of one thousand dollars by a committee of the American Association for the Advancement of Science as a "noteworthy paper representing an outstanding contribution to science." Professor John von Neumann of the Institute for Advanced Study gave his Retiring Presidential Address on the topic: "A Logical Theory of Automata."

Section A of the American Association for the Advancement of Science held a single session on Monday at which Professor R. L. Wilder of the University of Michigan delivered his Retiring Vice-Presidential Address on "The Origin and Growth of Mathematical Concepts."

The Association for Symbolic Logic held its sessions on Monday morning and afternoon.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of R. R. Middlemiss, Chairman, T. L. Downs, H. Margaret Elliott, H. M. Gehman, Francis Regan, Marlow Sholander, and J. W. T. Youngs.

Registration headquarters was in the entry hall of the Women's Building of Washington University. Those attending the meetings were housed in hotels, including the Chase, Melbourne, Roosevelt and Sheraton Hotels. Meals were served in the cafeteria of Lee Hall.

Washington University entertained with coffee on Saturday evening in the Women's Building and tea was served on Sunday, Monday and Tuesday afternoons. A sightseeing tour of Saint Louis was held on Monday morning. An Opera Workshop Performance was presented on Sunday evening in the Auditorium of Brown Hall by the Music Department of Washington University.

A dinner for the members of the mathematical organizations and their guests was held on Monday at 6:30 P.M. in the Gymnasium of the Women's Building. Professor T. L. Downs of Washington University acted as toastmaster. Vice-Chancellor L. J. Buchan of Washington University brought greetings from the University and spoke about the scientific development of the University during the past century. Professor G. T. Whyburn, President-elect of the American Mathematical Society, spoke of the common interest of the organizations represented at the meeting in the advancement of mathematics. Professor L. M. Graves, Vice-President of Section A of the A.A.A.S., asked for the support of that organization. Professor C. B. Allendoerfer, representing the Mathematical Association of America as its Editor-in-chief, related the plans of the Association for carrying out its primary job of improving undergraduate instruction in mathematics. President J. B. Rosser of the Association for Symbolic Logic told of the successful meeting of that organization.

A motion prepared by Professor J. S. Frame was adopted by a rising vote of those present, expressing sincere thanks to the officers of Washington University for placing at our disposal the excellent facilities of the University and expressing

deep appreciation to the members of the local committee on arrangements whose cooperative efforts had made this meeting so pleasant and congenial.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held October 31, 1952 at Oklahoma City University, Oklahoma City, Oklahoma.

There were sixty-two in attendance, including the following thirty-seven members of the Association:

E. F. Allen, R. V. Andree, Arthur Bernhart, J. C. Bradford, J. C. Brixey, R. L. Caskey, N. A. Court, R. B. Deal, D. C. Dragoo, N. A. Eisen, I. E. Glover, A. A. Grau, E. V. Greer, L. A. Guest, O. H. Hamilton, J. O. Hassler, E. E. Heimann, J. E. Hoffman, W. N. Huff, P. W. M. John, L. W. Johnson, J. T. Krattiger, J. E. LaFon, Gene Levy, H. W. Linscheid, Dora McFarland, G. E. Meador, Mrs. Dorothea Meagher, R. R. Murphy, C. M. Pirrong, J. W. Sehestedt, H. W. Smith, C. E. Springer, Vivian Spurgeon, R. W. Veatch, G. R. Vick, J. H. Zant.

The following officers were elected for the coming year: Chairman, Professor W. N. Huff, University of Oklahoma; Vice-Chairman, Professor I. E. Glover, Langston University; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The following program was presented:

1. *Analytic functions with an irregular linear measurable set of singular points*, by Professor I. E. Glover, Langston University.

This paper will be published in the *Canadian Journal of Mathematics.*

2. *A mapping of the n-dimensional euclidean space on plane*, by Mr. A. Zirakzadeh, Oklahoma Agricultural and Mechanical College, introduced by the Chairman.

A set of n points $x_1, x_2, \cdots, x_n$ lying on n coplanar parallel lines $X_1, X_2, \cdots, X_n$ respectively, is called a point in the n-dimensional parallel space P^n. By a suitable choice of origins on these n parallel lines, it is possible to establish a one-to-one correspondence between points $P_e(a_1, \cdots, a_n)$ of an n-dimensional euclidean space E^n and points $(X_1, \cdots, X_n)$ of space P^n.

Two points $P_e(a_1, \cdots, a_n)$ and $P_e(b_1, \cdots, b_n)$ of the space E^n map into $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ of space P^n. The $n-1$ lines X_iX_{i+1} meet the $n-1$ lines Y_iY_{i+1} in $n-1$ points $x_{i-1,i}$. This set of $n-1$ points is the image of the line joining $P_e(a_1, \cdots, a_n)$ and $P_e(b_1, \cdots, b_n)$ in the space E^n. Any other point $P_e(c_1, \cdots, c_n)$ of this line maps into a point $(Z_1, \cdots, Z_n)$ such that the line Z_iZ_{i+1} passes through the point $X_{i-1,i}$.

This mapping is one-to-one and preserves incidence and parallelism. In case $n=2$ it reduces to a special form of duality in plane.

It is also possible to map planes and hyperplanes since they can be defined in terms of points and lines.

This mapping makes it possible to prove the theorems concerning incidence in plane or higher spaces with the use of only the axioms of the plane euclidean geometry. It also gives easier solutions to many of the problems concerning point and line.

An application of this mapping leads to a new graphical method for the solution of a system of 4 non-homogeneous linear equations in 4 unknowns.

3. *A generalization of the theorem of Meusnier*, by Professor C. E. Springer, University of Oklahoma.

Professor Springer developed a formula for the angle between the principal normal to a curve through a point P on a surface and the line through P of a rectilinear congruence. The formula reduces to that of the theorem of Meusnier in case the congruence is normal to the surface. A geometric interpretation for the general case was presented.

4. *Determination of the form of an empirical equation by numerical methods*, by Mr. N. A. Eisen, Tulsa, Oklahoma.

Divided differences and partial divided differences are defined and applied to polynomials, exponential functions, and power functions.

5. *The Riemann metric in plane affine geometry*, by Mr. R. B. Deal, University of Oklahoma.

The purpose is to point out the pedagogical advantages in using a Riemannian metric and linear affine transformations in an advanced plane analytics course where the student has a simple picture to guide his intuition as he is introduced to certain fundamental tensors and scalars.

6. *Some topological aspects of the four color problem*, by Professor O. H. Hamilton, Oklahoma Agricultural and Mechanical College.

A necessary and sufficient condition for the truth of the four color conjecture is given in terms of chains of sub-maps. The original and the dual maps are compared and some sufficient conditions not necessary and necessary conditions not sufficient for the truth of the four color proposition are given. In the class of minimal maps, a sufficient condition is that all vertices lie on a single simple closed curve. A necessary condition is that all vertices lie on the sum of a finite number of disjoint simple closed curves.

From a consideration of the duals of minimal maps, it follows that each triangulation of the sphere can be converted into a net of quadrilaterals on the sphere by deleting one and only one side from each triangle, if the four-color proposition is true.

7. *Three-dimensional plane geometry*, by Professor Arthur Bernhart, University of Oklahoma.

An exposition of the geometry of the triangle is achieved by constructing a three-dimensional figure and then projecting its points and lines onto the plane.

8. *On binary relations*, by Mr. C. J. Clark, Oklahoma Agricultural and Mechanical College, introduced by the Chairman.

A definition is given for what is called a C-R algebra. This abstract algebraic system, which is an extension of the notion of closure algebra, is a Boolean algebra with operators. In this system the closure operation is extended from elements of a Boolean algebra to those of a relation algebra. This necessitates additional axioms on operations and elements peculiar to relation algebras. Definitions are given for domain element, range element, image of certain types of elements under other elements, and theorems are established concerning these notions. The concept of continuity of a function is generalized to that of a relation, namely one that is closed, and theorems are established similar to those for continuous functions.

9. *Cryptographic systems*, by Professor R. V. Andree, University of Oklahoma.

This paper will appear in *Scripta Mathematica*, vol. XVIII, No. 1.

10. *On a "United States Copyright Angle Trisection,"* by Professor N. A. Court, University of Oklahoma.

The perennial problem of angle trisectors was discussed with special reference to a recent and persistent individual trisector whose copyright trisection many of the hearers have seen. It was suggested that if you yourself are not willing to solve the "find my error" puzzle, that you should *not* take the liberty of referring the trisector to someone else. Most teachers are capable of finding errors in such arguments if they are willing to take the required time. If they are not willing, they should not impose an identical burden on others. In the case of the particular trisector for instance, it is readily shown that his "trisection" would imply that, in a triangle in which the ratio of two sides is 2:1, the same ratio holds for the respectively opposite angles. This is not valid for a triangle whose angles are 90°, 60°, 30°.

The program was followed by a joint luncheon with the Mathematics teachers of the Oklahoma Education Association at which Professor F. E. Grossnickle of New Jersey State Teachers College spoke. Members of the Association were invited to attend a panel discussion for high-school mathematics teachers after the dinner.

R. V. ANDREE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May 2, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS, Washburn Municipal University of Topeka, April 11, 1953.

KENTUCKY, University of Louisville, May 9, 1953.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Proving Ground, Dahlgren, Virginia, May 2, 1953.

METROPOLITAN NEW YORK, Teachers College, Columbia University, March 28, 1953.

MICHIGAN, Wayne University, Detroit, April 18, 1953

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, April 24, 1953.

NEBRASKA

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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 4

CONTENTS

APRIL 1953

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

SPECTRAL THEORY AND ITS APPLICATION TO DIFFERENTIAL EIGENVALUE PROBLEMS

W. H. McEWEN, University of Manitoba

1. Introduction. Eigenvalue problems such as those associated with the differential equation $y''+(\lambda-q(x))y=0$, in which $q(x)$ is real-valued and continuous on $a<x<b$, may be treated as concrete applications of the abstract theory of self-adjoint operators on Hilbert space. The discussion which follows will be in a sense an abridged account of this theory with an indication of how it may be applied to the differential problems mentioned. The treatment will be descriptive and suggestive rather than rigorous. In this form it is hoped the account may be of interest particularly to those who are approaching the study of this subject for the first time.

2. Abstract Hilbert space. An abstract Hilbert space $\mathfrak{H}$ is a set of undefined elements f, g, *etc.*, which conform to certain axioms. These axioms are chosen in such a way as to give to $\mathfrak{H}$ a formal structure similar to that of the vectors of ordinary 3-dimensional space. The axioms and their relationship to ordinary space will be discussed in this section.

The set $\mathfrak{E}_3$ of points of 3-space may be identified with a system $\mathfrak{V}_3$ of vectors, for by choosing at random a point 0 to serve as origin (and null vector), each point f may be identified with a vector extending from 0 to f. Adopting the usual rules of vector algebra for vector addition $(+)$ and multiplication by a scalar $(\cdot)$, $f+g$ and $a\cdot f$ are seen to be elements of $\mathfrak{V}_3$ whenever f and g are elements of $\mathfrak{V}_3$ and a is a real number. With respect to these operations $\mathfrak{V}_3$ is a *linear manifold* in the sense that if f and g are in $\mathfrak{V}_3$, then $af+bg\in\mathfrak{V}_3$, where a and b are any real numbers. With this fact in mind, and in view of our desire to use, in general, complex-valued scalars rather than real-valued ones, we adopt as the first axiom of Hilbert space the following:

AXIOM 1. *Operations* $(+)$ *and* $(\cdot)$ *exist with respect to which* $\mathfrak{H}$ *is a linear manifold, the scalar field being in general the set of complex numbers.* (*If only real-valued scalars are used the Hilbert space is said to be real.*)

$\mathfrak{V}_3$ possesses in addition a so-called *inner product* operation. If f and g are any two points (vectors), $(f, g)=\|f\|\cdot\|g\|\cos\theta$ defines their inner product. Here θ is the angle between the vectors, and $\|f\|$ and $\|g\|$ denote their positive lengths or *norms*. When $\theta=90°$, $(f, g)=0$ and the vectors are *orthogonal* (*i.e.* perpendicular). When $g=f$, $\|f\|=\sqrt{(f,f)}$ which indicates that the norm can be defined in terms of the inner product. If a rectangular coordinate system is used in $\mathfrak{V}_3$ the inner product may be expressed by $(f, g)=\sum_1^3 a_kb_k$, where (a_1, a_2, a_3) and (b_1, b_2, b_3) are the coordinates of f and g respectively. Clearly, $(f,f)=\|f\|^2=\sum_1^3 a_k^2$. When complex-valued scalars rather than real-valued ones are used a_k^2 must be replaced by $|a_k|^2$ or $a_k\bar{a}_k$, and in keeping with this change we must now write $(f, g)=\sum_1^3 a_k\bar{b}_k$. Thus in the complex case the inner product (f, g)

exhibits the following properties:

(f, g) is complex valued, and is uniquely determined for each pair of elements f, g;

$(f, g) = \overline{(g, f)}$;

$(af, g) = a(f, g)$ whereas $(f, ag) = \bar{a}(f, g)$;

$(f_1+f_2, g) = (f_1, g) + (f_2, g)$ and similarly $(f, g_1+g_2) = (f, g_1) + (f, g_2)$;

$(f, f) \geqq 0$, the equality sign holding only if $f=0$.

With these results in mind we now enunciate:

AXIOM 2. *$\mathfrak{H}$ is an inner product space, the inner product exhibiting the properties listed above. A norm is defined in $\mathfrak{H}$ by the formula $\|f\| = \sqrt{(f, f)}$, $f \in \mathfrak{H}$.*

Once the norm has been defined it is possible to introduce notions of convergence and limit (that is, a topology) in $\mathfrak{H}$. Since the norm $\|f-g\|$ is the analog of the 3-dimensional notion of "distance" between two elements f and g, a sequence $\{f_k\}$ of elements of $\mathfrak{H}$ is said to be convergent (in the Cauchy sense) if and only if $\|f_m - f_n\| \to 0$ as $m, n \to \infty$; f is a limit of such a sequence if $\|f-f_n\| \to 0$ as $n \to \infty$. In $\mathfrak{B}_3$ it is known that all Cauchy convergent sequences have limits in the space. This fact is expressed by saying that $\mathfrak{B}_3$ is complete, and it implies that $\mathfrak{B}_3$ contains all its limit points and is therefore a *closed* linear manifold. This property we now postulate for Hilbert space.

AXIOM 3. *$\mathfrak{H}$ is complete; it is therefore a closed linear manifold.*

An *orthonormal system* in $\mathfrak{B}_3$ is a set of mutually perpendicular vectors of unit length. Such a set can have at most 3 members, and if it has 3 members, say ϕ_1, ϕ_2, ϕ_3, it *spans* the entire space in the sense that every element $f \in \mathfrak{B}_3$ may be expressed in the form $f = \sum_1^3 a_k\phi_k$. The orthonormal property can be easily expressed in the inner product notation, $(\phi_k, \phi_m) = \delta_{km}$, where δ_{km} is 0 when $k \neq m$ and has the value 1 when $k=m$.

AXIOM 4. *$\mathfrak{H}$ is such that it can be spanned by an orthonormal set (i.e. one having the orthonormal property described above) of its own elements; if $\{\phi_k\}$ is such a set, every $f \in \mathfrak{H}$ may be expanded in the form $f = \sum a_k \phi_k$. If the set $\{\phi_k\}$ is infinite, $\mathfrak{H}$ is said to be infinite dimensional, and the series $\sum a_k \phi_k$ converges to f in the manner $\|f - \sum_1^n a_k\phi_k\| \to 0$ as $n \to \infty$. If the set $\{\phi_k\}$ is finite with n members, $\mathfrak{H}$ is finite dimensional (more precisely n dimensional).*

Axioms 1 to 4 define an abstract Hilbert space. Any particular system of elements with associated operations which satisfies these axioms may be regarded as a realization of Hilbert space. Thus $\mathfrak{B}_3$ is an example of a real Hilbert space of 3 dimensions.

The most notable example of an infinite dimensional Hilbert space is the *function class* $L_2[a, b]$. This class consists of all complex-valued functions $f(x)$ on the real interval $[a, b]$ for which $\int_a^b |f(x)|^2 dx < \infty$, the integration being taken in the sense of Lebesgue. In treating this as a Hilbert space the operations $(+)$

and $(\cdot)$ are given their ordinary point-wise meaning, $(f+g)(x)=f(x)+g(x)$, $(a\cdot f)(x)=a\cdot f(x)$, and the function which vanishes identically on $[a, b]$ is taken as the null element. f is considered identical with g in case $\int_a^b|f(x)-g(x)|dx=0$. The inner product is defined by $(f, g)=\int_a^b f(x)\overline{g(x)}dx$, and the norm of an element $f(x)$ is therefore given by $\|f\|^2=\int_a^b|f(x)|^2dx$. With these definitions $L_2[a, b]$ may be shown to satisfy axioms 1 to 4. It should be noted that convergence in the Hilbert space $L_2[a, b]$ is in the sense of the mean, for $\|f-f_n\|^2 =\int_a^b|f(x)-f_n(x)|^2dx$.

Two important inequalities may be shown to hold in abstract Hilbert space. These are Schwarz's inequality $|(f, g)| \leqq \|f\|\cdot\|g\|$, which in $\mathfrak{V}_3$ results from the fact that $|\cos\theta| \leqq 1$, and the triangle inequality $\|f+g\| \leqq \|f\|+\|g\|$, which in $\mathfrak{V}_3$ expresses an obvious geometric relation in any triangle.

3. Linear manifolds and subspaces. Any subset of a Hilbert space $\mathfrak{H}$ which is closed with respect to the operations $(+)$ and $(\cdot)$ is a linear manifold. If it is closed also with respect to the limiting process it is complete, and is therefore a closed linear manifold. Such closed manifolds are themselves Hilbert spaces, and so are called subspaces of $\mathfrak{H}$.

Any finite dimensional linear manifold in $\mathfrak{H}$ is, like $\mathfrak{V}_3$ itself, a closed manifold (a subspace). An infinite dimensional manifold, however, may or may not be closed. If not, its closure can be taken and thus a subspace obtained.

It should be noted that all linear manifolds in $\mathfrak{H}$ contain the null element 0. As an intuitive aid in studying linear manifolds it is helpful to recall that a straight line through the origin or a plane through the origin are examples of these sets in $\mathfrak{V}_3$. (So also are $\mathfrak{V}_3$ itself and the set consisting of the origin alone.)

4. Self-adjoint operators. In a Hilbert space $\mathfrak{H}$ a linear operator T defines a mapping of a linear manifold $\mathfrak{D}\subset\mathfrak{H}$ into an "image" set $\mathfrak{R}\subset\mathfrak{H}$. Moreover the mapping $\mathfrak{D}\rightarrow\mathfrak{R}$ is single-valued in that each f in $\mathfrak{D}$ has a unique image g in $\mathfrak{R}$, where g is denoted by Tf. Furthermore $T(a_1f_1+a_2f_2)=a_1Tf_1+a_2Tf_2$ for all elements f_1, f_2 in $\mathfrak{D}$ and for all complex numbers a_1, a_2. $\mathfrak{D}$ is called the *domain* of definition of T, and $\mathfrak{R}$, being the set $T(\mathfrak{D})$, is called the *range* (*i.e.* reach) of T. It is easily seen that $\mathfrak{R}$ also is a linear manifold. It should be noted that $T0=0$.

It may happen that two elements f_1, and f_2 have the same image g. If so, the equation $Ty=0$ has, in addition to the trivial solution $y=0$, a non-zero solution $y=f_1-f_2$, for clearly $T(f_1-f_2)=Tf_1-Tf_2=g-g=0$. This being so, every equation $Ty=g$ has at least two distinct solutions, for if $y=f$ is a solution so is $y=f+f_1-f_2$. The mapping $\mathfrak{D}\rightarrow\mathfrak{R}$ is then many-one throughout. In all cases then the mapping $\mathfrak{D}\rightarrow\mathfrak{R}$ is either one-to-one or many-one throughout. If it is one-to-one the reverse mapping $\mathfrak{R}\rightarrow\mathfrak{D}$ defines the *inverse operator* T^{-1} which is also linear. The non-existence of T^{-1} implies that the equation $Ty=0$ has a non-trivial solution; its existence implies that $Ty=0$ has no solution other than $y=0$ and that $Ty=g$ has a unique solution $y=T^{-1}g$.

A linear operator T is said to be *closed* if $T\lim f_n=\lim Tf_n$ whenever $\{f_n\}$

and $\{Tf_n\}$ are both convergent sequences in $\mathfrak{D}$. A linear operator T is said to be *bounded* if a constant $M>0$ exists such that $\|Tf\| \leqq M\|f\|$ for all elements f in $\mathfrak{D}$.

It may be shown that for every linear operator T whose domain is dense in $\mathfrak{H}$ (*i.e.* the closure of $\mathfrak{D}$ is $\mathfrak{H}$) there exists a uniquely determined linear operator T^*, called the *adjoint* of T, such that $(Tf_1, f_2) = (f_1, T^*f_2)$ for every f_1 in the domain of T and every f_2 in the domain of T^*. Moreover T^* is closed. When $T=T^*$, T is said to be *self-adjoint.* A self-adjoint operator T is thus a linear operator such that (i) $\mathfrak{D}$ is dense in $\mathfrak{H}$, (ii) T is closed, (iii) $(Tf_1, f_2) = (f_1, Tf_2)$ for all elements f_1, f_2 in $\mathfrak{H}$. The third condition imparts to T a symmetric character.

5. The spectrum of a self-adjoint operator. From this point on the operator T will be assumed to be self-adjoint. With a given T we can associate a family of operators T_λ, where $T_\lambda f = Tf - \lambda f$ for each complex λ. T_λ is linear (but not necessarily self-adjoint) and its domain of definition is obviously the same as that of T, *i.e.* $\mathfrak{D}$. Its range $\mathfrak{R}_\lambda$ will depend on the choice of λ.

The points λ of the complex λ-plane may now be classified according to the behavior of T_λ as follows:

I. A point λ at which T^{-1} fails to exist is said to belong to the *point spectrum* of the operator T.

II. A point λ at which T^{-1} exists and is an unbounded operator is said to belong to the *continuous spectrum* of T.

III. The points of the λ-plane that are not in the sets I or II are said to belong to the *resolvent set* of T. If λ is in this set, T_λ^{-1} exists and is a bounded operator. The points of the sets I and II together constitute the *spectrum* of T.

The points of the point spectrum are called eigenvalues (characteristic values). If λ_1 is an eigenvalue $T_{\lambda_1}^{-1}$ does not exist and hence the equation $T_{\lambda_1}y=0$ (*i.e.* $Ty-\lambda_1 y=0$) has a non-zero solution f_1. Such a solution is called an eigensolution (characteristic solution). The symmetric relation $(Tf_1, f_1) = (f_1, Tf_1)$ may then be written $\lambda_1(f_1, f_1) = \bar{\lambda}_1(f_1, f_1)$. But $(f_1, f_1) = \|f_1\|^2 \neq 0$ and hence $\lambda_1 = \bar{\lambda}_1$, so that λ_1 is real. This proves that the points of the point spectrum lie on the real axis of the complex λ-plane.

If λ is not on the real axis it cannot be an eigenvalue and hence T_λ^{-1} exists. The equation $T_\lambda y = g$, $g \in \mathfrak{R}$ must therefore have a unique solution $y = T_\lambda^{-1}g$. We shall show that T_λ^{-1} is a bounded operator, and thus infer that the non-real points of the λ-plane belong to the resolvent set. Since $(T_\lambda y, y) = (Ty - \lambda y, y) = (Ty, y) - \lambda(y, y)$ and $(y, T_\lambda y) = (y, Ty) - \bar{\lambda}(y, y)$, and since (Ty, y) and (y, Ty) are identical we find that $(T_\lambda y, y) - (y, T_\lambda y) = (\bar{\lambda} - \lambda)(y, y) = (\bar{\lambda} - \lambda)\|y\|^2$. On applying the triangle inequality and Schwarz's inequality to this we obtain $|\bar{\lambda} - \lambda| \, \|y\|^2 \leqq 2\|T_\lambda y\| \cdot \|y\|$, from which it follows that $\|y\| \leqq M\|Ty\|$ where $M = 2/|\bar{\lambda} - \lambda|$. On substituting $y = T_\lambda^{-1}g$ and $T_\lambda y = g$ in this inequality we obtain finally $\|T^{-1}g\| \leqq M\|g\|$ which proves that T_λ^{-1} is bounded. Since the non-real values of λ are in the resolvent set we conclude that the continuous spectrum as well as the point spectrum must lie on the real axis of the λ-plane.

We shall show next that the eigenvalues are at most denumerable. Corresponding to a given eigenvalue λ_k there will exist a linear manifold of eigensolutions satisfying $T_{\lambda_k}y=0$, for clearly any finite linear combination of solutions of this homogeneous equation is also a solution. ($y=0$ is included in this manifold although it is not strictly an eigensolution). The closed manifold of these solutions will be denoted by $\mathfrak{M}_k$. Now consider two distinct eigenvalues λ_k and λ_m with manifolds $\mathfrak{M}_k$ and $\mathfrak{M}_m$ respectively. If $f_k \in M_k$ and $f_m \in M_m$ then $Tf_k=\lambda_k f_k$ and $Tf_m=\lambda_m f_m$ and hence the symmetric relation $(Tf_k, f_m)=(f_k, Tf_m)$ may be written $\lambda_k(f_k, f_m)=\bar{\lambda}_m(f_k, f_m)$. But $\bar{\lambda}_m=\lambda_m \neq \lambda_k$ and hence $(f_k, f_m)=0$. This proves that the eigensolution manifolds form a mutually orthogonal system. The manifolds can therefore be at most denumerable in number (in view of the fact that $\mathfrak{H}$ has a countable number of dimensions), and hence the eigenvalues themselves are at most denumerable. These points therefore may be arranged as a sequence $\{\lambda_k\}$. As one might expect from its name the continuous spectrum is usually an interval or set of intervals although in some cases it may degenerate into a single point.

6. Spectral representation of T, and related formulae. Let $\mathfrak{M}$ denote the closed linear manifold spanned by the entire set of eigensolution manifolds $\mathfrak{M}_k$. Every element f in $\mathfrak{M}$ can then be uniquely decomposed into components in the subspaces $\mathfrak{M}_k$ (by orthogonal projection). Thus if $f \in \mathfrak{M}$ we can write $f=\sum f_k$, $f_k \in \mathfrak{M}_k$. Here, and later where similar expansions occur, it is to be understood that the summation is taken over the entire set of subspaces $\mathfrak{M}_k$.

Let $\mathfrak{N}$ be the orthogonal complement of $\mathfrak{M}$. By this we mean that $\mathfrak{N}$ is the closed linear manifold of elements in $\mathfrak{H}$ which are orthogonal to $\mathfrak{M}$. (In $\mathfrak{V}_3$ we could think of the x and y-axes as being the manifolds $\mathfrak{M}_k$, the xy-plane as being $\mathfrak{M}$, and the z-axis as being $\mathfrak{N}$.) If ψ is in $\mathfrak{H}$ it can be uniquely decomposed in the form $\psi=\psi_1+\psi_2$ where $\psi_1 \in \mathfrak{M}$ and $\psi_2 \in \mathfrak{N}$.

Let T_1 denote the operator T when applied solely in $\mathfrak{M}$, and let T_2 denote it when applied solely in $\mathfrak{N}$. Then $T\psi=T_1\psi_1+T_2\psi_2$ if $\psi \in \mathfrak{D}$. The domains $\mathfrak{D}_1$, $\mathfrak{D}_2$ of T_1, T_2 are then the sets of all elements ψ_1, ψ_2 (respectively) obtained when $\psi=\psi_1+\psi_2$ runs through the set $\mathfrak{D}$.

Let us consider first the operator T_1. If $y \in \mathfrak{D}_1$, $y=\sum y_k$ and $T_1 y=T_1 \sum y_k = \sum T_1 y_k = \sum \lambda_k y_k$. This shows, incidentally, that the range of T_1 is also in $\mathfrak{M}$. The interchange of $\sum$ and T_1 implies that $\sum T_1 y_k = \sum \lambda_k y_k$ is convergent, and this in turn implies that $\sum |\lambda_k|^2 \|y_k\|^2 < \infty$. This latter condition is in fact the defining condition for the domain $\mathfrak{D}_1$. The formula proved above,

$$T_1 y = \sum \lambda_k y_k, \tag{1}$$

gives the spectral representation of T_1.

Another useful formula may be deduced from (1). Suppose λ is not in the point spectrum of T_1 (*i.e.* $\lambda \neq \lambda_k$ for all k). Then $T_{1\lambda}^{-1}$ exists ($T_{1\lambda}y$ denotes $T_1 y - \lambda y$), and the equation $T_{1\lambda}y=p$, where p is a given element in the range of $T_{1\lambda}$, has a unique solution $y=T_{1\lambda}^{-1}p$. But $y=\sum y_k$, and a similar expansion $p=\sum p_k$ holds

for p. Hence $T_{1\lambda}y=p$ (*i.e.* $T_1y-\lambda y=p$) may be written $\sum\lambda_k y_k-\lambda\sum y_k=\sum p_k$. Since the decomposition into $\mathfrak{M}_k$ components is unique these series can be equated term by term to give $(\lambda_k-\lambda)y_k=p_k$. On solving this for y_k and summing the result we obtain

$$y = \overset{-1}{T_{1\lambda}} p = -\sum \frac{p_k}{\lambda-\lambda_k}. \tag{2}$$

Formula (2) indicates that $y=y(\lambda)$ regarded as a function of the complex variable λ is analytic throughout the λ-plane except for poles at the eigenvalues $\lambda=\lambda_k$. Moreover, $-p_k$ is the residue of the pole at $\lambda=\lambda_k$ and hence the expansion $p=\sum p_k$ may be expressed as a summation involving the negatives of the residues of the poles λ_k. In the applications it frequently happens that $y(\lambda)$ can be computed by direct means; the eigenvalues of T_1 and the related eigensolution expansion of p can then be obtained from this result. From (2) it may be deduced also that $\overset{-1}{T_{1\lambda}}$ is bounded when $\lambda\neq\lambda_k$, and hence it may be inferred that the continuous spectrum of T_1 is empty. Thus we conclude that T_1 is a self-adjoint operator in the subspace $\mathfrak{M}$ and its spectrum is a pure point spectrum.

When it comes to the consideration of the operator T_2 in the Hilbert space $\mathfrak{N}$ we find the conditions quite changed. There are no eigenvalues now since these were all accounted for in the subspace $\mathfrak{M}$, and the arguments used above therefore have little meaning. Formulae (1) and (2) if re-stated in Stieltjes integral form and re-interpreted may, however, be used to describe T_2. That this is so is the assertion of one of the fundamental theorems in the theory of self-adjoint operators. For an outline of a proof of this theorem the reader is referred to [4]. We shall be content here merely to describe the result.

Let $E_1(t)$ be a family of linear operators defined on $-\infty<t<\infty$ with domain $\mathfrak{M}$ by the relation $E_1(t)y=\sum_{\lambda_j\leq t} y_j$, $y\in\mathfrak{M}$. Here the notation means that $E_1(t)y$ is the partial sum of the components y_j corresponding to the eigenvalues λ_j which lie to the left or are coincident with t. $E_1(t)y$ has the nature of a step-function, with steps y_k occurring at the points $t=\lambda_k$. From the theory of Stieltjes integrals it follows that $\int_{-\infty}^{\infty}dE_1(t)y=\sum y_k=y$, and for this reason $E_1(t)$ is called a ***resolution of the identity*** in the subspace $\mathfrak{M}$. Formulae (1) and (2) may now be written

$$T_1 y = \int_{-\infty}^{\infty} t\,dE_1(t)y, \qquad y\in\mathfrak{D}_1, \tag{1$'$}$$

$$y = \overset{-1}{T_{1\lambda}} p = -\int_{-\infty}^{\infty}\frac{dE_1(t)p}{\lambda-t}, \qquad p\in\mathfrak{R}_{1\lambda}. \tag{2$'$}$$

In dealing with T_2 of the subspace $\mathfrak{N}$ we shall need a new family of operators $E_2(t)$ with domain $\mathfrak{N}$. The operators $E_2(t)$ must be such that $\int_{-\infty}^{\infty}dE_2(t)y=y$

for all elements y in $\mathfrak{N}$. Also, the real numbers t which are not in the spectrum of T_2 must be points of constancy of $E_2(t)$ (a point of constancy being defined as one in a neighborhood of which $E_2(t)$ is constant), so that these points contribute nothing to the integral representation of y. On the other hand, every neighborhood Δ of a point of the (continuous) spectrum of T_2 must be one in which $E_2(t)$ varies. The vector $\int_\Delta dE_2(t)y$ may be interpreted as the projection of y on a closed linear manifold $\mathfrak{N}_\Delta$ associated with Δ and it may be shown that any two such manifolds corresponding to distinct intervals Δ_1, Δ_2 are orthogonal. There is thus a strong analogy between this case and that of the operator T_1, in which, for a neighborhood Δ of a single eigenvalue λ_k, $\int_\Delta dE_1(t)y = y_k \in \mathfrak{M}_k$, and manifolds $\mathfrak{M}_k$, $\mathfrak{M}_m$ corresponding to distinct eigenvalues λ_k, λ_m are orthogonal. Formulae (1)′ and (2)′ may now be re-stated for the subspace $\mathfrak{N}$ by changing the subscripts from 1 to 2 and replacing $p \in \mathfrak{R}_{1\lambda}$ by $q \in \mathfrak{R}_{2\lambda}$.

The results concerning T_1 and T_2 may now be combined to apply to the operator T in $\mathfrak{H}$. We define $E(t)$ with domain $\mathfrak{H}$ by the statement $E(t)\psi = E_1(t)\psi_1 + E_2(t)\psi_2$, $\psi \in \mathfrak{H}$, where ψ_1 and ψ_2 are the components of ψ in the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ respectively. We then have $\int_{-\infty}^{\infty} dE(t)\psi = \int_{-\infty}^{\infty} dE_1(t)\psi_1 + \int_{-\infty}^{\infty} dE_2(t)\psi_2 = \psi_1 + \psi_2 = \psi$, and thus we see that $E(t)$ is a resolution of the identity in $\mathfrak{H}$. We can now write*

$$(3) \qquad Ty = \int_{-\infty}^{\infty} t\,dE(t)y, \qquad y \in \mathfrak{D}.$$

$$(4) \qquad y = y(\lambda) = T_\lambda^{-1} g = -\int_{-\infty}^{\infty} \frac{dE(t)g}{\lambda - t}, \qquad g \in \mathfrak{R}_\lambda.$$

The domain $\mathfrak{D}$ consists of all elements y for which $\int_{-\infty}^{\infty} t^2 d\|E(t)y\|^2 < \infty$. (This is in keeping with the observation made earlier in the case of T_1 that $\sum T_1 y_k = \sum \lambda_k y_k$ must converge and therefore $\sum |\lambda_k|^2 \|y_k\|^2 < \infty$.) The range $\mathfrak{R}_\lambda$ is a proper subset of $\mathfrak{H}$ when λ is in the continuous spectrum but is identical with $\mathfrak{H}$ when λ is in the resolvent set. In connection with (4) it should be noted that the singularities of $y(\lambda)$ determine the spectrum of T, the poles being the eigenvalues and other singularities determining the continuous spectrum.

We have considered the eigensolution expansion $p = \sum p_k$ in the subspace $\mathfrak{M}$ and have discovered that it may be expressed in terms of the residues of the poles of $y(\lambda)$ the unique solution of $T_{1\lambda}y = p$. We now ask, what is the nature of the "expansion" of an element q in the subspace $\mathfrak{N}$? The answer is—it is a continuous summation (*i.e.* an integral) rather than a series. To obtain a formula for this expansion let us look again at the analogous problem in the subspace $\mathfrak{M}$.

* The integral operators such as T, T_λ^{-1} are not Stieltjes integrals in the ordinary sense of course, but may be defined in terms of such integrals. Thus T may be defined as the (unique) linear operator for which $(Tf, g) = \int_{-\infty}^{\infty} t\, d(E(t)f, g)$ for all pairs of elements $f \in \mathfrak{D}$, $g \in \mathfrak{H}$. A similar definition may be given for T_λ^{-1}.

If $y(\lambda)$ of formula (2) is integrated around a contour Γ_α in the λ-plane enclosing the poles λ_k lying between $-\alpha$ and α and excluding all the others, the result is $\int_{\Gamma_\alpha} y(\lambda)d\lambda = 2\pi i \sum_{-\alpha<\lambda_j<\alpha}(-p_j)$. From this we deduce the formula

$$p = \sum p_k = \lim_{\alpha\to\infty} -\frac{1}{2\pi i}\int_{\Gamma_\alpha} y(\lambda)d\lambda.$$

Γ_α may be taken to be a rectangle with vertical sides through the points $-\alpha$ and α and horizontal sides through the points $-\beta i$ and βi, $\beta>0$. The vertical pieces of the contour may be neglected if we let $\beta\to 0+$. On the upper line we will have $\lambda=\mu+\beta i$ and $d\lambda=d\mu$ where the real variable μ changes from α to $-\alpha$; on the lower line we will have $\lambda=\mu-i\beta$, $d\lambda=d\mu$, where μ varies from $-\alpha$ to α. It follows then that

$$p = \lim_{\alpha\to\infty}\lim_{\beta\to 0}\frac{1}{2\pi i}\int_{\Gamma_\alpha}[y(\mu+i\beta)-y(\mu-i\beta)]d\mu. \tag{5}$$

This formula carries over without essential change to the case of T_2 in the subspace $\mathfrak{N}$, and therefore to the case of T in the space $\mathfrak{H}$. In the first instance, for the expansion of an element q, $y(\lambda)$ must be identified with the unique solution of $T_{2\lambda}y=q$. In the general case, for the expansion of an element g, $y(\lambda)$ must be identified with the unique solution of $T_\lambda y=g$. The expansion of g in $\mathfrak{H}$ will in fact be the sum of the expansions of its components, p in $\mathfrak{M}$ and q in $\mathfrak{N}$.

7. Application to differential problems. Consider the differential equation $y''+(\lambda-q(x))y=0$, in which $q(x)$ is real-valued and continuous in $a<x<b$, its behavior at a and b being arbitrary. Let the operator T be defined by the relation $Ty=-y''+q(x)y$, its domain of definition $\mathfrak{D}$ being chosen so that y and Ty both are in the Hilbert space $L_2[a,b]$ when $y\in\mathfrak{D}$. Let $\mathfrak{D}$ be restricted further if necessary so as to insure that T is self-adjoint.

What this restriction involves may be seen by imposing the symmetric condition $(Ty_1, y_2)=(y_1, Ty_2)$, in which $y_1(x)$, $y_2(x)$ are any two functions in $L_2[a,b]$. We have

$$\begin{aligned}(Ty_1, y_2)-(y_1, Ty_2) &= \int_a^b(-y_1''+qy_1)\bar{y}_2dx-\int_a^b y_1\overline{(-y_2''+qy_2)}dx\\ &= \int_a^b(y_1\bar{y}_2''-y_1''\bar{y}_2)dx = W_b(y_1\bar{y}_2)-W_a(y_1\bar{y}_2),\end{aligned}$$

where $W_x(y_1\bar{y}_2)=y_1\bar{y}_2'-y_1'\bar{y}_2$ is the Wronksian of y_1 and $\bar{y}_2$ evaluated at the point x. To insure the vanishing of this result we impose on the functions $y(x)\in\mathfrak{D}$ a pair of linear homogeneous boundary conditions $\gamma y(a)+\gamma' y'(a)=0$, $\delta y(b)+\delta' y'(b)=0$, in which γ, γ', δ, δ' are preassigned real constants. The first condition insures that $W_a(y_1\bar{y}_2)=0$, and the second that $W_b(y_1\bar{y}_2)=0$. With $\mathfrak{D}$

so restricted it can be verified that $\mathfrak{D}$ is dense in $\mathfrak{H}$ and that T is a closed operator. It follows then that T is self-adjoint.

It should be noted that when $a=-\infty$, or when a is finite and $q(a+0)$ is infinite, the L_2 requirements in $\mathfrak{D}$ are enough to make $W_a(y_1\bar{y}_2)=0$. No boundary condition is needed at the point a in this case. A similar observation can be made about the point b. Thus, for example, in the case when $q(x)$ is continuous in $-\infty<x<\infty$ and $L_2[-\infty, \infty]$ is the Hilbert space no boundary conditions are needed to make T self-adjoint.

8. Three illustrative examples.

Example 1. Suppose $q(x)\equiv 0$ on $0\leqq x\leqq 1$, and let $T\equiv -d^2/dx^2$, and let $\mathfrak{D}$ be restricted by the boundary conditions $y(0)=y(1)=0$. T is then a self-adjoint operator in the Hilbert space $L_2[0, 1]$, and the equation $T_\lambda y=g=-f(x)$ is equivalent to the differential system $y''+\lambda y=f(x)$, $y(0)=y(1)=0$. An easy computation shows that the unique solution of this system is

$$y(\lambda) = y(x, \lambda) = \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}\sin\sqrt{\lambda}}\int_0^x f(t)\sin\sqrt{\lambda}\,t\,dt + \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}\sin\sqrt{\lambda}}\int_x^1 f(t)\cos\sqrt{\lambda}(1-t)dt.$$

The only singularities of $y(\lambda)$ are the poles $\lambda_k=k^2\pi^2$, $k=1, 2, 3, \cdots$. The continuous spectrum is therefore empty and the poles are the eigenvalues. The residue of the pole $\lambda_k=k^2\pi^2$ is easily computed, and is found to be $-2\sin k\pi x\int_0^1 f(t)\sin k\pi t dt$. The eigensolution expansion of $f(x)$ is therefore

$$f(x) = 2\sum_1^\infty \sin k\pi x\int_0^1 f(t)\sin k\pi t dt.$$

This will be recognized as the Fourier sine expansion of $f(x)$ on the half-period interval $(0, 1)$. The convergence of the series is of course in the sense of the mean.

Example 2. Suppose $q(x)\equiv 0$ on $-\infty<x<\infty$, and let $T\equiv -d^2/dx^2$. No boundary conditions are needed; the L_2 requirements are enough to insure that T is a self-adjoint operator in the Hilbert space $L_2[-\infty, \infty]$.

The equation $T_\lambda y=g=-f(x)$ is equivalent to the differential system $y''+\lambda y=f(x)$ $y\in\mathfrak{D}$, and the unique solution of this is given by

$$y(\lambda) = y(x, \lambda) = -\frac{1}{2i\sqrt{\lambda}}\left[\int_{-\infty}^x f(t)e^{i\sqrt{\lambda}(x-t)}dt + \int_x^\infty f(t)e^{-i\sqrt{\lambda}(x-t)}dt\right].$$

This function has no poles and so the point spectrum is empty. The expansion of $g=-f(x)$ may be obtained by an application of (5) in which $y(\lambda)$ is as given above.

$$f(x) = -\lim_{\alpha\to\infty}\lim_{\beta\to 0}\frac{1}{2\pi i}\int_{-\alpha}^{\alpha}[y(x, \mu + i\beta) - y(x, \mu - i\beta)d\mu.]$$

When $\mu<0$ the integrand$\to 0$ as $\beta\to 0$, and hence only the range of integration from 0 to α need be considered. When $\mu>0$ we find, on taking the limits as $\alpha\to\infty$, $\beta\to 0$, and substituting $\mu=\xi^2$, that

$$f(x) = \frac{1}{\pi}\int_0^\infty d\mu[\cos\xi x\int_{-\infty}^{\infty}f(t)\cos\xi t dt + \sin\xi x\int_{-\infty}^{\infty}f(t)\sin\xi t dt].$$

This is the familiar Fourier Integral expansion of $f(x)$ on $-\infty<x<\infty$.

Example 3. We consider finally a case of the more general type in which $q(x)\neq 0$. Suppose $q(x)=x^2$ on $-\infty<x<\infty$ and let T be defined by $Ty=-y''+x^2y$, with $\mathfrak{D}$ appropriately chosen (as described above) in the Hilbert space $L_2[-\infty, \infty]$. The equation $T_\lambda y=-f(x)$ is then equivalent to $y''+(\lambda-x^2)y=f(x)$, $y\in\mathfrak{D}$. The unique solution of the latter is expressible in terms of the function

$$\phi(x, \lambda) = e^{-\frac{1}{2}x^2}\int_\infty^{(0+)} e^{-x\xi-\frac{1}{4}\xi^2}\xi^{-\frac{1}{2}\lambda-\frac{1}{2}}d\xi,$$

[see [3], p. 61] as follows:

$$y = y(x, \lambda) = \frac{1}{\omega(\lambda)}\int_{-\infty}^{x} f(t)\phi(x, \lambda)\phi(-t, \lambda)dt + \frac{1}{\omega(\lambda)}\int_x^\infty f(t)\phi(t, \lambda)\phi(-x, \lambda)dt,$$

where $\omega(\lambda)=W_x[\phi(-x, \lambda), \phi(x, \lambda)] = -(1+e^{-\pi i\lambda})^2 2^{\frac{3}{2}-\frac{1}{2}\lambda}\pi^{\frac{1}{2}\lambda}\Gamma(\frac{1}{2}-\frac{1}{2}\lambda)$. The only singularities of this function are the simple poles $\lambda_k=2k+1$, $k=0, 1, 2, \cdots$. These are the eigenvalues. The residue of the pole $\lambda_k=2k+1$ is found to be

$$\gamma_n e^{-\frac{1}{2}x^2}H_n(x)\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}H_n(t)f(t)dt,$$

where $\gamma_n^{-1}=2^n n!\pi^{\frac{1}{2}}$, and the functions $H_n(x)$ are the Hermite polynomials. The eigensolution expansion of $f(x)$ is therefore

$$f(x) = \sum_0^\infty \gamma_n e^{-\frac{1}{2}x^2}H_n(x)\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}H_n(t)f(t)dt.$$

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AN EXTENDED INVERSIVE GEOMETRY

E. F. ALLEN, Oklahoma A and M College

1. Introduction. The Inversive Geometry of Morley and Morley [8], Weaver, and others is based on the complex number $z=x+iy$ and its conjugate $\bar{z}=x-iy$, where i is defined by the relation $i^2=-1$. Using this notation the above authors have proved theorems concerning the geometry of a triangle inscribed in a circle and the Line of Images of a point on a circle with respect to the sides of an inscribed triangle. By extending the definition of the variable z it is possible to state and prove corresponding theorems relating to triangles inscribed in central conics, and also to show that the Line of Images is a special case of a more general Line of Images of a point on a conic with respect to the sides of an inscribed triangle.

It is the purpose of this paper, first, to extend the concept of perpendicular diameters of circles by showing them to be special instances of conjugate diameters of central conics; second, to define a nine-point conic of a triangle inscribed in a central conic, to obtain its equation, and to exhibit a pseudo Euler Line which passes through four points analogous to the four points of the Euler Line of a triangle, third, to define a general Line of Images, and a general Simson Line.

2. The notation. Let the variables z and $\bar{z}$ be

$$z = x + ry \quad \text{and} \quad \bar{z} = x - ry, \tag{1}$$

where r is defined by one of the relations

$$r^2 = -1, \qquad r^2 = -k^2, \qquad r^2 = 1, \qquad r^2 = k^2 \tag{2}$$

where k is a real number. Then the equation $z\bar{z}=1$ becomes in ordinary rectangular Cartesian coordinates either

$$\begin{aligned} x^2 + y^2 &= 1, \text{ the unit circle,} \\ x^2 + k^2y^2 &= 1, \text{ an ellipse,} \\ x^2 - y^2 &= 1, \text{ a rectangular hyperbola, or} \\ x^2 - k^2y^2 &= 1, \text{ a hyperbola.} \end{aligned} \tag{3}$$

3. Pseudo altitudes of triangle. Let t_1, t_2, t_3 where $t_i = x_i + ry_i$ and $\bar{t}_i = x_i - ry_i$, be the vertices of a triangle inscribed in the central conic $z\bar{z} = 1$. First consider the case where $r^2 = -1$. Now z is the ordinary complex number and the geometry the Inversive Geometry of Morley and Morley. The equation of the side t_1, t_2 of the triangle is [7]

$$z + t_1t_2\bar{z} = t_1 + t_2, \tag{4}$$

and the equation of the altitude from t_3 upon this side is [7]

$$z - t_1t_2\bar{z} = t_3 - t_1t_2\bar{t}_3. \tag{5}$$

The foot of the altitude from t_3 upon this side is obtained by solving these equations for z. Thus adding,

$$2z = t_1 + t_2 + t_3 - t_1t_2\bar{t}_3, \tag{6}$$

or

$$z = S_1/2 - t_1t_2/(2t_3), \tag{7}$$

where $S_1 = t_1 + t_2 + t_3$. In like manner the equations of the altitudes from t_1 and t_2 are, respectively,

$$z - t_2t_3\bar{z} = t_1 - t_2t_3\bar{t}_1 \tag{8}$$

and

$$z - t_3t_1\bar{z} = t_2 - t_3t_1\bar{t}_2. \tag{9}$$

The feet of these altitudes are, respectively,

$$S_1/2 - t_2t_3/(2t_1) \quad \text{and} \quad S_1/2 - t_3t_1/(2t_2). \tag{10}$$

To find the point of intersection of altitudes (8) and (9), multiply the first equation by t_1, the second by t_2 and subtract. This gives, remembering that $t_1\bar{t}_1 = t_2\bar{t}_2 = 1$,

$$z(t_1 - t_2) = t_1^2 - t_2^2 + t_3t_1 - t_2t_3. \tag{11}$$

Dividing through by $t_1 - t_2$

$$z = t_1 + t_2 + t_3, \tag{12}$$

or

$$z = S_1. \tag{13}$$

By substitution we find that this value of z satisfies equation (5). Hence the three altitudes of the triangle are concurrent at the point S_1. This point is called the orthocenter of the triangle.

Now let us examine the geometry of equations (4) and (5) for the other definitions of r. The algebraic operations are identical. The equation of the side

t_1, t_2 of a triangle inscribed in the conic corresponding to the particular definition of r is (4) [7]. In order to identify the line represented by (5), consider the equations

$$z + t_1 t_2 \bar{z} = 0 \tag{14}$$

and

$$z - t_1 t_2 \bar{z} = 0. \tag{15}$$

These are the equations of two diameters of the conic which are parallel to lines (4) and (5), respectively. Equation (15) is satisfied by $(t_1+t_2)/2$, the mid-point of the chord t_1, t_2 of the conic. Line (14) is parallel to this chord. Hence by definitions found in Analytic Geometry (14) and (15) are conjugate diameters of the conic. Thus lines (4) and (5) are parallel, respectively, to a pair of conjugate diameters of the conic. These lines are called conjugate lines with respect to the conic.

Any pair of lines L_1 and L_2 which are parallel respectively to a pair of conjugate diameters l_1 and l_2 of the conic will be called conjugate lines. Line L_1 is said to be a conjugate of line L_2 with respect to the conic. Thus the altitude of a triangle inscribed in a circle is a conjugate of the corresponding side of the triangle. Hence if $r^2 = -1$ any two perpendicular lines may be regarded as conjugate lines with respect to the circle.

For values of r other than $r^2 = -1$, line (5) is called a pseudo altitude from the vertex t_3 upon the side t_1, t_2 of the triangle. A pseudo altitude and the corresponding side of the inscribed triangle form a pair of conjugate lines with respect to the circumscribing conic. Evidently every triangle inscribed in a conic has three pseudo altitudes. The feet of the pseudo altitudes upon the corresponding sides of the triangle are

$$\begin{aligned} &S_1/2 - t_2t_3/(2t_1), \qquad &&\text{from } t_1,\\ &S_1/2 - t_3t_1/(2t_2), &&\text{from } t_2,\\ &S_1/2 - t_1t_2/(2t_3), &&\text{from } t_3. \end{aligned} \tag{16}$$

The pseudo altitudes are concurrent at a point H' whose coordinate is S_1. H' is called the pseudo orthocenter of the inscribed triangle with respect to the circumscribing conic. A pseudo altitude or any line parallel to it is a conjugate of the corresponding side of a triangle inscribed in a conic.

4. The nine-point conic. The reader will recall that the nine-point circle of a triangle passes through the following sets of points, (a) the mid-points of the sides, (b) the feet of the altitudes, (c) the mid-points of the line segments joining the orthocenter H to the vertices of the triangle. Similarly, there is a nine-point conic of a triangle inscribed in a central conic, which passes through, (a) the mid-points of the sides, (b) the feet of the pseudo altitudes, (c) the mid-points of the line segments joining the pseudo orthocenter H' to the vertices of

the triangle. This statement is easily proved by using the Extended Inversive Geometry described in the Introduction. Let

$$z\bar{z} = 1, \tag{17}$$

be the equation of a central conic and let t_1, t_2, t_3 be the coordinates of the vertices of the triangle. The coordinates of the mid-points of the sides are

$$(t_1 + t_2)/2, \quad (t_2 + t_3)/2 \quad \text{and} \quad (t_3 + t_1)/2. \tag{18}$$

The coordinates of the feet of the pseudo altitudes on the opposite sides are given by (16) in Section 3. Since the coordinate of H' is S_1, the mid-points of the line segments joining H' to each vertex are

$$(S_1 + t_1)/2, \qquad (S_1 + t_2)/2, \qquad (S_1 + t_3)/2. \tag{19}$$

These nine points all lie on the conic whose equation is

$$(z - S_1/2)(\bar{z} - \bar{S}_1/2) = 1/4. \tag{20}$$

This can be verified by substituting the coordinates of each point in the equation. The coordinate of the center N of the nine-point conic is $S_1/2$. Its axes are parallel to and one half the length of the axes of the circumscribing conic. The eccentricities are equal.

5. The pseudo Euler line. The center O of the conic, the centroid G of the triangle, the center N of the nine-point conic and the pseudo orthocenter H' are collinear. This follows from the fact that the coordinates of these points are, respectively, 0, $S_1/3$, $S_1/2$, and S_1. Due to its similarity to the Euler Line of a triangle we are calling this line the pseudo Euler Line.

6. Projective transformation. In the case of the nine-point conic of a triangle inscribed in an ellipse the configuration can also be obtained by the projective transformation $x = \bar{x}$ and $y = k\bar{y}$, applied to the nine-point circle configuration.

7. The line of images. The concept of conjugate lines can be used to unify the theory of the line of images. The image of a point T of a circle with respect to the side t_2, t_3 of an inscribed triangle has been defined as a point T_1 such that the side of the triangle is the perpendicular bisector of the line segment TT_1 [4]. The images of the point T with respect to the three sides of the triangle are collinear. The equation of this line is [5]

$$Tz - S_3\bar{z} = TS_1 - S_2, \tag{21}$$

where $S_2 = t_1t_2 + t_2t_3 + t_3t_1$, and $S_3 = t_1t_2t_3$. Recently the image of a point T of a hyperbola with respect to the sides of an inscribed triangle has been defined in terms of slant lines [5]. In this case the line segment TT_1 was bisected by the side of the triangle. Again the three points so obtained were shown to be collinear. By recognizing that the perpendicular lines of the first paper and the slant lines of the second paper are conjugate lines of the sides of the triangle, it is

possible to generalize the theorems to a triangle inscribed in any central conic. Thus a well known theorem can be stated: *Lines of images of two diametrically opposite points of a conic with respect to the sides of an inscribed triangle are conjugate lines of the conic.*

8. The generalized Simson line. Since the line segments TT_1, TT_2, TT_3 are bisected by the corresponding sides of the triangle it follows that the feet of these lines from T are collinear. If the conic is a circle we have the Simson line. Otherwise a generalized Simson line.

9. Conclusion. The above theory applies to any central conic. The author's 1941 article [7] applied only to a triangle inscribed in a rectangular hyperbola. The wider generalization is obtained by using the concept of conjugate lines instead of that of anti-parallel lines which were defined in terms of the asymptotes of the hyperbola.

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THE NOTS REAC

F. H. YOUNG, U. S. Naval Ordnance Test Station, Inyokern, China Lake, California

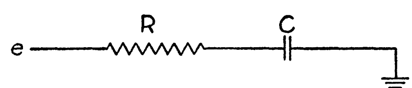
In October, 1951, the Naval Ordnance Test Station at China Lake put into operation a modest REAC installation. A REAC, by the way, is a Reeves electronic analog computer. It is probable that most mathematicians are more familiar with digital computers, for they are much more common. The digital machines are essentially arithmetical, performing very rapidly the elementary operations of arithmetic. When it comes to calculus operations, digital machines proceed by small steps, solving difference equations rather than differential equations. Computers of this type are capable of an accuracy limited only by the number of digits carried through a problem. Since this number is usually very large, extreme accuracy can be obtained. In problems such as the inversion

of matrices of higher order, such accuracy is essential. On the other hand, on digital machines the investigation of the variation of the parameters involved in a given problem is usually time-consuming and not too simple.

The REAC, like many other electronic analog computers, was designed to investigate those problems in which slide rule accuracy is sufficient and in which flexibility in the variation of parameters is important. Specifically, the REAC is a machine made to solve systems of ordinary differential equations, linear or non-linear. In modern research one encounters a great variety of problems that can be set up as a system of such equations, and frequently the initial conditions and parameters involved are known to only two or three significant figures. Once such a problem is plugged into the REAC, a solution can be plotted in much less than a minute. A parameter can be changed in a few seconds by the turn of a knob, and a new solution can be obtained shortly.

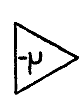
Such a machine must be able to perform integrations, algebraic additions, multiplications, and divisions. Also, it must be able to generate functions determined empirically and must be able to produce solutions as plotted graphs, sets of cards, or meter readings. In the REAC, a variable is represented as an electric potential whose magnitude and sign vary as the value of the variable. A very well regulated ± 100 volt power supply provides the frame of reference for all calculations.

Let us consider the problem of integration. Suppose a fixed voltage, e, were to be applied to the following resistor-condenser network.



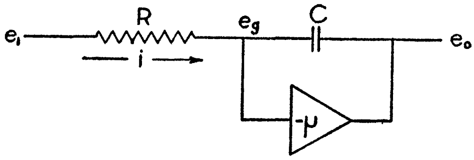
The applied voltage, e, will cause a current to pass through the resistor, R, and to accumulate as a charge on the condenser, C. The effect of this charge will be to oppose further flow of current. As a consequence, the potential across C will start to rise and will approach the value e asymptotically. At first, when the opposing voltage across the condenser is small, the rise in this voltage will be nearly linear and approximately proportional to the applied voltage. Thus, for a short time the condenser acts like an integrator. If the effect of the opposing voltage across C could be eliminated, the current stored in C would continue to rise at a rate proportional to the applied voltage. This desideratum is achieved in the REAC by means of a direct current amplifier having very high gain and using negative feedback.

Such an amplifier will be represented by the triangle,



where $-\mu$ denotes the gain. In the REAC, μ is of the order of 10^7. Let us place

such an amplifier in parallel with the condenser, C, as follows.



With an applied voltage e_i, let the resulting current be denoted by i. Since the amplifier draws no current (its input is applied to the grid of a tube), all of the current will be stored in the capacitor. In the computer, the amplifier works in such a manner that any current drawn from the output of the circuit above will be supplied by the amplifier. We shall let e_g denote the voltage at the junction indicated. If R is measured in ohms and C in farads, then we have, by Ohm's Law,

$$i = (e_i - e_g)/R.$$

Also,

$$Q = C(e_g - e_0),$$

where Q is the charge in coulombs on the condenser. Also,

$$Q = \int_0^t i dt + Q_0,$$

Q_0 being the charge on the capacitor at time $t=0$. Thus,

$$C(e_g - e_0) = \int_0^t (e_i - e_g)/R \, dt + Q_0.$$

Also, $e_0 = -\mu e_g$, so that

$$e_0 + e_0/\mu = -(1/RC)\int_0^t e_i dt - (1/\mu RC)\int_0^t e_0 dt - Q_0/C.$$

If μ is very large, $e_0 \doteq -(1/RC)\int_0^t e_i dt - Q_0/C$. In particular, if R is one megohm and C is one microfarad,

$$e_0 \doteq -\int_0^t e_i dt - V_0.$$

The quantity V_0, the voltage across C at time $t=0$, is available as an initial condition for the integration. It may be noted that the effect of the amplifier is to make e_g a virtual ground, for $e_g = -e_0/\mu \doteq 0$.

In the REAC at NOTS, the capacitors are fixed at 1 microfarad, but the input resistors are available in a variety of sizes from 0.01 to 10 megohms. A moment's consideration will show that if the input resistor is R megohms and

C is 1 microfarad, then

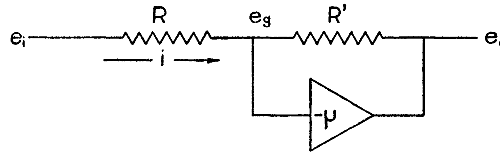
$$e_0 \doteq -(1/R)\int_0^t e_i dt - V_0.$$

Furthermore, if several voltages, say e_1, e_2, and e_3, are fed into the capacitor C through resistances of R_1, R_2, and R_3 megohms respectively, then

$$-e_0 \doteq \int_0^t (e_1/R_1 + e_2/R_2 + e_3/R_3)dt + V_0.$$

Thus, the integration of a sum is possible in a single step.

If the condenser C were replaced by a resistor, as below,



a different operation would be performed.

Since the current through R and R' is the same,

$$i = (e_i - e_g)/R = (e_g - e_0)/R'.$$

Since $e_0 = -\mu e_g$, we have

$$(e_i + e_0/\mu)/R = -(e_0/\mu + e_0)/R',$$

or

$$R'e_i + R'e_0/\mu + Re_0/\mu + Re_0 = 0,$$

or

$$e_0 = -R'e_i/(R + R'/\mu + R/\mu) \doteq -e_i(R'/R).$$

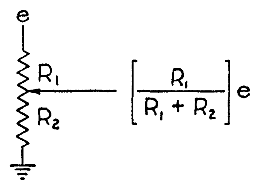
Again, the last approximation is very good because of the very large magnitude of μ.

Similarly, if we have several inputs, say e_1, e_2, and e_3 fed in through resistors R_1, R_2, and R_3, we find that

$$e_0 \doteq -[(R'/R_1)e_1 + (R'/R_2)e_2 + (R'/R_3)e_3].$$

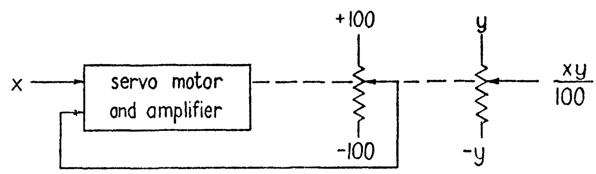
Thus, such a network can be used to add several variables and to manipulate their scale factors, or it can be used as a simple sign changer.

In the REAC, only certain values of resistance are available for such input scaling. Hence, it becomes necessary to have a way of making finer scale changes. This can be done by the following simple device, utilizing a 30,000 ohm potentiometer.



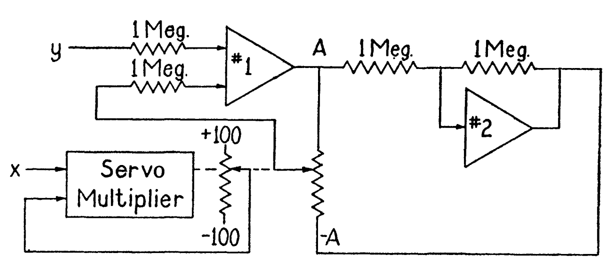
Thus, if a voltage e is fed to the top of the potentiometer, any proper fraction of e can be taken off to be used elsewhere in the computer.

Multiplication is done by means of servomechanisms as illustrated below.



When x is fed into the servo amplifier, the servo motor seeks a point on the follow-up potentiometer (hereafter called "pot") that has a voltage x. The motor shaft is then positioned so that all of the pots driven by the shaft are $x/100$ of the distance from the center to one end. If a second signal, y, is put on the upper end of such a multiplying pot and $-y$ is put on the bottom, the resulting voltage tapped off will be $xy/100$.

Division is accomplished by means of a servo multiplier and an amplifier connected as below.



Since x is fed into the servo, the pots are positioned at $x/100$ of the way from the center to the top. If we designate by A the output of amplifier 1, then the voltage tapped from the second pot will be $xA/100$. Since this voltage and y are fed into amplifier 1 through equal resistors, and the gain of the amplifier is $-\mu$, we have

$$-\mu(y + xA/100) = A, \quad \text{or} \quad y + xA/100 = -A/\mu.$$

Since μ is extremely large in comparison with A, this relationship becomes

$$y + xA/100 \doteq 0, \quad \text{or} \quad A \doteq -100y/x.$$

The REAC also has provision for the introduction of empirical functions. One such device is the electronic function generator that utilizes lantern slides.

A beam from a cathode ray tube rides on top of a lantern slide curve and is sensed by a photoelectric cell. The voltage output is proportional to the vertical deflection of the beam. Another input device consists of a servo-operated rotating drum on which an arbitrary function can be represented by a fine wire glued to the surface. A resistor card riding on the wire permits the functional voltage to be picked off for use elsewhere in the computer.

Output devices consist of a plotting table on which any variable can be plotted against any other, and a four-channel hot wire recorder on which any four variables can be plotted against time. In addition, three of the rotating drums described as input devices can also be used to plot any variable against any other. Altogether, then, the NOTS REAC can plot out eight variables at a time. If more are needed, it becomes necessary to replug several wires and re-run the problem, a process requiring about a minute.

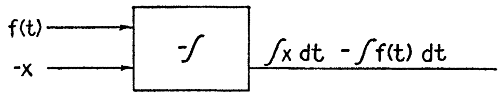
As an illustration of REAC operation, let us consider the second order linear differential equation:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = f(t).$$

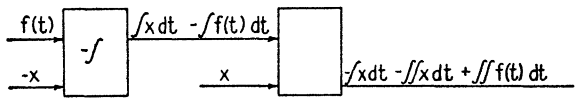
This we shall rewrite as

$$x = -\int x dt - \int\int x dt + \int\int f(t) dt.$$

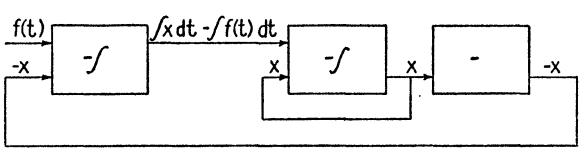
We shall suppose that the forcing function, $f(t)$, is available. In addition, suppose we had the negative of our solution, $-x$. Let us feed these into an integrator, each with scale factor 1.



Let us feed this output, together with x, into another integrator.



We observe that the output is precisely x. Then, if we feed x through an inverting amplifier with a gain of -1, we have both the x and $-x$ required to feed into the two integrators.



In order for the problem to be meaningful, initial conditions must be specified.

These initial conditions are introduced by charging the capacitors of the two integrators prior to starting the solution. The starting mechanism is merely a switch that simultaneously removes the voltage sources applied to the capacitors and connects the inputs to the amplifiers. The solution, x, can now be taken from the second integrator and plotted against t, which can be obtained by feeding a small negative voltage into a one-megohm input of an integrator. The value of the small voltage depends only on the scale desired and the length of time required for the solution.

The installation at NOTS has, at present, 14 integrators. That is, it is possible to solve seven simultaneous second-order differential equations, not necessarily linear.

Although there are about 300 REAC facilities in this country, it is doubtful if any two are alike. They are a bit like the old Model T Ford in that the original machine is generally regarded merely as a point of departure. It is a challenge to the operator's ingenuity, and modifications usually accumulate rapidly. RAND Corporation of Santa Monica, California, has probably been the source of more extensive alterations than anyone, and many of these modifications have now been adopted by Reeves.

SOME INTERESTING SERIES RESULTING FROM A CERTAIN MACLAURIN EXPANSION

M. R. SPIEGEL, Rensselaer Polytechnic Institute

The expansion of (arc sin $t)^2$ into a Maclaurin Series can be quite a tedious calculation since it is necessary to obtain $[(d^n/dt^n)\,(\text{arc sin } t)^2]_{t=0}$. The approach in the present paper—although somewhat roundabout—does give the Maclaurin series a very simple form and we shall in addition be able to arrive at several interesting "by-products."

We consider

$$I = \int_0^{\pi/2} \frac{dx}{1 - t\cos x} \qquad |t| < 1. \tag{1}$$

This integral may be evaluated in the following manner.

$$I = \int_0^{\pi/2} \frac{1 + t\cos x}{1 - t^2\cos^2 x}\,dx = \int_0^{\pi/2} \frac{dx}{1 - t^2\cos^2 x} + \int_0^{\pi/2} \frac{t\cos x dx}{1 - t^2\cos^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x dx}{1 - t^2 + \tan^2 x} + \int_0^{\pi/2} \frac{t\cos x dx}{1 - t^2 + t^2\sin^2 x}$$

$$= \frac{1}{\sqrt{1-t^2}}\left[\text{arc tan}\left(\frac{\tan x}{\sqrt{1-t^2}}\right) + \text{arc tan}\left(\frac{t \sin x}{\sqrt{1-t^2}}\right)\right]\Bigg|_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}}\,\text{arc tan}\left(\frac{t}{\sqrt{1-t^2}}\right).$$

Thus we have

$$I = \frac{\pi}{2\sqrt{1-t^2}} + \frac{\text{arc sin } t}{\sqrt{1-t^2}}\,. \tag{2}$$

Consider (1) once again. We may write

$$I = \int_0^{\pi/2} \{1 + t\cos x + t^2\cos^2 x + \cdots\}dx$$

or

$$\begin{aligned} I = {}& \frac{\pi}{2}\left\{1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \cdots\right\} \\ & + t\left\{1 + \frac{2}{1\cdot 3}t^2 + \frac{2\cdot 4}{1\cdot 3\cdot 5}t^4 + \cdots\right\} \end{aligned} \tag{3}$$

since

$$\int_0^{\pi/2} \cos^{2p} x dx = \frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots(2p)}\,\frac{\pi}{2} \tag{4}$$

and

$$\int_0^{\pi/2} \cos^{2p-1} x dx = \frac{2\cdot 4\cdots(2p-2)}{1\cdot 3\cdots(2p-1)}\,. \tag{5}$$

Noting that

$$\frac{\pi}{2\sqrt{1-t^2}} = \frac{\pi}{2}\left\{1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \cdots\right\}$$

we have upon comparison of (2) and (3) the result

$$\frac{\text{arc sin } t}{\sqrt{1-t^2}} = t + \frac{2}{1\cdot 3}t^3 + \frac{2\cdot 4}{1\cdot 3\cdot 5}t^5 + \cdots\,. \tag{6}$$

Integration of (6) from 0 to t, yields the result

$$(\text{arc sin } t)^2 = t^2 + \frac{2}{1\cdot 3}\left(\frac{t^4}{2}\right) + \frac{2\cdot 4}{1\cdot 3\cdot 5}\left(\frac{t^6}{3}\right) + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\left(\frac{t^8}{4}\right) + \cdots \tag{7}$$

which is the required Maclaurin expansion.

It follows from (7) that

$$\frac{d^n}{dt^n}(\arc \sin t)^2\bigg|_{t=0} = \frac{(2p)!}{p}\left(\frac{2\cdot 4\cdots(2p-2)}{1\cdot 3\cdots(2p-1)}\right) \tag{8}$$
$$= 2[2\cdot 4\cdots(2p-2)]^2$$

if $n=2p$, and zero otherwise.

From (7), upon letting $t=1$ we obtain the interesting result

$$\frac{\pi^2}{4} = 1 + \frac{2}{1\cdot 3}\left(\frac{1}{2}\right) + \frac{2\cdot 4}{1\cdot 3\cdot 5}\left(\frac{1}{3}\right) + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\left(\frac{1}{4}\right) + \cdots. \tag{9}$$

Another interesting result, but much better known, may be obtained as follows.

Let $t=\sin\theta$ in (7) so that we obtain

$$\theta^2 = \sin^2\theta + \frac{2}{1\cdot 3}\,\frac{\sin^4\theta}{2} + \frac{2\cdot 4}{1\cdot 3\cdot 5}\,\frac{\sin^6\theta}{3} + \cdots. \tag{10}$$

Integrating from 0 to $\pi/2$, using the result

$$\int_0^{\pi/2} \sin^{2p}\theta d\theta = \frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots((2p)}\,\frac{\pi}{2}, \tag{11}$$

and making the appropriate simplifications we have

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.^* \tag{12}$$

Still another interesting result may be obtained as follows.

We have

$$\arc \sin t = \sum_{p=0}^{\infty} \frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots(2p)}\,\frac{t^{2p+1}}{2p+1} \tag{13}$$

where the term corresponding to $p=0$ is interpreted to be equal to t. Squaring (13) we have

$$(\arc \sin t)^2$$
$$= \sum_{p,q=0}^{\infty}\left\{\frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots(2p)}\right\}\left\{\frac{1\cdot 3\cdots(2q-1)}{2\cdot 4\cdots(2q)}\right\}\frac{t^{2p+2q+2}}{(2p+1)(2q+1)}. \tag{14}$$

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* The author wishes to thank the referee for pointing out that the method used here was the method whereby the series in (12) was summed for the first time by Euler. At present it is, of course, considered easier to use Fourier Series.

Thus

$$(\text{arc sin } t)^2 = \sum_{n=0}^{\infty} a_{2n} t^{2n} \tag{15}$$

where

$$a_{2n} = \sum_{p=0}^{n} \left\{ \frac{1\cdot 3 \cdots (2p-1)}{2\cdot 4 \cdots (2p)} \right\} \left\{ \frac{1\cdot 3 \cdots (2n-2p-3)}{2\cdot 4 \cdots (2n-2p-2)} \right\} \cdot \frac{1}{(2p+1)(2n-2p-1)} \tag{16}$$

or

$$a_{2n} = \frac{1}{2^{2n-2}} \sum_{p=0}^{n} \binom{2p}{p} \binom{2n-2p-2}{n-p-1} \frac{1}{(2p+1)(2n-2p-1)}. \tag{17}$$

Comparison of (7), (14) and (17), yields the result

$$\sum_{p=0}^{n} \binom{2p}{p} \binom{2n-2p-2}{n-p-1} \frac{1}{(2p+1)(2n-2p-1)} = \left\{ \frac{2\cdot 4 \cdots (2n-2)}{1\cdot 3 \cdots (2n-1)} \right\} \frac{2^{2n-2}}{n}. \tag{18}$$

Several other series of interest may be summed. Multiplication of (10) by $\sin \theta$, integrating from 0 to $\pi/2$ and making use of

$$\int_0^{\pi/2} \sin^{2p+1} \theta d\theta = \frac{2\cdot 4 \cdots (2p)}{1\cdot 3 \cdots (2p+1)}, \tag{19}$$

we find

$$\frac{\pi}{2} = \frac{4}{3} + \left(\frac{2}{1\cdot 3}\right)^2 \frac{1}{5} + \left(\frac{2\cdot 4}{1\cdot 3\cdot 5}\right)^2 \frac{1}{7} + \left(\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\right)^2 \frac{1}{9} + \cdots. \tag{20}$$

Multiplication of (6) by t and integrating from 0 to t, we have

$$t - (\text{arc sin } t)(\sqrt{1-t^2}) = \frac{1}{3} t^3 + \frac{2}{1\cdot 3\cdot 5} t^5 + \frac{2\cdot 4}{1\cdot 3\cdot 5\cdot 7} t^7 + \cdots. \tag{21}$$

Letting $t=1$ we then have the result

$$\frac{2}{1\cdot 3\cdot 5} + \frac{2\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \cdots = \frac{2}{3}. \tag{22}$$

If we put $t=\frac{1}{2}$ in (21) we have

$$(23)\quad \frac{2}{1\cdot 3\cdot 5}\left(\frac{1}{2^5}\right)+\frac{2\cdot 4}{1\cdot 3\cdot 5\cdot 7}\left(\frac{1}{2^7}\right)+\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7\cdot 9}\left(\frac{1}{2^9}\right)+\cdots=\frac{11-2\pi\sqrt{3}}{24}.$$

Letting $t=\sin\theta$ in (21) and integrating from 0 to $\pi/2$ we obtain the curious result

$$(24)\quad \pi = 4-\left\{\left(\frac{2}{1\cdot 3}\right)^2+\left(\frac{2\cdot 4}{1\cdot 3\cdot 5}\right)^2\frac{1}{2}+\left(\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\right)^2\frac{1}{3}+\cdots\right\}.$$

Another curious result may be obtained as follows. Integrating (7) from 0 to t we obtain

$$(25)\quad \begin{aligned} &t(\arcsin t)^2+2(\arcsin t)(\sqrt{1-t^2})-2t\\ &\qquad=\frac{t^3}{3}+\left(\frac{2}{1\cdot 3\cdot 5}\right)\frac{t^5}{2}+\left(\frac{2\cdot 4}{1\cdot 3\cdot 5\cdot 7}\right)\frac{t^7}{3}+\cdots.\end{aligned}$$

Letting $t=\sin\theta$ in (25), integrating from 0 to $\pi/2$ and making use of (19), we find

$$(26)\quad \pi=3+\left\{\left(\frac{1}{1\cdot 3}\right)^2+\left(\frac{2}{1\cdot 3\cdot 5}\right)^2+\left(\frac{2\cdot 4}{1\cdot 3\cdot 5\cdot 7}\right)^2+\left(\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7\cdot 9}\right)^2+\cdots\right\}.$$

As a last result analogous in many respects to a recently proposed advanced problem of this MONTHLY (problem 4483, April 1952, p. 254), we note that upon placing $t=1$ in (25) we obtain after some simple alterations

$$(27)\quad \frac{\pi^2}{2}-4=\left(\frac{2}{1\cdot 3}\right)\frac{1}{1^2}+\left(\frac{2\cdot 4}{1\cdot 3\cdot 5}\right)\frac{1}{2^2}+\left(\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\right)\frac{1}{3^2}+\cdots.$$

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

CHARACTERISTIC ROOTS OF A SET OF MATRICES

W. V. PARKER, Alabama Polytechnic Institute

1. Introduction. A square matrix A of order n with complex elements may be written uniquely as

$$(1)\qquad A = H + iK$$

where H and K are Hermitian matrices. Hirsch [1] and Bromwich [2] have

DEFINITION OF THE LEBESGUE INTEGRAL

CASPER GOFFMAN, University of Oklahoma

Many definitions of the Lebesgue integral have been given. However, the following definition, which is a simple generalization of the Cauchy integral for continuous functions on a closed interval, is not well-known in this country.*

For convenience, only real functions defined on the closed interval $[0, 1]$ will be considered, and all sets will be understood to be subsets of this interval. Suppose S is a closed set and $f(x)$ is continuous on S relative to S. Then the complement $C(S)$ of S consists of a finite or denumerable set of disjoint intervals, $I_1, I_2, \cdots, I_n, \cdots$. Let a_n and b_n be the ends points of I_n, for every $n=1, 2, \cdots$. Moreover, $f(x)$ may be extended to a continuous function $F(x)$ on $[0, 1]$ such that $F(x)=f(x)$ for $x \in S$. The Cauchy integral of $f(x)$ on S may be defined as

$$\int_S f(x)dx = \int_0^1 F(x)dx - \sum_{n=1}^{\infty} \int_{a_n}^{b_n} F(x)dx.$$

The proof that $\int_S f(x)dx$ is independent of the choice of $F(x)$, subject to the condition that $F(x)$ is a continuous extension of $f(x)$, is easy and is left to the reader.

Now, let $f(x)$ on $[0, 1]$ belong to the class $\mathfrak{M}$ if there is an increasing sequence of closed sets $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$ such that, for every $\epsilon > 0$, there is an m for which the sum of the lengths of the disjoint intervals whose union is $C(S_m)$ is less than ϵ, and such that $f(x)$ is continuous on S_n, for every n.

If $f(x) \in \mathfrak{M}$ is non-negative, and $\{S_n\}$ is a sequence of closed sets having the above property relative to $f(x)$, the sequence $\{\int_{S_n} f(x)dx\}$ is non-decreasing. If this sequence converges, then $f(x)$ is summable, and the integral of $f(x)$ is defined as

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_{S_n} f(x)dx.$$

If the sequence diverges, then $f(x)$ is non-summable.

The sequence $\{S_n\}$ of closed sets corresponding to a given $f(x) \in \mathfrak{M}$ is not unique. That the summability or non-summability of $f(x)$ and the value of its integral are not dependent on the choice of the closed sets is again an easy exercise which is left for the reader. The definition of summability and integral for arbitrary $f(x) \in \mathfrak{M}$ may be accomplished in the usual way by writing $f(x)$ as the difference between two non-negative functions. The proof that $f(x)$, $g(x)$ summable implies $f(x)+g(x)$ summable and $\int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$ is now essentially a corollary to the corresponding theorem for continuous functions. Indeed, the whole theory of Lebesgue integration may be

* The author has been informed by Professor E. J. Mickle that a definition similar to this one is commonly used in Italy.

obtained from this definition.

Finally, although the details of the integration theory may be obtained without reference to this fact, the above class $\mathfrak{M}$ is precisely the class of measurable functions. This follows by Lusin's Theorem that a function $f(x)$ on $[0, 1]$ is measurable if and only if, for every $\epsilon > 0$, there is a closed set S of measure exceeding $1-\epsilon$ such that $f(x)$ is continuous on S relative to S.

A PROPERTY OF RANDOMNESS OF AN ARITHMETICAL FUNCTION

N. METROPOLIS and S. ULAM, Los Alamos Scientific Laboratory

Let $f(x)$ be a transformation of a set E into itself. One can decompose E into minimal invariant subsets or "trees" by considering for each point x the smallest set X with the following properties: 1. $x \epsilon X$, 2. If $y \epsilon X$ then $f(y) \epsilon X$, 3. If $y \epsilon X$ all the points x such that $f(x) = y$ also belong to X. One will obtain a decomposition of $E = X_1 + \cdots + X_\xi + \cdots$ into disjoint trees X_ξ. These characterize $f(x)$ up to a conjugating transformation h; *i.e.*, all $h(f(h^{-1}))$ where h is an arbitrary one-one transformation of E into itself.*

Suppose that $f(x)$ is a random function defined on E with values in E, *i.e.*, for each x a value y called $f(x)$ was chosen from E, say with a uniform distribution of probability in E (assumed to be a measure space).§ In case of E finite, one could ask about the "probable" or expected number of trees in the decomposition given by f. This expected number, not easy to estimate, is likely of the order of $\log n$; n is the number of points of E. For the case where f is postulated as a one to one transformation, *i.e.*, a permutation, the trees become cycles. It is well known that the expected number of cycles is $\log n$.

We examined the number of trees for some specific *not a priori* random functions. One function, often used in various Monte Carlo problems to produce random *digits* (by iteration), is the following: let x be an integer ranging from 0 to $2^k - 1$ written in the binary development $x = \alpha_0 2^0 + \cdots \alpha_{k-1} 2^{k-1}$. Let $f(x)$ be defined as follows: we take x^2 written again in the binary notation and define $f(x)$ as the number given by the *middle* k (out of $2k$) digits of x^2. It will, of course, again range between 0 and $2^k - 1$.

With this $f(x)$, for $k = 12$, *i.e.*, x ranging from 0 to 4095, the number of trees is 7.

Imagine that, as mentioned before, for each element x of a set E, one performs a selection of an element from E, with, say, uniform probability, *i.e.*, each time, every element has an equal chance $(=1/n)$ of being selected. One can ask of such a "random function," or transformation, for the probable number λ_0 of points y in the set E which are *not* of the form $y = f(x)$. Also one can ask for the number λ_1 of points y for which there is exactly *one* x such that $y = f(x)$—more generally, λ_i of points for which there are exactly i distinct

* See S. Ulam, Bull. Am. Math. Soc., 1943, Abstract, p. 49.

§ We consider the product space E^E. Logically, the study of a random function is, of course, the study of the probability distribution in E^E, the totality of all $f(x)$ on E into E.

values of x so that $f(x_1) = \cdots = f(x_i) = y$. For large n, the numbers λ_i should have a Poisson distribution, *i.e.*, $\lambda_i = ne^{-1}/i!$ (*e.g.*, λ_0 is the number of points not selected. But, given a number, at each choice, the probability of it *not* being selected is $1-1/n$, and this experiment is repeated n times independently—similarly for $i=1, 2, \cdots$.)

It is perhaps amusing that for some *specific* functions, for instance $f(x)$ as defined above by the "middle of the square," the actual distribution of the λ_i is extremely close to the one expected for a "random transformation." Sample values are given below:

For $k=12$; *i.e.* $n=4096$

i	λ_i/n observed	λ_i/n Poisson
0	.366	.368
1	.377	.368
2	.178	.184
3	.061	.061
4	.012	.015
5	.004	.003

For $k=16$, $n=65{,}536$

i	λ_i/n observed	λ_i/n Poisson
0	.370	.368
1	.367	.368
2	.183	.184
3	.062	.061
4	.015	.015
5	.003	.003
6	.0006	.0006

The only sizable deviation from what one would expect in a "random" f are, of course the sets $f^{-1}(0)$ and $f^{-1}(2^{k/2})$.

The problem of enumeration for the case $k=12$ was done with the aid of a desk calculator. The larger problem of $k=16$, $n=65{,}536$ was performed on the recently completed Los Alamos electronic computer. The computing problem consisted of generating the particular set $f(E)$ and of counting how many times each y occurred.

It was possible to keep track in the "memory" of 4096 numbers at any one time, so that it was necessary to regenerate the whole set E sixteen times to get a complete enumeration and check. The total computing time was one and one-half hours.

We propose sometime to examine, empirically on the computer, certain other arithmetical functions for their "tree" distribution and other combinatorial properties. We express our thanks to Miss Lois Cook for aid in doing both the hand computation and "coding" for the automatic calculation.

A NOTE ON ORTHOGONAL MATRICES

L. CARLITZ, Duke University

1. J. L. Brenner [1] showed that the only 2×2 orthogonal matrices with elements in $GF[p, x]$ are constant matrices for $p>2$, while for $p=2$ they are of the form

$$\begin{pmatrix} 1+P(x) & P(x) \\ P(x) & 1+P(x) \end{pmatrix} \qquad (P(x)\epsilon GF[2, x]).$$

The result followed as a corollary of a theorem on polynomial parametrization. It is not difficult to give a direct proof.

In the present note we construct some special $p \times p$ orthogonal matrices with polynomial elements. Consider first the special matrix

$$S = (i - j) \qquad (i, j = 1, \cdots, p).$$

Then it is clear that $S^2 = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{p} (i - k)(k - j) = - \sum_{k=1}^{p} (k^2 - (i + j)k + ij),$$

from which it follows that

$$c_{ij} \equiv \begin{cases} -1 \pmod{p} & (p = 3) \\ 0 \pmod{p} & (p > 3). \end{cases}$$

In other words $S^2 = 0$ for $p > 3$; for $p = 3$ it is easily verified that $S^3 = 0$.

Let A' denote the transpose of the matrix A. Then $S' = -S$ and

$$(I + xS)(I + xS)' = (I + xS)(I - xS) = I$$

for $p > 3$, so that $I + xS$ is orthogonal for $p > 3$. For $p = 3$, put $A = I + xS - x^2S^2$; then

$$AA' = (I + xS - x^2S^2)(I - xS - x^2S^2) = I,$$

so that A is orthogonal.

2. The above result can be extended by considering the matrices

$$S_r = ((i - j)^r) \qquad (i, j = 1, \cdots, p; r = 0, 1, 2, \cdots).$$

Then $S_r' = (-1)^r S_r$ so that S_r is skew-symmetric for r odd. (Note that for $r = \frac{1}{2}(p-1)$, $S_r = ((i-j/p))$, where (a/p) is the Legendre symbol). Now

$$S_r S_s = (a_{ij}), \qquad a_{ij} = \sum_{k=1}^{p} (i - k)^r (k - j)^s.$$

Using the well-known formula

$$\sum_{k=1}^{p} k^r \equiv \begin{cases} 0 & (p - 1 \nmid r) \\ -1 & (p - 1 \mid r) \end{cases} \qquad (r > 0),$$

we see that

$$a_{ij} \equiv \begin{cases} 0 & (r + s < p - 1) \\ -1 & (r + s = p - 1). \end{cases}$$

Thus it follows that

$$S_rS_s = \begin{cases} 0 & (r+s<p-1) \\ -S_0 & (r+s=p-1); \end{cases} \tag{1}$$

we remark that all the elements of $S_0=1$.

Now consider the matrix

$$A = I + x_1S_1 + \cdots + x_kS_{2k-1},$$

where $k<\frac{1}{2}(p-1)$ and $x_1, \cdots, x_k$ are indeterminates. It follows at once from (1) that

$$AA' = (I + x_1S_1 + \cdots + x_kS_{2k-1})(I - x_1S_1 - \cdots - x_kS_{2k-1}) = I,$$

so that A is orthogonal.

If $p=4m-1$, we put

$$B = I + x_1S_1 + \cdots + x_mS_{2m-1} + \tfrac{1}{2}x_m^2S_0;$$

then by the second of (1)

$$BB' = I - x_m^2(S_{2m-1}^2 + S_0) = I,$$

so that B is orthogonal. Additional examples are easily constructed.

Reference

1. J. L. Brenner, Polynomial parametrizations, this MONTHLY, vol. 58, 1951, pp. 327–329.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE DERIVATIVE OF cos x

A. G. ANDERSON, Oberlin College

The derivation of the expression

$$\frac{d}{dx}\cos x = -\sin x \tag{1}$$

in most elementary calculus texts is based on the identity $\cos x=\sin(\frac{1}{2}\pi-x)$. The purpose of the present note is to advocate an alternative derivation based on the fundamental identity $\sin^2 x+\cos^2 x=1$.

Let $\sin x=u$ and $\cos x=v$. Then $u^2+v^2=1$. Differentiation of this expression

with respect to x yields

$$2u\frac{du}{dx} + 2v\frac{dv}{dx} = 0.$$

Thus

$$\frac{dv}{dx} = -\frac{u}{v}\frac{du}{dx},$$

and (1) is seen to hold.

There is of course nothing in this presentation which is unknown to any reader, but it is my feeling that the method is often overlooked. The derivation given here can be presented without modification either in an ordinary calculus class or a freshman survey course including the elements of calculus. It has been my experience that students, particularly those with sketchy backgrounds in trigonometry, grasp this derivation more readily and that it is more useful in serving as a reminder that basic techniques are applicable in the case of trigonometric functions as well as for the algebraic functions in connection with which they were first obtained.

GEOMETRIC CONVERGENCE

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In discussing the convergence of a real sequence of the form $u_{n+1} = F(u_n)$, it is often useful to look at the geometry associated with such a sequence.

Let us first consider the sequence $u_{n+1} = \sqrt{k+u_n}$. We wish to determine the values of k and u_0 such that the sequence $\{u_n\}$ converges, and also to find its limit.

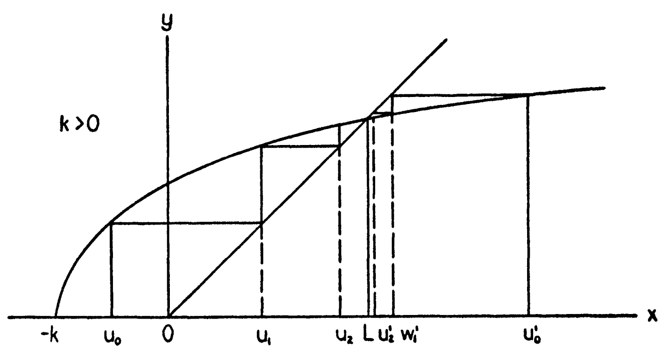


FIG. 1

We first graph $y = \sqrt{k+x}$ and $y = x$ (see Fig. 1), and then construct a ladder starting from $x = u_0$ between the parabola $y = \sqrt{k+x}$ and the straight line $y = x$

by horizontal and vertical displacements as shown. If the ladder tends to a point, then the sequence is convergent and will approach this point as a limit. This limit point will be one of the intersections points of $y=x$ and $y=\sqrt{k+x}$ in case there is more than one. Since both the parabola and line are monotonic increasing curves, it is evident from the plot that for $-k<u_0<L$, the sequence $\{u_n\}$ is monotonically increasing and bounded. Similarly, for $u_0 \geqq L$, the sequence is monotonically decreasing and bounded. Thus in either case, the sequence approaches L as a limit. For $u_0<-k$, we do not get a real sequence. These results can easily be verified analytically.

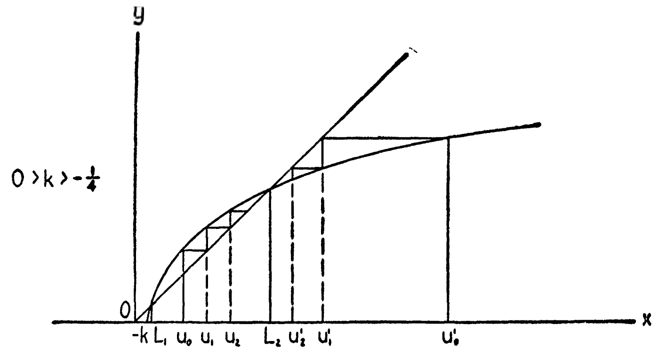


FIG. 2

For $0>k>-1/4$, the graph is given by Figure 2. If $u_0 \geqq L_2$, $\{u_n\}$ is a monotonic decreasing sequence approaching L_2 as a limit. For $L_1<u_0<L_2$, $\{u_n\}$ is monotonically increasing and also approaches L_2. For $u_0<L_1$, u_n is eventually complex.

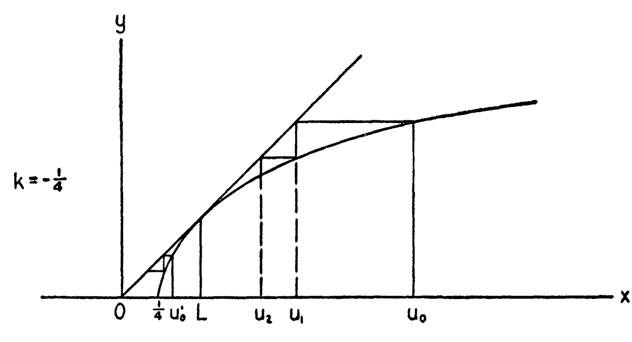


FIG. 3

When $k=-1/4$, the line $y=x$ is tangent to the parabola (see Fig. 3). If $u_0 \geqq 1/2$, the sequence converges. If $u_0<1/2$, the sequence is eventually complex. The value of $k=-1/4$ is obtained by making $x=\sqrt{k+x}$ have a double root.

Finally, for $k<-1/4$, the parabola doesn't intersect the straight line $y=x$.

This leads again to a complex sequence.

A more involved problem is given by the sequence $u_{n+1}=1/(2+u_n)$. We first plot $y=1/(2+x)$ and $y=x$ (see Fig. 4).

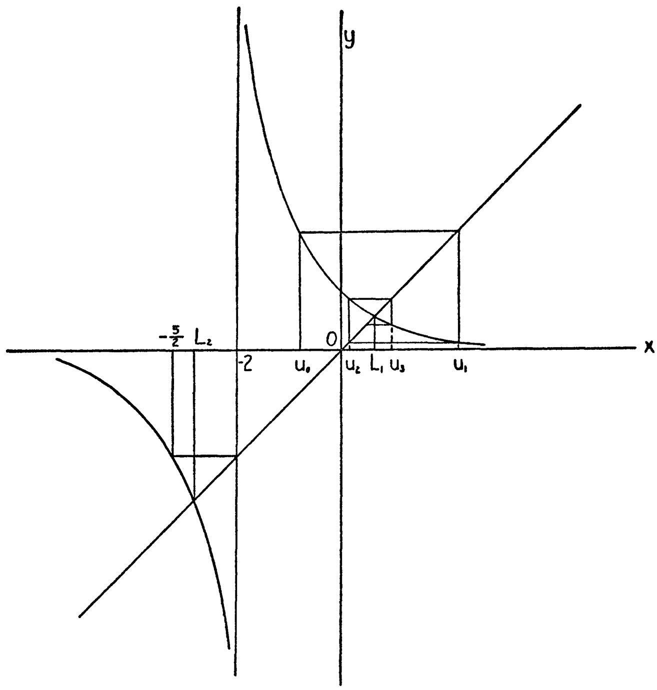


FIG. 4

If $u_0 > -2$, then $\{u_{2n}\}$ is either a monotonic increasing or a monotonic decreasing bounded sequence, while correspondingly, $\{u_{2n-1}\}$ is monotonically decreasing or increasing and bounded. In either case, the sequence $\{u_n\}$ approaches L_1 as a limit. This gives a picture of a rectangular converging spiral. If $u_0 < -5/2$, we go from the negative branch of the hyperbola to the positive branch and the discussion is the same as for $u_0 > -2$.

However, for $-5/2 < u_0 < -2$, the sequence may or may not converge. If any $u_n = -2$, the sequence is not defined from there on and will be considered as non-convergent. The values of u_0 that lead to a $u_n = -2$ can be obtained by considering the inverse sequence $\{a_n\}$ satisfying the equation $a_n = 1/(2+a_{n+1})$ with $a_0 = -2$. Then if $u_0 = a_r$, $u_r = -2$. Thus if $u_0 \neq a_r$ in the given range, the spiral will first unwind about L_2 and then eventually wind around and converge to L_1. Incidentally, we can make the unwinding process take as many steps as desired by taking u_0 sufficiently close to L_2.

In the case $u_r = -2$ the sequence will converge if we define $u_{r+2} = 0$.

An example from the 1952 William Lowell Putnam Competition lends itself readily to the type of geometric analysis illustrated. This was to show that $\lim u_n$, (where $u_{n+1}=\cos u_n$) is independent of u_0 (u_0 real) (see Fig. 5).

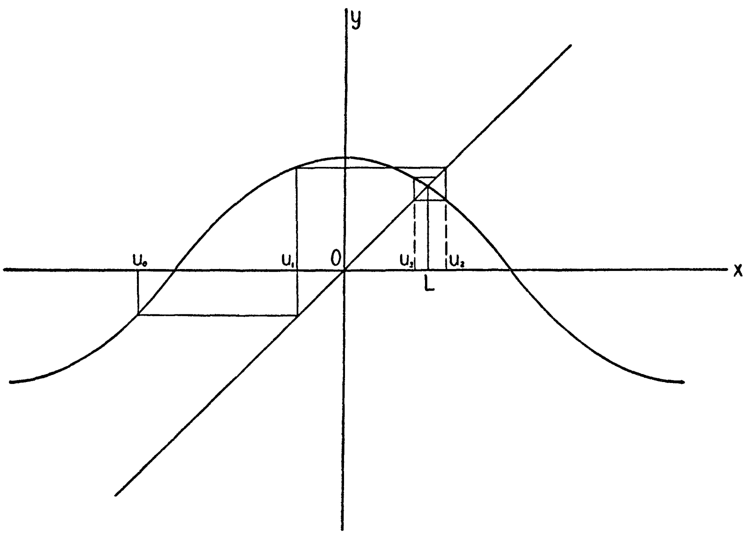


FIG. 5

ON THE DEFINITION OF FUNCTIONS*

H. P. THIELMAN, Iowa State College

1. Fundamental concepts. We shall begin with certain undefined concepts such as set and element of a set. We might give some synonyms of these terms, but the synonyms would then have to be left undefined. Thus a set might be described as a collection of definite, distinct objects associated in thought. By the word "definite" we mean here the following. Given a set S and an object, one and only one of the next two statements is true: the object is an element of the given set; the object is not an element of the given set. It is not required that we, nor anyone else, need to know which of these statements is true. Thus the collection of ladies present that are over sixteen years of age at this moment is a set, even though we or anybody may not know whether a given lady belongs to this set or not. The word "distinct" is used here to indicate that no object can be considered as an element of a given set more than once. For example, the collection of letters a, b, a, c, b constitutes the set a, b, c. In contradistinction to the expression "set of elements" (or set of sets) we shall use the terms "collection of elements" (or collection of sets) to indicate that it is not

* An excerpt from an address presented under the title *Types of Functions* to the Minnesota Section of the Mathematical Association of America at the invitation of the Executive Committee on May 10, 1952. Another excerpt from this address is published in the March issue of this MONTHLY.

implied that the elements (or the sets) be all distinct.

When a symbol such as a letter, say x, stands for an unspecified element of a set, then this symbol x is said to *vary* over the set, and the symbol x is called a *variable* on the set. If x varies over a set which consists of only one element then x is called a constant.

An *ordered pair of elements* is a collection which consists of two elements one of which has been designated as the first. If an ordered pair consists of the elements a and b of which a has been designated as the first, we indicate this by the symbol (a, b). The word "ordered" refers only to the order in which the elements appear within the parentheses. Two ordered pairs (a, b) and (c, d) are equal if and only if $a=c$, and $b=d$.

A *relation* is a set of ordered pairs. For example, the relation $>$ (greater) is the set of all ordered pairs (x, y) of real numbers such that $x-y$ is a positive real number. The *domain of definition of a relation* is the set which consists of all the first elements, and the *range of the relation* is the set which consists of all the second elements of the ordered pairs.

2. Function. A *function* is a set of ordered pairs such that no two ordered pairs have the same first element. The set which consists of all the first elements of the ordered pairs of the given function is called the *domain of definition of the function.* The set which consists of all the second elements of the ordered pairs is called the *range of the function.*

Let f be a given function with domain of definition X, and with range Y. If x stands for an unspecified element of X, x is called the *independent variable* of the given function. If y stands for an unspecified element of the range Y, then y is called the *dependent variable* of the given function. For a given ordered pair (x, y) of f, y is called the *image* of x under f, while x is called the *counter image* or *source* of y under f. The image of x under f is also called the *value of the function f at x*, and is denoted by $f(x)$. A function f whose domain of definition is X, and whose range is Y is frequently denoted by $f: X \to Y$, and is referred to as a function *on* X *onto* Y. If X is a subset of X_1, and Y is a subset of Y_1, then f is said to be a function *from* X_1 *to* Y_1, or *from* X_1 *into* Y_1.

3. Inverse function. Let f be a function on X onto Y. Thus f is a set of ordered pairs (x, y) where x is an element of X, and y is an element of Y. If f is such that no two of its ordered pairs have the same second element, then the function obtained from f by interchanging in each ordered pair (x, y) the places of x and y is called the *inverse function* of f. This function is indicated by f^{-1}. Its domain of definition is Y, and its range is X.

The symbols $f(x)$, $f^{-1}(y)$, or $y=f(x)$, $x=f^{-1}(y)$ should be used in referring to a function and its inverse only when it is clear from the context what the domains of definition and the ranges of these functions are.

4. Inverse relations. Let f be a function on X onto Y. If f is such that two or more of its ordered pairs have the same second element, then the set of ordered pairs obtained from f by interchanging in each ordered pair (x, y) the places of

y and x does not constitute a function from Y to X, but it does constitute a relation. The symbol $\{f^{-1}(y)\}$ can be used to denote the set which consists of all those elements of X which are counter images of y under f. Thus for every y in the range Y of f, $\{f^{-1}(y)\}$ represents a subset of X. If the inverse function f^{-1} exists, then for each y of Y, $\{f^{-1}(y)\}$ represents the set which consists of the unique element $f^{-1}(y)$.

The definition of function leaves the nature of the sets which constitute the domain of definition and the range of a function unspecified. These sets may be sets of sets, sets of real or complex numbers, sets whose elements are vectors or other arbitrary sets.

If the range of a function is a subset of the set of real numbers, the function is called a *real function*, if the domain of definition of a function is a subset of the real numbers we have a function of a *real variable*. If the domain of definition of a function is a set of sets, we have a *set function*. A set function whose range is a subset of the set of real numbers is called a *real set function*.

5. One-to-one correspondence. If f is a function on X onto Y which has an inverse f^{-1} on Y onto X, then f and f^{-1} are said to constitute a ***one-to-one correspondence*** between X and Y.

6. Unrestricted functions. Let f be a real function of a real variable. No other restrictions are put on this function. What can be said of such a function? This question has been partially answered in recent years, and the answer is quite different from what had been surmised by some mathematicians. Even E. W. Hobson in his 1921 edition of the Theory of Functions of Real Variables wrote: "No elaborate theory is required for functions which retain their complete generality, . . . since few deductions of importance can be made from the definition which will be valid for all functions." This conjecture was proven to be false by the late Professor Henry Blumberg who has developed an extensive theory which reveals that every unrestricted real function of real variables possesses many properties which are far from obvious and yet have a beauty of simplicity that has attracted the interest of many mathematicians. Since this theory has been presented by its originator in a number of nontechnical addresses before the American Mathematical Society, and since these addresses have been published and are easily accessible we shall not elaborate on this very interesting topic here.*

Some of these results on unrestricted functions of real variables have very recently been extended to relations in general neighborhood spaces.†

It should perhaps be mentioned that the term multiple-valued function which is frequently met in mathematical literature means a relation, that is,

* Blumberg, Henry, Properties of unrestricted real functions, Bull. Amer. Math. Soc., vol. 32, pp. 132–148, 1926; Methods in point sets and the theory of real functions, Bull. Amer. Math. Soc., vol. 36, pp. 809–830, 1930.

† Block, H. D., and Cargal, B., Arbitrary mappings, Proc. Amer. Math. Soc., vol. 3, 1952, pp. 937–941.

it is a name for a set of ordered pairs in which two distinct ordered pairs may have the same first element. If one admits the term multiple-valued function, then the concept *function* as given in section 2 of the present paper would be described as a single-valued function. In recent years the tendency in mathematical literature has been to limit the term function so as to mean a single-valued function. An equation such as $x^2+y^2=1$, where $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, does not define a function. There exist infinitely many functions whose images and counter images satisfy the condition imposed on them by the equation of the preceding sentence. Thus if $y=\sqrt{1-x^2}$ if x is a rational number, and $y=-\sqrt{1-x^2}$ if x is an irrational number, there is defined a function for which $x^2+y^2=1$. By a proper reassignment of the values of y for given values of x, arbitrarily many functions can be constructed whose images and counter images satisfy the given equation in x and y.

It is gratifying to see that the unambiguous definition of a function, which is advocated here, has been included in a new elementary textbook.‡

‡ Randolph, John F., Calculus, p. 10, Macmillan Co., 1952.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1061. *Proposed by Walter Penney, Washington, D.C.*

Solve for n, given that the equation

$$\sum_{i=1}^{n} (x+i-1)(x+i) = 10n$$

has roots r and $r+1$.

E 1062. *Proposed by Leo Moser, University of Alberta*

(1) Find six positive integers, not exceeding 24, such that the sums of the numbers in the possible subsets of those numbers will all be different.

(2) Prove that no seven positive integers, not exceeding 24, can have sums of all subsets distinct.

E 1063. *Proposed by J. V. Whittaker, U.C.L.A.*

Show that if $a \geqq 2$ and $x > 0$, then $a^x + a^{1/x} \leqq a^{x+1/x}$, equality holding if, and only if, $a = 2$ and $x = 1$.

E 1064. *Proposed by Jacob Samoloff and Albert Wilansky, Lehigh University*

Let $f(x)$ be continuous and $f'(x)$ exist in a neighborhood of $x = c$. Suppose that there exists a continuous function $\theta(h)$, with $0 < \theta(h) < 1$, satisfying the equation

$$f(c + h) - f(c) = hf'[c + h\theta(h)].$$

Does it follow that $f'(x)$ is continuous at $x = c$?

E 1065. *Proposed by C. S. Ogilvy, Syracuse University*

Find the largest plane section of a given solid right circular cylinder.

SOLUTIONS

The Game of Nim

E 1031 [1952, 551]. *Proposed by A. J. Friedland, Brookhaven National Laboratory, Upton, L.I.*

A certain game is played between two participants in the following manner. A set of numbers is written down and the players alternately cross out any one of the numbers and substitute any smaller number until nothing remains but zeros. The player putting down the last zero is the winner. What method of play should one follow in order to win this game?

Solution by C. A. Swanson, University of British Columbia. The game is nothing but *Nim* in disguise. A complete treatment is given by C. L. Bouton, *Annals of Mathematics*, Series 2, Vol. III, pp. 35–39 (October 1901). To win the game from an uninitiated opponent, one writes all the numbers in the binary scale, adds the coefficients of each power of two so obtained, and at each play reduces one of the given numbers so that the sum of the coefficients of each power of two becomes an even number.

There is a machine in the Department of Physics at the University of British Columbia which plays this game for a particular set of given numbers. The machine not only plays to win, but it detects an opponent who tries to cheat, and then refuses to play.

Also solved by Julian Braun, Richard Courter, E. T. Frankel, Michael Goldberg, R. E. Greenwood, S. W. Hahn, B. A. Hausmann, Patricia James, John Jones, Jr., M. S. Klamkin, Octave Levenspiel, D. M. Mandelbaum, Leo Moser, W. J. Pervin, D. L. Silverman, G. W. Walker, and the proposer.

Some ready references where this game is discussed are: Ball-Coxeter, *Mathematical Recreations*, 11th ed., 1939; Uspensky and Heaslet, *Elementary Number Theory*, 1939; Hardy and Wright, *An Introduction to the Theory of*

Numbers, 1938; Northrop, *Riddles in Mathematics*, 1944; Kraitchik, *Mathematical Recreations*, 1942; Jones, *Elementary Concepts of Mathematics*, 1947.

A digital computer has been designed to play the game. See *Electronics*, XXV, no. 11, pp. 155–157 (November, 1952): "Digital Computer Plays Nim."

The MONTHLY has published a number of articles on the game of Nim. See for example, D. P. McIntyre, "A New System for Playing the Game of Nim" [1942, 44–45]; E. U. Condon, "The Nimatron" [1942, 330–332]; L. S. Recht, "The Game of Nim" [1943, 435].

Silverman proposed to study the variation wherein each player has the option of playing on n numbers, where $n>1$.

A Differential Equation

E 1032 [1952, 551]. *Proposed by E. W. Marchand, Eastman Kodak Company, Rochester, N.Y.*

Solve the differential equation $(dy/dx)^3-3y(dy/dx)^2+4y^3=a$.

Solution by C. R. Sparks, Hampton Institute. By differentiating the given equation with respect to x we obtain

$$(dy/dx)(d^2y/dx^2 - dy/dx - 2y)(dy/dx - 2y) = 0, \tag{1}$$

which must be satisfied by any solution y of the original equation. It follows that any solution y must have one of the following three forms:

$$y = C_1, \qquad y = C_2e^{2x} + C_3e^{-x}, \qquad y = C_4e^{2x}. \tag{2}$$

It is readily seen that the third form does not satisfy the original equation (unless $a=0$). By substituting each of the first two forms of (2) into the original equation we easily find, respectively,

$$y = (a/4)^{1/3}, \qquad y = (a/27K^2)e^{2x} + Ke^{-x},$$

where K is an arbitrary constant.

Also solved by A. N. Aheart, A. P. Boblétt, J. J. Corliss, H. M. Feldman, R. R. Gutzman, Sherman Kingsbury, M. S. Klamkin, W. H. McKenzie, R. V. Muguercia, Herbert Schalz, O. E. Stanaitis, F. Underwood, J. E. Wilkins, Jr., and the proposer.

The Balancing Beam

E 1033 [1952, 551]. *Proposed by G. B. Robison, University of Connecticut*

A horizontal plank is balanced on a cylindrical surface whose elements are perpendicular to the length of the plank. What is the cross section of the cylindrical surface if the plank has neutral equilibrium?

Solution by P. F. Hultquist, Boulder, Colo. With the plank in a horizontal position take the x-axis parallel to the plank and through the center of gravity, and take the y-axis vertically downward through the center of gravity and

point of support. Let the coordinates of the point of support when the plank is horizontal be $(0, a)$, and the coordinates of any other point of support be (x, y). Since the plank remains always tangent to the curve, and the center of gravity remains always on the x-axis vertically in line with the point of tangency, we can write $y=a \sec \theta$, where θ is the angle of inclination of the plank. Then

$$y = a\sqrt{1 + (dy/dx)^2},$$

whence we obtain the explicit equation,

$$y = a \cosh (x/a),$$

the equation of a *catenary*.

The solution is valid until the plank is tilted enough to cause slipping. This will occur when the component of weight in the direction of the tangent exceeds the coefficient of friction times the normal force between plank and curved surface. The value of the critical angle is $\theta = \tan^{-1} \mu$, where μ is the coefficient of friction.

Also solved by J. W. Baldwin, C. W. Bruce, Vern Hoggatt and Octave Levenspiel (jointly), M. S. Klamkin, C. S. Ogilvy, F. Underwood, and the proposer.

Four Digit Numbers and Their Reverses

E 1034 [1952, 551]. *Proposed by M. Narasimhamurthy, Presidency College, Madras, India*

Add to its own reverse the difference between any four digit number and its reverse. What are the possible results and what are the probabilities of getting each of these results?

Solution by G. W. Walker, Buffalo, N.Y. For the sake of neatness we assume that any of the digits may be 0. There are then 10^4 possible four digit numbers. Let the original number be $1000a+100b+10c+d$.

Case I. Suppose $a=d$, and $b=c$. The probability for this is 1/100.

In this case, the difference between the original number and its reverse will be 0000; and the sum of this number and its reverse will also be 0000.

Case II. Suppose $a \neq d$, and $b=c$. The probability for this is 9/100.

In this case, the difference between the original number and its reverse will be $999(a-d)$ or $999(d-a)$, whichever is positive, the number in parentheses being some integer between 1 and 9, inclusive, which we will call e. Then the four digits making up the difference between the original number and its reverse will be, respectively, $(e-1)$, 9, 9, and $(10-e)$. Reversing this and adding will yield in every case 10989.

Case III. Suppose $a=d$, and $b \neq c$. The probability for this is 9/100.

In this case, the difference between the original number and its reverse will

be $90(b-c)$ or $90(c-b)$, whichever is positive, the number in parentheses being some integer between 1 and 9, inclusive, which we will call f. Then the four digits making up the difference between the original number and its reverse will be, respectively, 0, $(f-1)$, $(10-f)$, and 0. Reversing this and adding will yield in every case 0990.

Case IV. Suppose $a>d$, and $b>c$, or else $a<d$, and $b<c$. The probability for this is 81/200.

In this case, the difference between the original number and its reverse will be $999e+90f$. The four digits making up this number will be, respectively, e, $(f-1)$, $(9-f)$, and $(10-e)$. Reversing this and adding will yield 10890.

Case V. Suppose $a>d$, and $b<c$, or else $a<d$, and $b>c$. The probability for this is 81/200.

In this case, the difference between the original number and its reverse will be $999e-90f$. The four digits making up this number will be, respectively, $(e-1)$, $(10-f)$, $(f-1)$, and $(10-e)$. Reversing this and adding will yield 9999.

Summing up: 1% of all possible cases will yield 0000, 9% will yield 10989, 9% will yield 0990, $40\frac{1}{2}$% will yield 10890, and $40\frac{1}{2}$% will yield 9999.

If, in order to qualify as a four digit number, the first digit has to be other than zero, the solution will not be as neat. If, besides, initial zeros are dropped out before reversing the difference, the solution will become quite messy, and not interesting.

Also solved by I. A. Dodes, J. D. Haggard, S. W. Hahn, B. A. Hausmann, Vern Hoggatt and Octave Levenspiel (jointly), M. S. Klamkin, P. W. Allen Raine, Azriel Rosenfeld, C. A. Swanson, and the proposer.

Hahn generalized the problem to the situation where the four digit numbers are expressed in an arbitrary scale m. The proposer considered the extension to numbers of n digits.

The Circle Through the Feet of the Symmedians of a Triangle

E 1035 [1952, 551]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that if the circle passing through the feet of the symmedians of a non-isosceles triangle of sides a, b, c is tangent to one side, then the quantities b^2+c^2, c^2+a^2, a^2+b^2, arranged in some order, are consecutive terms of a geometric progression.

Solution by Chih-yi Wang, Hampton Institute. Let a, b, c be the sides BC, CA, AB of a non-isosceles triangle ABC, and A', B', C' the feet of the corresponding symmedians. For definiteness let the circle passing through A', B', C' be tangent to BC at A' and intersect CA, AB at M and N respectively. It is well known that

$$(1) \qquad BA'/A'C = c^2/b^2, \qquad CB'/B'A = a^2/c^2, \qquad AC'/C'B = b^2/a^2.$$

By the aid of (1) we obtain

$$A'C = ab^2/(b^2+c^2), \qquad BA' = ac^2/(b^2+c^2),$$

and analogous relations for CB', $B'A$, AC', and $C'B$, which together with the relations $(BN)(BC')=(BA')^2$, $AN=AB-NB$; $(CM)(CB')=(A'C)^2$, $AM=AC-MC$ imply

$$AN = c(b^4+b^2c^2+c^4-c^2a^2)/(b^2+c^2)^2$$

and

$$AM = b(b^4+b^2c^2+c^4-a^2b^2)/(b^2+c^2)^2.$$

By using the relation $(AB')(AM)=(AC')(AN)$, we get, after simplification,

$$(b^2-c^2)(b^2+c^2)^2 = (b^2-c^2)(a^2+b^2)(a^2+c^2).$$

Since the triangle ABC is non-isosceles, the desired result follows.

Also solved by Arthur Gregory, Vern Hoggatt and Octave Levenspiel (jointly), L. M. Kelly, B. R. Leeds, Roscoe Woods, and the proposer.

Hoggatt showed that $(a^2+b^2)/(b^2+c^2)$, the ratio of the progression, is equal to $A'B'/A'C'$. Gregory's solution employed cartesian analysis.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4533. *Proposed by R. Kissling, Student, University of California, Berkeley*

Given $a^n+b^n=c^n$, with a, b, c, n integers and $a>b>1$, $n\geqq 2$; $\sigma_n(k)$ being the sum of the nth powers of all divisors of k, prove

$$(1) \qquad \left|1-\frac{\sigma_n(c)}{\sigma_n(a)+\sigma_n(b)}\right| < \frac{2n-1}{n(n-1)} < \frac{2}{n-1},$$

$$(2) \qquad \lim_{k\to\infty} \frac{\sigma_k(c)}{\left\{[\sigma_k(a)]^{n/k}+[\sigma_k(b)]^{n/k}\right\}^{k/n}} = 1.$$

4534. *Proposed by Frank Harary, University of Michigan*

If n is an odd prime and M is a symmetric n by n matrix each of whose rows is a permutation of $1, \cdots, n$, then the main diagonal of M is also such a permutation.

4535. *Proposed by M. S. Robertson, Rutgers University*

Let the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, be regular for $|z| \leqq 1$ and let it map $|z| \leqq 1$ onto a simply-connected domain D, star-like when viewed from the origin. Show that

$$2\left(\sum_{n=1}^{\infty} n \mid a_n \mid^2\right)^2 \leqq \sum_{n=1}^{\infty} (n+1) \mid a_1 a_n + a_2 a_{n-1} + \cdots + a_n a_1 \mid^2,$$

with equality only for $f(z) = a_1 z$.

4536. *Proposed by C. E. Stanaitis, St. Olaf College, Northfield, Minn.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \sin n\theta}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos n^2\theta \sin n\theta}{n}$$

are uniformly convergent in any interval, and that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \cos n\theta}{n}$$

is divergent.

4537. *Proposed by Albert Wilansky, Lehigh University*

Given that $\sum_n \sum_k a_{nk} x_k$ converges (as an iterated sum) whenever $\{x_k\}$ is a sequence such that $\sum |x_k| < \infty$. Show that $\sum_n \sum_k a_{nk} x_k = \sum_k \sum_n a_{nk} x_k$ for each such $\{x_k\}$.

SOLUTIONS

Three Related Triangles

4470 [1952, 46]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, three lines a', b', c' drawn through the vertices A, B, C determine by their intersections a triangle $A'B'C'$, and their isogonals a'', b'', c'' determine a triangle $A''B''C''$. (1) Show that the orthic triangles of $A'B'C'$ and $A''B''C''$ have equal perimeters. (2) If, further, a', b', c' are equally inclined to AB, BC, CA, show that the circles $A'B'C'$ and $A''B''C''$ are symmetric with respect to the line joining the symmedian point to the circumcenter.

Solution by the Proposer. (1) Let Δ, Δ', Δ'' denote the areas of triangles ABC, $A'B'C'$, $A''B''C''$, and R, R', R'' their circumradii. Now, if a', b', c' divide angles A, B, C into angles α and α', β and β', γ and γ', one can show that

$$\frac{\Delta'}{\Delta} = \frac{(\sin\alpha\sin\beta\sin\gamma - \sin\alpha'\sin\beta'\sin\gamma')^2}{\sin A\sin B\sin C\sin A'\sin B'\sin C'} = \frac{4R^2R'^2D^2}{\Delta\Delta'},$$

where D^2 is the numerator of the middle fraction. Similarly,

$$\Delta''/\Delta = 4R^2R''^2D^2/\Delta\Delta''.$$

It now follows that $\Delta'/R' = \Delta''/R''$. But, if triangles $A'B'C'$ and $A''B''C''$ are acute, $2\Delta'/R'$ and $2\Delta''/R''$ are the perimeters of their orthic triangles.

If either of the triangles $A'B'C'$, $A''B''C''$ is not acute, the statement of the theorem needs modification. For example, if triangle $A'B'C'$ is obtuse angled at A', then the perimeter $2p'$ of its orthic triangle must be replaced by the quantity $2(p'-s_{A'})$, where $s_{A'}$ is the side of the orthic triangle corresponding to vertex A'.

(2) If $\alpha=\beta=\gamma$ then $A'=B''$, $B'=C''$, $C'=A''$, whence, from above, $\Delta'/\Delta = \Delta''/\Delta$, or $R'=R''$. One can also show, using barycentric coordinates, that the radical axis of the circles $A'B'C'$, $A''B''C''$ coincides with the Brocard diameter of triangle ABC. This completes part (2).

Also solved by Joseph Langr.

The Moduli of the Zeros of Polynomials

4474 [1952, 109]. *Proposed by Ky Fan, University of Notre Dame*

Let the zeros $z_i(1 \leqq i \leqq n)$ of the polynomial

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1} + z^n$$

be so arranged that $|z_1| \geqq |z_2| \geqq \cdots \geqq |z_n|$. It is a classical result due to Carmichael and Mason that

$$(1) \qquad |z_1| \leqq \sqrt{1 + \sum_{i=0}^{n-1} |a_i|^2} = B.$$

Using Jensen's inequality, M. Fujiwara obtained a stronger inequality

$$(2) \qquad |z_1z_2\cdots z_k| \leqq B, \qquad (1 \leqq k \leqq n).$$

Prove that this result (2) can still be slightly improved as follows:

$$(3) \qquad |z_1z_2\cdots z_k|^2 \leqq \tfrac{1}{2}(B^2 + \sqrt{B^4 - 4|a_0|^2}), \qquad (1 \leqq k \leqq n).$$

Solution by the Proposer. Using the coefficients of the polynomial $f(z)$, we form the matrix of order n:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

Then the characteristic equation of this matrix A is $f(z)=0$. In other words, the zeros of our polynomial $f(z)$ are exactly the eigenvalues of A. Let A^* denote the complex conjugate transpose of A, i.e., the (i, j)-element of A^* is the complex conjugate of the (j, i)-element of A. Form the product AA^*:

$$AA^* = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -\bar{a}_1 \\ 0 & 1 & 0 & \cdots & 0 & -\bar{a}_2 \\ 0 & 0 & 1 & \cdots & 0 & -\bar{a}_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -\bar{a}_{n-1} \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & \sum_{i=0}^{n-1} | a_i |^2 \end{pmatrix}.$$

Then a simple calculation gives the characteristic equation of the matrix AA^*:

$$(w-1)^{n-2}\left\{w^2 - \left(1 + \sum_{i=0}^{n-1} | a_i |^2\right) w + | a_0 |^2\right\} = 0.$$

In this we have assumed $n \geqq 2$, the case $n=1$ of our problem being trivial.

Hence the eigenvalues of AA^* are:

$$w_1 = \frac{1}{2}\left\{1 + \sum_{i=0}^{n-1} | a_i |^2 + \sqrt{\left(1 + \sum_{i=0}^{n-1} | a_i |^2\right)^2 - 4 | a_0 |^2}\right\},$$

$$w_2 = w_3 = \cdots = w_{n-1} = 1,$$

$$w_n = \frac{1}{2}\left\{1 + \sum_{i=0}^{n-1} | a_i |^2 - \sqrt{\left(1 + \sum_{i=0}^{n-1} | a_i |^2\right)^2 - 4 | a_0 |^2}\right\}.$$

It can be easily seen that $w_1 \geqq 1 \geqq w_n$.

Now, our inequality (3) is an immediate consequence of the following result of H. Weyl, Inequalities between two kinds of eigenvalues of a linear transformation, *Proc. National Acad. Sci. U.S.A.*, v. 35, 1949, pp. 408–411: *For an arbitrary $n \times n$ matrix A, if the eigenvalues z_i of A and the eigenvalues w_i of AA^* are so*

arranged that $|z_i| \geqq |z_{i+1}|$, $w_i \geqq w_{i+1}$, *then*

$$|z_1 z_2 \cdots z_k|^2 \leqq w_1 w_2 \cdots w_k, \qquad (1 \leqq k \leqq n).$$

It is hoped that a direct, matrix-free proof of (3) may yet be submitted.

Generalization of the Fundamental Theorem of Algebra

4475 [1952, 110]. *Proposed by D. J. Newman, Harvard University*

Let $f(z)$ be continuous throughout the complex z-plane, and suppose that $f(z)/z \to 1$ as $z \to \infty$. Show that $f(z)$ must have a zero.

I. *Solution by Ramon Moore, Ballistic Research Laboratory, Aberdeen, Md.* From the hypothesis there is a real number $R > 0$, sufficiently large, such that

$$\left|\frac{f(z)}{z} - 1\right| < 1 \quad \text{or} \quad |f(z) - z| < |z|$$

for $|z| > R$. Hence as z traces out a circle C with center at the origin of the complex plane and radius $t_0 > R$, $f(z)$ will trace out a closed curve Γ in whose interior the origin lies. But $f(0)$ is some point ζ in the complex plane and since $f(z)$ is a continuous function of z, as the radius t of circles traced out by z various continuously from t_0 to 0 the closed curve traced out by $f(z)$ will vary continuously from Γ to the single point ζ. Hence there is some $\bar{t}$ for which the curve traced out by $f(z)$, as z traces out the circle of radius $\bar{t}$, passes through the origin of the complex plane.

II. *Solution by J. W. Gaddum, National Bureau of Standards, Los Angeles.* Let $g(z) \equiv z - f(z)$. Then $g(z)$ is a continuous function mapping the whole z-plane into itself. Also, $g(z)/z$ approaches zero as $z \to \infty$. Let C_r be the closed region bounded by the circle of radius r and with center at the origin. Now for some r, g maps C_r into itself. Otherwise there would be an infinite sequence $z_1, z_2, \cdots,$ with $|g(z_n)|/|z_n| > 1$, and $g(z_n)/z_n$ would not have the limit zero. For a C_r which is mapped by g into itself there is a fixed point, that is, a z_0 such that $g(z_0) = z_0$. Thus $f(z_0) = 0$.

Also solved by L. M. Kelly, J. H. Michael, E. J. Mickle, L. A. Ringenberg, and the Proposer.

Editorial Note. It is evident that the argument in I above holds also if the hypothesis is changed by replacing $f(z)/z$ by $f(z)/z^n$. $f(z)$ a polynomial is a special case.

Number of Unrestricted Partitions of n

4476 [1952, 110]. *Proposed by T. M. Apostol, California Institute of Technology*

Let $p(n)$ be the number of unrestricted partitions of n, and let $\sigma(n)$ be the sum of the divisors of n. Prove

$$p(n) = \sum_{m=1}^{n} \sum_{\substack{n_1, n_2, \cdots, n_m \\ n_1 + \cdots + n_m = n}} \frac{1}{m!} \frac{\sigma(n_1)\sigma(n_2) \cdots \sigma(n_m)}{n_1 \cdot n_2 \cdots n_m}.$$

For each fixed m, the inner sum is to be taken over all positive integers $n_1, n_2, \cdots, n_m$ whose sum is n, the order of the summands being taken into consideration.

Solution by Leonard Carlitz, Duke University. We have

$$\prod_{m=1}^{\infty} (1 - x^m) = 1 + \sum_{n=1}^{\infty} p(n) x^n.$$

Also

$$-\log \prod_{m=1}^{\infty} (1 - x^m) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^{mk}}{k} = \sum_{m=1}^{\infty} \frac{\sigma(n)}{n} x^n,$$

whence

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} p(n) x^n &= \exp \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n \\ &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left[\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n \right]^m. \end{aligned}$$

Since

$$\left[\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n \right]^m = \sum_{n_1, \cdots, n_m = 1}^{\infty} \frac{\sigma(n_1) \cdots \sigma(n_m)}{n_1 \cdots n_m} x^{n_1 + \cdots + n_m},$$

it follows that

$$p(n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{n_1 + \cdots + n_m = n} \frac{\sigma(n_1) \cdots \sigma(n_m)}{n_1 \cdots n_m}.$$

In the same way, if we put

$$\prod_{m=1}^{\infty} (1 + x^m) = 1 + \sum_{n=1}^{\infty} q(n) x^n,$$

then

$$q(n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{n_1 + \cdots + n_m = n} \sigma'(n_1) \cdots \sigma'(n_m),$$

where

$$\sigma'(n) = \sum_{k \mid n} \frac{(-1)^{k-1}}{k}.$$

Also solved by the proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

The Nature of Number. An Approach to Basic Ideas of Modern Mathematics. By Roy Dubisch. New York, The Ronald Press Co. 1952. xii+159 pages. $4.00.

"The objectives of this book may be stated as follows: (1) To portray the development of a single line of mathematical thought from its most primitive beginnings of contemporary times, and (2) to make the abstractions of advanced mathematics easier to grasp by pointing out that the concreteness of elementary mathematics is largely illusory and that, actually, all basic mathematics is abstract. The particular line of procedure employed is that of the development of the number concept." With these objectives in mind, the author has written a book that should be stimulating reading for young students and will present some modern points of view to older people who have not followed recent developments. The book would also be an excellent foundation for a reading course—a type of instruction which, with the shortage of teachers, might become more popular in colleges. A few minor objections: The definition of transcendental number (p. 77) should be corrected and clarified. On p. 73, the author might well have proved that the square of an odd number is odd instead of saying merely that "the fact can be rigorously proved." A word about the scope of symbols would have been in line with the author's emphasis on notational questions. For instance, he might have explained that $2+3\cdot 4$ stands for $2+(3\cdot 4)$ rather than for $(2+3)\cdot 4$. On p. 34 the author quotes an (anonymous) reader of the manuscript as saying that geometry actually is a mixture of postulates, geometric intuition, and abracadabra. This statement might better have remained unquoted since it must have been made in a moment of absent-mindedness when that reader forgot about the work of Pasch, Pieri, Hilbert, and others. But these are minor points. All in all, one would wish that more books of this type were written and that they were widely read.

KARL MENGER
Illinois Institute of Technology

Society of Actuaries' Textbook on Life Contingencies. By C. W. Jordan. Chicago, Society of Actuaries, 1952. xi+331 pages. $8.00.

"Life Contingencies" is the name applied to a branch of mathematics which deals with probabilities of human survivorship and which provides the basis for financial calculations involving a combination of these probabilities with functions of compound interest. Through the latter, the technical basis for a life

insurance policy is established; the life insurance applications of Life Contingencies theory are, in fact, its most important applications and the subject has developed for the most part along lines dictated by the needs of the life insurance companies.

Until now, the standard reference work in this field has been the textbook by E. F. Spurgeon, published by the British Institute of Actuaries. For many years there has been a need for a treatment of the subject from the American point of view, similarly comprehensive in scope but reflecting life insurance practices in this country rather than the practices which obtain in Great Britain. The Society of Actuaries therefore commissioned Mr. C. W. Jordan, a Fellow of the Society and Associate Professor of Mathematics at Williams College, to write the present volume.

Mr. Jordan has combined broad knowledge of his subject with great pedagogic ability derived from his experience as a teacher, and has written a book of which the Society of Actuaries is understandably proud and which students will respect. He has stated clearly the fundamental principles upon which the theory is built and has developed these principles in a way which emphasizes throughout the essential unity of the different parts of the theory.

Actuaries naturally comprise the group most interested in this branch of mathematics, but the subject should also be of interest to others both on its merits as a practical application of probability theory and because the methods employed may find application in other fields.

D. H. Harris
Equitable Life Assurance
Society of the United States

Theory of Numbers. By B. M. Stewart, New York, The Macmillan Company, xiii+267 pages, $5.50.

The preface of this book opens with the words "The present work makes no pretense at being more than a textbook; the subject matter is classical and the only attempt at originality is in the choice of topics and the manner of presentation." In this attempt the author has been highly successful.

The subject matter is motivated in detail and the explanations are full. There is none of that forbidding conciseness which often makes a textbook unpleasant to students. A simple illustration is the statement *verbatim* of Theorem G, 8.1. "If $(a, m)=d$, then $ax\equiv b \bmod m$ has no solutions when d is not a divisor of b; but if d divides b, there are exactly d solutions." There are exercises and illustrations to satisfy all tastes; they vary from games to group-theoretical questions. Unlike other American writers the author does not hesitate to introduce simple concepts from group and set theory, and to use the advantages they have to offer.

In addition to the standard topics (divisibility, congruences, quadratic residues, forms, additive number theory) there is an introduction to algebraic

numbers *via* the question of unique factorization, and a full development of the real number system from Peano's axioms. One can take exception to a strict interpretation of the author's assertion (p. 205) that these axioms are categorical. On the other hand he avoids the classical blunder of an incomplete definition of addition from the axioms (p. 201).

One can also object to the assertion (p. 205) that "Ordinarily it is rather restricted and uninteresting to study a mathematical system that is categorical." Finite dimensional vector spaces and separable Hilbert space seem to me adequate counterexamples. But this is a small matter compared to the author's achievement.

HARRY POLLARD
Cornell University and Institute for Advanced Study

NEW BOOKS RECEIVED

A Short Table for the Bessel Functions $I_{n+1/2(x)}$, $(2/\pi)K_{n+1/2(x)}$. By C. W. Jones. Published for the Royal Society at the University Press, Cambridge, 1952. 20 pages. 6s. 6d.

Advanced Mathematics in Physics and Engineering. By Arthur Bronwell. New York, McGraw-Hill Book Company, 1953. xvi+475 pages. $6.00.

Anschauliche Einführung in die Mehrdimensionale Geometrie. By Walther Lietzmann. Germany, R. Oldenbourg, Publisher, Munich. 1952.

Computing Manual. By Fred Gruenberger. Madison, Wisconsin. The University of Wisconsin Press, 1952. 123 pages. $3.00.

Mathematics of Finance. By L. L. Smail. New York, McGraw-Hill Book Company, 1953. x+282 pages. $4.50.

Calculus, A Modern Approach. By Karl Menger. Printed by Illinois Institute of Technology (Mimeographed form). 1952. xxv+255 pages. $4.50.

Associated Measurements. By M. H. Quenouille. New York, Academic Press, Inc., 1952. x+242 pages. $5.80.

CLUBS AND ALLIED ACTIVITIES

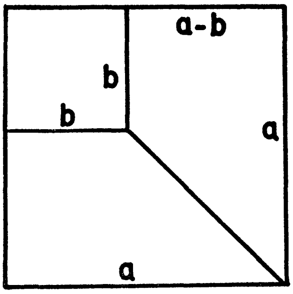
EDITED BY H. D. LARSEN, Albion College

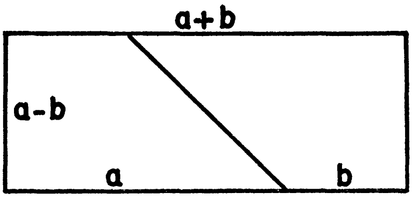
Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

(a^2-b^2): A RECREATION

The familiar formula for factoring a^2-b^2 can be made the basis of an elementary recreation, perhaps suitable for secondary school students for whom the available recreational material in algebra is not too plentiful.

1. A geometric proof of the formula, $a^2-b^2=(a+b)(a-b)$.
Behold!





2. A proof that a rectangle of given perimeter has the greatest area when its sides are equal.

Let the perimeter of the rectangle be $4a$, with a constant, and let the sides be $a+b$ and $a-b$. Then the area of the rectangle is $(a+b)(a-b)=a^2-b^2$. Clearly, this is a maximum when $b=0$.

3. A short method of squaring numbers near 50, 100, 500, 1000, *etc.*

We use the rule, $a^2=(a+b)(a-b)+b^2$. This rule is very useful if one has memorized the squares of the first 25 integers. For example,

$$83^2 = (83 + 17)(83 - 17) + 17^2 = (100)(66) + 289 = 6889.$$

$$978^2 = (978 + 22)(978 - 22) + 22^2 = (1000)(956) + 484 = 956{,}484.$$

4. A method of finding the product of certain small numbers.

If, as in (3), one has memorized the squares of the first 25 integers (or more), he can reduce many multiplications to the difference of two squares. For example,

$$19\cdot 13 = (16 + 3)(16 - 3) = 256 - 9 = 247,$$
$$26\cdot 14 = (20 + 6)(20 - 6) = 400 - 36 = 364.$$

5. Interesting number forms.

$$6^2 - 5^2 = 11$$
$$56^2 - 45^2 = 1111$$
$$556^2 - 445^2 = 111111$$
$$5556^2 - 4445^2 = 11111111$$
$$55556^2 - 44445^2 = 1111111111$$

etc.

These and other forms are derived easily. Thus, let us choose $a+b=1001$ and $a-b=333$. Then $a=667$, $b=334$, and $667^2-334^2=333333$. Indeed,

$$7^2 - 4^2 = 33$$
$$67^2 - 34^2 = 3333$$
$$667^2 - 334^2 = 333333$$
$$6667^2 - 3334^2 = 33333333$$
$$66667^2 - 33334^2 = 3333333333$$

etc.

6. A method for finding square roots.
We use the rule, $a^2=b^2+k(a+b)$, where $k=a-b$. For example, to find $\sqrt{330}$:

$$\begin{array}{rcll} 18^2 &=& 324 & \\ .2(18+18.2) &=& \underline{7.24} & (k=.2) \\ 18.2^2 &=& 331.24 & \\ -.04(18.2+18.16) &=& \underline{-1.4544} & (k=-.04) \\ 18.16^2 &=& 329.7856 & \\ .005(18.16+18.165) &=& \underline{.181625} & (k=.005) \\ 18.165^2 &=& 329.967225 & \\ .0009(18.165+18.1659) &=& \underline{.03269781} & (k=.0009) \\ 18.1659^2 &=& 329.99992281 & \end{array}$$

Can any reader add to this list?

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York, Items must be submitted at least two months before publication can take place.

INTERNATIONAL PRIZE IN HONOR OF GUIDO FUBINI

In memory of Guido Fubini, l'Unione Matematica Italiana will award an international prize in mathematics which has been donated by his friends and admirers.

There will be a single award made in Italian currency equivalent to about 550 grams of gold, and it will be for important contributions in the field of dif-

ferential geometry published between January 1, 1946 and December 31, 1953.

Applicants are invited to submit lists of such works to the Presidency of the Unione Matematica Italiana before March 31, 1954, but the awarding committee, whose decision will be final, may also consider works published within this period by mathematicians who do not so apply.

If there are no suitable works in differential geometry, the committee may also grant the prize for contributions made in the same period to the theory of automorphic functions and related topics.

The committee will consist of: Professor Salomon Bochner, Princeton University; Professor Charles Ehresmann, University of Strasbourg, France; Professor Alessandro Terracini, University of Turin, Italy.

MATHEMATICS INSTITUTE IN RIO DE JANEIRO

An Institute for Pure and Applied Mathematics has been founded in Rio de Janeiro by the National Research Council of Brazil. The new institution will sponsor the publication of the principal Brazilian mathematical journal *Summa Brasiliensis Mathematicae*. The Director of the Institute is the noted mathematician and astronomer, Dr. Lelio I. Gama, Caixa Postal 46, Rio de Janeiro, Brazil.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1953:

Boston College. June 24 to August 4: Professor Calabi, group theory; Professor Eiardi, advanced calculus and differential equations.

The Catholic University of America. June 29 to August 8: Professor Ramler, college geometry, analytic projective geometry, differential equations; Professor Finan, fundamental concepts; Professor Rice, advanced calculus II; Professor Moller, higher algebra I; Miss Handrich, advanced calculus I.

Columbia University, Teachers College. July 6 to August 14: Professor Fehr, teaching arithmetic in the elementary school; Professors Fehr and Rosskopf, research and departmental seminar in teaching mathematics; Professors Fehr and Yates, professionalized subject matter in advanced secondary school mathematics, part I; Professor Rosskopf, foundations of mathematics for teachers; Professors Rosskopf and Shuster, teaching geometry in secondary schools; Professors Rosskopf and Yates, teaching of elementary college mathematics; Miss Schult, supervision and teaching of mathematics in the junior high school; Professor Shuster, field work in mathematics, business arithmetic and mathematics. July 20 to 31: Professors Fehr, Yates and special lecturers, workshop in the theory, construction and use of models and materials in mathematical education; daily from 1:30 to 4 p.m.

De Paul University. June 15 to August 4: Professor De Cicco, tensor analysis and Riemannian geometry, matrices and linear transformations; Professor Caton, theory of oscillations, college geometry.

Kent State University. June 22 to July 31: Professor Jenkins, theory of numbers; Professor Dressler, theory of equations; Professor Stapleford, advanced methods of teaching mathematics in high school. August 3 to September 4: Professor Iwanchuk, vector analysis.

Massachusetts Institute of Technology. July 6 to 10: Professors Norbert Wiener and Y. W. Lee will conduct a one week special program entitled "Mathematical Problems of Communication Theory."

Michigan State College. June 23 to July 31: Professor Bell, matrices and groups, higher algebra II; Mr. Coy, elements of statistics, correlation analysis; Professor Herzog, complex variable III; Dr. J. Kelly, theory of equations, differential equations; Professor L. Kelly, topology, foundations of mathematics; Professor Powell, complex variable I, advanced calculus I. June 23 to August 21: Professor Arnold, analysis of variance, theory of probability; Professor Hill, real variable, advanced calculus II; Professor Olkin, advanced topics in statistics; Professor Parkus, potential theory, partial differential equations; Professor Stelson, solid analytic geometry.

Northwestern University. June 19 to August 22: differential equations; theory of statistics; introduction to the theory of numbers; algebra for teachers; introduction to the theory of groups; vector analysis; analysis seminar.

Purdue University. June 15 to August 8: partial differential equations and applications; advanced calculus I; vector analysis; introduction to theoretical statistics; higher algebra; fundamental concepts of mathematics; functions of a complex variable; Laplace and Fourier transformations; theory of groups; statistical methods; machine methods in statistics; advanced statistical methods. June 22 to August 1: special program for the General Electric Mathematics Fellows; fifty fellowships for secondary school teachers have been established at the University by the General Electric Company.

Syracuse University. June 29 to August 7: Professor Kibbey, introduction to mathematical logic; Professor Exner, analysis of elementary mathematics; Mr. Paul Smith, teaching of high school mathematics, workshop in mathematics education (these courses are specially adapted for secondary school teachers).

University of Buffalo. June 29 to August 8: Professor Schneckenburger, topics in higher geometry; Professor Gehman, foundations of mathematics; Professor Montague, advanced methods for teachers of mathematics.

University of California, Berkeley. June 22 to August 1: Professor Kendall, stochastic processes associated with population growth and with the theory of queues; Professor Neyman, individual research leading to higher degrees. August 3 to September 12: Professor Neyman, individual research leading to higher degrees.

University of Chicago. June 22 to August 29: Professor Halmos, algebra IV (theory of groups, commutative rings), topics in set theory; Professor Hartung, fundamental concepts of mathematics for teachers; Professor Kaplansky, differential geometry; Professor Segal, point set topology, representation of topo-

logical and Lie groups; Professor Stone, Hilbert space; Professor Weil, elementary number theory, seminar on current literature; Professor Zygmund, ordinary differential equations.

University of Colorado. June 15 to August 25: Professor Chowla, advanced calculus, introduction to modern algebra; Mr. Osserman, topology; Professor Britton, vector analysis, functions of a complex variable. June 15 to July 21: Professor Jones, fundamental concepts of arithmetic and algebra; Professor Pingry (University of Illinois), teaching of secondary mathematics, mathematics workshop in curriculum problems. July 22 to August 25: Professor Kendall, history of mathematics.

University of Florida. June 15 to August 15: Professor Cowan, Fourier series; Professor Dostal, vector analysis; Professor Ellis, theory of groups of finite order, differential geometry; Professor Gager, history of elementary mathematics; Professor Gormsen, synthetic projective geometry, advanced college geometry; Professor Lang, functions of a complex variable II; Professor Phipps, foundations of geometry; Professor Smith, advanced topics in calculus I and II; Professor South, theory of probability and theory of sampling.

University of Kentucky. June 22 to August 15: Professor Cowling, advanced calculus; Professor Pence and Miss Baskett, college geometry; Professor Leser, introduction to applied mathematics; Professor Goodman, introduction to the theory of numbers; Professor Ward, higher algebra.

University of Maryland, Department of Mathematics. June 22 to July 31: Professor Good, theory of equations; Professor Jackson higher geometry; Mr. Spencer, vector analysis; Dr. Payne, selected topics in applied mathematics.

University of Michigan. June 22 to August 14: Professor Bartels, hydrodynamics; Professor Bott, operational mathematics; Professor Carver, theory of statistics II and finite differences; Professor Christofferson, teaching of geometry; Professor Copeland, probability and theory of games; Professor Craig, theory of statistics I and multivariate analysis; Professor Dwyer, computational methods; Professor Jones, history of algebra; Professor Leisenring, higher geometry for teachers; Professor Nevanlinna, advanced functions of complex variables; Professor LeVeque, theory of numbers; Professor Lohwater, elementary functions of a complex variable with applications; Professor Myers, functions of a real variable and calculus of variations; Professor Nesbitt, mathematics of life insurance; Professor Nyswander, Galois theory and theory of matrices; Professor Rainville, intermediate differential equations; Professor Rothe, Fourier series and methods in partial differential equations; Professor Tornheim, arithmetic of rings; Professor Young, foundations of mathematics and unified topology.

University of North Carolina. June 11 to July 17: Professor Winsor, elementary algebra from an advanced viewpoint; Professor Garner, history of mathematics; Professor Cameron, fundamental concepts; Professor Linker, differential equations; Professor Mann, topics in analysis; Professor Brauer, some recent results in number theory. July 20 to August 22: Professor Mackie, theory of

equations; Professor Lasley, synthetic projective geometry; Professor Jones, topics in analysis; Professor MacNerney, summability.

University of Pittsburgh. June 8 to July 17: Professor Bryson, differential equations; Professor Blumberg, advanced calculus; Professor Taylor, functions of a complex variable; Professor Laush, functions of a real variable; Professor Elyash, advanced function theory; Professor Bompiani, advanced differential equations. July 20 to August 28: Professor Bryson, differential equations; Professor Blumberg, advanced calculus; Professor Taylor, functions of a complex variable; Professor Laush, functions of a real variable, Lie theory of groups; Professor Bompiani, advanced differential equations (all courses in the July 20 to August 28 session except Lie theory of groups are continuations of courses of the preceding six weeks). June 15 to August 7 (evenings): Professor Barsotti, partial differential equations and Fourier series. June 29 to August 7: Professor Teats, history of mathematics; Professor Blumberg, recreational mathematics for teachers, theory of equations; Mr. Kachun, navigation for teachers.

University of Southern California. June 22 to August 1: Professor Steed, secondary mathematics from an advanced standpoint; Professor Sherman, foundations of mathematics; Professor Whaples (Indiana University), advanced theory of numbers, seminar in algebra; (evenings) advanced calculus; introduction to theory of complex variables; seminar.

University of Washington. June 22 to July 23 and July 24 to August 21: Professor Peterson, linear algebra; Professors Ball and Cramlet, differential equations; Professors McFarlan and Avann, vector analysis; Professor Hewitt, Fourier series; Professor Dekker, advanced Euclidean geometry.

PERSONAL ITEMS

Professor L. S. Hill of Hunter College represented the Association at the inauguration of President B. G. Gallagher of the City College of the City of New York on February 19, 1953.

Professor A. A. Albert of the University of Chicago has been elected to a corresponding membership in the Academia Brasileira de Ciências.

Professor H. B. Curry of Pennsylvania State College has been granted the title of "Visiting Professor Honoraire" of the University of Louvain, Belgium.

Dr. Leila A. Dragonette, research associate at the University of Chicago, is the recipient of the 1952 Research Award of $500 of Sigma Delta Epsilon. This award was made for a paper entitled "Some Asymptotic Formulae for the Mock Theta Series of Ramanujan."

Professor W. L. Duren, Jr. of Tulane University has been appointed to a Faculty Fellowship by the Ford Foundation and is on leave of absence from the University for the year 1952–53.

Associate Professor Maurice L'Abbé of the University of Montreal has been awarded a post-doctoral fellowship by the Government of Canada and is spending the year in France.

Professor C. O. Oakley has received a Ford Foundation grant and is on leave

from Haverford College for the year 1952–53.

Dr. E. R. Reifenberg, Fellow of Trinity College, Cambridge, holds a Commonwealth Fund Fellowship for 1952–53 and is spending the year at the University of California, Berkeley.

Professor Moses Richardson of Brooklyn College has a Ford Foundation Fellowship for the year 1952–53 and is at Princeton University.

Dr. I. M. Rose of the University of Massachusetts has been granted a Ford Foundation Fellowship and is on leave of absence for the current academic year.

Assistant Professor E. H. Spanier and Professor André Weil of the University of Chicago have received Guggenheim Fellowships for 1952–53.

Boston University announces the following appointments to instructorships: Miss Elizabeth A. Shuhany and Mr. V. R. Staknis.

Brooklyn College reports the following: Professor R. A. Johnson, formerly chairman of the Department of Mathematics, has retired; Associate Professor Samuel Borofsky has been appointed Chairman of the Department; Assistant Professor Moses Richardson has been promoted to an associate professorship; Dr. Melvin Hausner, formerly a mathematician with the Rand Corporation, has been appointed to an instructorship; Mrs. Mary D. Boeker has retired.

Carleton College announces the following: Associate Professor K. O. May has been promoted to a professorship; Dr. Anne W. Calloway and Dr. J. M. Calloway have been appointed to assistant professorships; Mr. F. L. Wolf has been appointed to an instructorship.

At Colorado Agricultural and Mechanical College: Professor A. G. Clark, formerly head of the Department of Mathematics, now has the position of Dean of the College and Professor of Mathematics; Professor H. T. Guard has been promoted to the position of Head of the Department.

Creighton University announces the promotions of Associate Professor A. K. Bettinger to a professorship and of Instructor J. P. Markoe to an assistant professorship.

Hampton Institute reports the following: Professor J. D. Eshleman has retired with the title of Professor Emeritus; Miss Rosalind M. Eagleson has been promoted to an assistant professorship.

Iowa State College makes the following announcements: Dr. H. E. Goheen, who has been Assistant Professor of Electrical Engineering at the University of Pennsylvania, has been appointed to an associate professorship; Mr. G. W. Peglar and Mr. David Wend have been appointed to instructorships; Dr. A. M. Feyerherm has been promoted to an assistant professorship; Associate Professor Gertrude A. Herr has retired; Assistant Professor Carl Langenhop has returned after a year's leave of absence at Princeton University; Associate Professor C. G. Maple has returned after a two years' leave of absence with the United States Navy in Washington.

Massachusetts Institute of Technology announces the following appointments: Dr. Kenkichi Iwasawa, formerly a member of the Institute for Advanced Study while on leave from his position as Assistant Professor at the University

of Toronto, has been appointed to an assistant professorship; Dr. Joseph Sampson of Princeton University and University of Paris, Dr. J. B. Serrin, Jr., Princeton University, and Dr. Gerard Washnitzer, also of Princeton University, have been appointed to C. L. E. Moore instructorships.

Northwestern University announces: Dr. Meyer Dwass of the United States Census Bureau has been appointed to an assistant professorship; Dr. Daniel Resch of the University of Michigan and Dr. Maxwell Rosenlicht of Princeton University have been appointed to assistant professorships; Dr. Jacob Dekker of the University of Chicago and Dr. Alex Rosenberg, Air Force Project, University of Michigan have been appointed to instructorships; Dr. M. P. Gaffney is on leave of absence until September 1, 1953 and is at Princeton University where he is working on a research project for the Army; Professor E. J. Moulton has retired with the title of Professor Emeritus.

Pennsylvania State College reports the following: Dr. R. D. Ayoub of Harvard University has been appointed to an assistant professorship; Assistant Professor Aline H. Frink has been promoted to an associate professorship; Instructors J. H. Kinney and William Craig have been promoted to assistant professorships.

Purdue University announces the following: Associate Professor H. F. S. Jonah has been promoted to a professorship; Visiting Professor Lamberto Cesari has been appointed to a professorship; Dr. J. H. B. Kamperman of Amsterdam, Holland has been reappointed Visiting Professor; Dr. D. B. Owen and Dr. A. J. Perlis have been appointed to assistant professorships; Mr. Leonard Gillman, formerly of the Navy Evaluations Group, has been appointed to an instructorship; Associate Professor R. B. Stone, formerly registrar, has retired with the title of Professor Emeritus.

Stevens Institute of Technology announces the following appointments to instructorships: Mr. Emanuel Fischer, formerly research assistant at Institute of Mathematics and Mechanics, New York University; Mr. Henry Polowy, previously instructor at Nutley High School, New Jersey.

Trinity College, Hartford, Connecticut announces the establishment of a new part-time graduate program leading to the degree of Master of Science in Mathematics.

Tulane University announces the following appointments: Dr. Morris Friedman of the University of Chicago has been appointed Visiting Assistant Professor; Mr. R. E. Allan and Dr. A. L. Shields, formerly graduate fellow at Massachusetts Institute of Technology, have been appointed to instructorships.

At the United States Naval Academy: Assistant Professor J. W. Popow has been promoted to an associate professorship; Senior Professor J. N. Galloway and Professor John Tyler have retired.

University of Alberta reports the following: Assistant Professor E. Phibbs has been promoted to an associate professorship; Lecturer T. M. Fostvedt, Dr. G. K. Horton, and Lecturer R. C. Jacka have been promoted to assistant professorships.

University of British Columbia announces: Dr. G. E. Latta of Princeton University has been appointed to an assistant professorship; Professors R. D. James and S. A. Jennings were holders of summer fellowships of the Canadian Mathematical Congress at Queen's University, Kingston, Ontario, during the summer of 1952.

At University of California at Berkeley: Mr. Kurt Bing has been appointed Associate; Professor Salomon Bochner of Princeton University is Visiting Professor during the second semester of 1952–53; Miss Anne C. Davis has been appointed to the position of junior research mathematician on a research project of the Office of Ordnance; Dr. Ulf Grenander was Visiting Associate Professor during the first semester of 1952–53; Dr. Lucien LeCam and Dr. Maurice Sion have been appointed to instructorships; Dr. R. F. Tate has been appointed Lecturer; Associate Professor Frantisek Wolf has been promoted to a professorship; Assistant Professors E. W. Barankin and Edmund Pinney have been promoted to associate professorships; Dr. H. M. Hughes and Dr. R. M. Lakness have been promoted to assistant professorships; Associate Professor E. W. Barankin is on leave of absence during 1952–53 and is at the Institute for Numerical Analysis; Professor D. H. Lehmer is also on leave during 1952–53 and is serving as Head of the Institute for Numerical Analysis; Professor Edmund Pinney was on sabbatical leave during the fall of 1952; Professor Sophia L. McDonald was on sabbatical leave in residence during the fall of 1952; Professor G. C. Evans is on sabbatical leave in residence, spring, 1953; Professor Thomas Buck has retired.

University of Chicago announces that Assistant Professors Irving Kaplansky and I. E. Segal have been promoted to associate professorships.

University of Maryland reports: Dr. Stuart Haywood, formerly supervisor of mathematics in the University of Maryland program in Europe, and Dr. D. M. Young, previously at Aberdeen Proving Ground, have been appointed to assistant professorships; Dr. H. S. Collins, previously a graduate student at Tulane University, Mrs. Elizabeth Cuthill of Purdue University, Dr. Charles McArthur, formerly a graduate assistant at Tulane University, Miss Jacqueline Penez, formerly research assistant at the University of Minnesota, Dr. W. R. Thickstun, who has been a graduate assistant at the Institute of Fluid Dynamics of the University, have been appointed to instructorships; Mr. B. R. Cato of the University of Arizona has been appointed to a junior instructorship.

At the University of Texas: Dr. W. T. Guy has been promoted to an assistant professorship; Dr. F. N. Edmonds, Jr., previously assistant professor of astronomy at the University of Missouri, has been appointed to an assistant professorship in applied mathematics and astronomy; Associate Professor R. E. Greenwood has returned after two years' military leave for service as Lieutenant Commander in the United States Naval Reserve.

University of Toronto reports: Dr. G. F. D. Duff of Massachusetts Institute of Technology has been appointed to an assistant professorship; Dr. G. A. Dirac of King's College, London, has been appointed Lecturer; Lecturers A. J. Cole-

man, R. G. Stanton, and W. T. Tutte have been promoted to assistant professorships; Dean S. Beatty who has served also as Head of the Department of Mathematics has retired with the title of Dean Emeritus after forty-four years at the University; I. Guttman, B. J. Kirby, K. Okashimo, G. W. Schaefer, S. Schuster, D. A. Sprott, J. R. Walter, and M. S. Watkins have been appointed to teaching fellowships.

University of Western Ontario announces the following: Lecturer G. P. Henderson has been promoted to an assistant professorship; Professor H. R. Kingston, Principal of University College and Dean of Arts and Science, has retired.

Washington University announces the appointments of Mr. W. A. Couch and Mr. G. C. Cree to instructorships.

Wheaton College, Illinois, makes the following announcements: Associate Professor Fannie W. Boyce has been promoted to a professorship; Assistant Professor Angeline J. Brandt has been promoted to an associate professorship.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 125 persons have been elected to membership by the Board of Governors on applications duly certified.

D. W. Allan, Student, University of Toronto.

S. I. Allen, M. A. (Harvard) Instr., University of Massachusetts.

Nachman Aronszajn, D.Sc. (Paris) Professor, University of Kansas.

R. T. Barnes, Student, University of Buffalo.

J. S. Becker, B.S. (Illinois) Conductor, New York Central Railroad, South Bend, Ind.

W. A. Beyer, M.S. (Illinois) Instr., Pennsylvania State College.

R. L. Blair, Ph.D. (Iowa) National Science Foundation Fellow, University of Chicago.

R. D. Boswell, Jr., M.S. (Mississippi S. C.) Professor, Reinhardt College, Waleska, Ga.

A. R. Brown, Jr., Ph.D. (Yale) Mathematician, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

G. C. Bush, Student, McMaster University.

D. I. Caplan, M.S. (M.I.T.) Engineering Assistant, Burroughs Adding Machine Company, Philadelphia, Pa.

Rev. R. E. Carias, B.A. (Javeriana U., Bogota) Grad. Student, Weston College, Mass.

M. N. Chase, M.A. (Illinois) Mathematician, Battelle Memorial Institute, Columbus, Ohio.

C. L. Childress, M.A. (Washington S. C.) Engineer, Boeing Airplane Company, Seattle, Wash.

T. Y. Chow, Ph.D. (Cornell) Asst. Professor, Rensselaer Polytechnic Institute.

John Christopher, Ph.D. (Oregon) Instr., Knox College.

W. J. Coles, M.A. (Duke) Grad. Fellow, Duke University.

I. G. Connell, Student, University of Manitoba.

H. W. CONNORS, M.A. (South Dakota) Instr., University of South Dakota.

ALBERTO CORAZAO, D.Sc. (Peru) Professor, Escuela Militar de Chorrillos, Lima, Peru.

J. B. CORNELISON, B.A. (Berea) Part-time Instr., University of Kentucky.

R. T. DAMES, M.S. (Michigan) Research Assistant, Willow Run Research Center, Ypsilanti, Mich.

R L. DAVIS, M.S. (Chicago) Research Assistant, University of Michigan.

R. Y. DEAN, Ph.D. (C.I.T.) Mathematician, General Electric Company, Hanford Works, Richland, Wash.

T. F. DROEGE, B.S.E.E. (Kentucky) Grad. Assistant, University of Kentucky.

E. L. DUBOWSKY, B.S. in Educ. (Northwest Missouri S. C.) Grad. Assistant, Kansas State College.

R. W. EMMERT, M.A. (Ohio State) Instr., Miami University, Oxford, Ohio.

VELTA ERDMANIS, Student, University of Richmond.

MARY E. ESTILL, Ph.D. (Texas) Instr., Duke University.

O. J. FARRELL, Ph.D. (Harvard) Professor, Union College.

W. R. FERRANTE, B.S. (Rhode Island) Instr., Lafayette College.

R. J. FIORE, Student, University of Buffalo.

GLORIA C. FORD, M.A. (Pennsylvania) Instr., Virginia State College.

W. T. FORD, B.A. (Oklahoma City) Grad. Assistant, University of Oklahoma.

M. P. FRIEDMAN, B.S.C.E. (N.Y.U.) Instr., Brooklyn College.

T. A. GEORGEVITCH, Consultant, Department of Defense.

S. I. GOLDBERG, Ph.D. (Toronto) Instr., Lehigh University.

J. C. GOUSE, Student, University of Rochester.

J. J. GREEVER, III, Student, University of Richmond.

A. E. HALTEMAN, M.A. (Oregon) Asst. Professor, University of Idaho.

MRS. ALBERTA H. HENRY, Ph.D. (Michigan) Instr., Brooklyn College.

L. H. HERBACH, M.A. (Columbia) Instr., Brooklyn College.

MYRON HOCHHEISER, Student, City College of the City of New York.

MISS TOBALEE ISAACS, B.A. (Western Maryland) Part-time Instr., Duke University.

R. F. JACKSON, Ph.D. (Harvard) Asso. Professor, University of Delaware.

VERN JAMES, Ph.D. (Stanford) Professor, San Jose State College.

DOUGLAS JONES, Student, Oklahoma University.

J. G. JORDAN, M.A. (Michigan) Head of Department, Board of Education, Maumee, Ohio.

J. H. KAPLAN, Student, Temple University.

F. C. KOCH, Student, University of California.

LOUIS KRAMER, D.D.S. (Buffalo) Student, University of Buffalo.

R. E. KRUCKLIN, B.S. (M.I.T.) Senior Engineer, A. H. Johnson and Company, New York City.

ELENORE M. LAZANSKY, M.A. (California) Teacher, Oakland Public Schools, Oakland, Calif.

R. J. LEMELIN, M.A. (Bowling Green S. U.) Research Assistant, Willow Run Research Center, Ypsilanti, Mich.

J. L. LEWIS, Student, University of Notre Dame.

VIKTORS LINIS, M.Sc. (McGill) Instr., University of Saskatchewan.

C. H. LITTLE, JR., M.A. (North Carolina) Assistant Professor, North Carolina State College.

T. J. LITTLE, Student, University of Richmond.

L. S. LOCKINGEN, B.S. (Houston) Instr., University of Houston; Research Engineer, M. D. Anderson Hospital for Cancer Research.

A. R. LOVAGLIA, Ph.D. (California) Asst. Professor, San Jose State College.

W. M. LOWNEY, B.S. (Great Falls) Grad. Student, Montana State College.

J. S. MACNERNEY, Ph.D. (Texas) Asst. Professor, University of North Carolina.

H. V. MADISON, Student, Oklahoma City University.

W. G. MADOW, Ph.D. (Columbia) Professor, University of Illinois.

WILLIAM MARCACCIO, M.S. (Rhode Island) Asso. Professor, Xavier University, Cincinnati, Ohio.

L. V. MEAD, M.A. (Ohio State) Asso. Professor, Montana School of Mines.

G. H. MEISTERS, Student, Iowa State College.

R. L. MENTZER, M.A. (Illinois) Instr., North Dakota Agricultural College.

H. E. MURRAY, B.S. (Hampton Inst.) Teacher, American College, Tarsus, Turkey.

W. M. MYERS, JR., Ph.D. (Ohio State) Asst. Professor, Montana State University.

N. D. NEWBY, JR., M.A. (Harvard) Instr., Ohio University.

HUGH NOLAND, B.A. (Reed) Teaching Fellow, University of Washington.

M. P. O'DONNELL, M.Sc. (Queensland) Lecturer, University of Queensland, Brisbane, Australia.

P. L. OGLESBY, Student, University of Richmond.

KATSUMI OKASHIMO, B.A. (McMaster) Teaching Fellow, University of Toronto.

E. M. OLSON, M.A. (Columbia) Lecturer, Columbia University.

W. R. ORTON, JR., Ph.D. (Illinois) Instr., Oberlin College.

F. P. PALERMO, M.S. (Brown) Grad. Assistant, Brown University.

O. O. PARDEE, Ph.D. (Stanford) Asst. Professor, Syracuse University.

M. J. PASCUAL, M.A. (Columbia) Instr., Siena College, Loudonville, N. Y.

D. L. PATTEN, B.A. (Oklahoma City) Grad. Assistant, University of Oklahoma.

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HENRY POLOWY, M.S. (N.Y.U.) Instr., Stevens Institute of Technology.

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V. J. RAELSON, LL.B. (Valparaiso) Instr., Valparaiso University.

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MRS. MARGARET B. SEELBINDER, M.A. (North Carolina) Part-time Instr., University of North Carolina.

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E. G. WALSH, M. A. (Michigan S. C.) Engi-

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J. J. WALSH, B.A. (Pomona) Instr., Pomona College.

MICHAEL WARSHAW, Senior Design Engineer, Electronic Engineering Company, Los Angeles, Calif.

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A. ZIRAKZADAH, M.S. (Oklahoma A. & M.) Instr., Oklahoma Agricultural and Mechanical College.

REPORT OF THE TREASURER FOR THE YEAR 1952

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1952. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1952

M & T Trust Co., Buffalo	$ 9,884.77	Current Fund	$ 9,884.77
Securities	61,594.98	Carus Fund	11,927.44
		Chace Fund	11,240.89
		Houck Fund	8,612.24
		Chauvenet Fund	653.40
		General Fund	29,161.01
	$71,479.75		$71,479.75

II. CURRENT FUND

Balance, January 1, 1952	$ 9,884.77	MONTHLY	
Dues	20,962.10	Publication	$17,472.60
Initiation fees	770.00	Reprints (net)	397.89
Subscriptions	4,475.34	Editor's Office	1,365.27
Sale of back numbers	1,239.66	Secretary-Treasurer's Office	
Advertisements	3,646.00	Clerical help	7,479.86
Sale of exchange periodicals	93.20	Postage and printing	1,877.50
Interest on General Fund	1,146.70	Office expenses	632.41
Income from Hardy Fund	120.00	Bank fees	65.21
Contributions	2.80	Auditing fee	25.00
Charges against Funds	858.95	Board of Governors	1,776.90
		Meetings	583.65
		Representatives	224.07
		Subventions	526.61
		Transfer to General Fund	1,401.74
		Balance, December 31, 1952	9,370.81

	III. Carus Fund	IV. Chace Fund	V. Houck Fund
Balance, January 1, 1952	$11,927.44	$11,240.89	$ 8,612.24
Sale of publications	2,695.59	641.40	5.00
Interest	470.09	442.39	339.67
Increase in value of securities	381.62	360.02	276.02
Less charges	675.51	163.81	18.40
Less Symposium expenses	—	—	600.00
Balance, December 31, 1952	$14,799.23	$12,520.89	$ 8,614.53

	VI. Chauvenet Fund	VII. Dunkel Fund	VIII. General Fund
Balance, January 1, 1952	$ 653.40	—	$29,161.01
Cash and securities from Dunkel Estate	—	$13,453.12	—
Interest	26.63	343.20	—
Increase in value of securities	21.60	429.63	931.25
Transfer from Current Fund	—	—	1,401.74
Less charges	1.23	—	—
Balance, December 31, 1952	$ 700.40	$14,225.95	$31,494.00

IX. Total Funds of the Association on December 31, 1952

Current Fund	$ 9,370.81	M & T Trust Co., Buffalo	$ 9,370.81
Carus Fund	14,799.23	Securities	82,355.00
Chace Fund	12,520.89		
Houck Fund	8,614.53		
Chauvenet Fund	700.40		
Dunkel Fund	14,225.95		
General Fund	31,494.00		
	$91,725.81		$91,725.81

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Delaware, Newark, Delaware, on November 29, 1952. Professor F. L. Manning, Chairman of the Section, presided at the morning and afternoon sessions.

Sixty persons were present, including the following thirty-nine members of the Association:

R. D. Anderson, R. J. Bickel, F. Marion Clarke, James Elmer Davis, F. L. Dennis, N. J. Fine, Mariano Garcia, E. D. Glenney, Samuel Goldberg, Emil Grosswald, V. H. Haag, Carl Hammer, Katharine E. Hazard, J. R. Holzinger, C. E. Kerr, R. A. C. Lane, Marguerite Lehr, E. V. Lewis, F. L. Manning, Edith A. McDougle, S. S. McNeary, C. A. Nelson, J. C. Oxtoby, C. F. Pinzka, G. E. Raynor, C. J. Rees, B. E. Rhoades, Judith A. Richman, J. D. Rutledge, C. W. Saalfrank, R. D. Schafer, Sister Mary Raphael, E. P. Starke, Rothwell Stephens, Alexander Tartler, J. I. Thigpen, H. G. Tucker, G. C. Webber, D. W. Western.

period during the program, and (3) a discussion with management and supervision of the use of these methods in their firm. The benefits of the program to both industry and the University were discussed and suggestions were offered for programs in other fields.

B. The essential features of several types of research effort were illustrated by analogies. Examples of problems which have remained unsolved while mathematics has grown around and beyond them were discussed. In the case of each example, useful approximation methods exist, but the absence of an exact solution is a barrier to progress in a related science. These specific problems were then considered as examples of ways in which other opportunities for research may be uncovered.

R. D. SCHAFER, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Howard University, Washington, D. C., on December 6, 1952. Professor Marian M. Torrey, Chairman of the Section, presided at the morning and afternoon sessions.

There were one hundred fifteen persons in attendance, including the following eighty-two members of the Association:

J. C. Abbott, M. I. Aissen, D. F. Atkins, T. J. Benac, T. A. Botts, Evelyn Boyd, B. H. Buikstra, W. E. Byrne, H. H. Campaigne, L. H. Chambers, H. J. Cheston, Jr., F. Marion Clarke, G. R. Clements, L. S. Dederick, J. W. Drew, J. A. Duerksen, E. L. Eagle, W. L. Fields, G. C. Francis, C. H. Frick, S. I. Gass, Michael Goldberg, R. A. Good, R. D. Gordon, E. S. Grable, E. C. Gras, Donald Greenspan, D. W. Hall, M. Gweneth Humphreys, S. B. Jackson, F. E. Johnston, Sidney Kaplan, L. M. Kells, F. B. Key, Blair Kinsman, H. L. Kinsolving, Mary A. Lee, D. C. Lewis, Jr., D. B. Lloyd, D. B. Lowdenslager, Elizabeth C. Lukacs, C. H. McCall, Jr., Carol V. McCamman, E. J. McShane, J. F. Manogue, Ella Marth, Florence M. Mears, Joseph Milkman, George Millman, R. W. Moller, Dewey Moore, T. W. Moore, W. K. Morrill, W. R. Murray, E. J. Musch, W. H. Norris, Jr., M. W. Oliphant, Hyman Orlin, J. W. Ponds, J. W. Popow, O. J. Ramler, R. W. Rector, J. D. Riley, T. J. Rivlin, J. B. Scarborough, W. F. Shenton, C. H. Sisam, H. C. Stotz, E. H. Swafford, Mildred E. Taylor, Feodor Theilheimer, O. M. Thomas, P. D. Thomas, J. A. Tierney, Marian M. Torrey, John Tyler, C. Y. Wang, M. E. White, P. M. Whitman, J. W. Wrench, Jr., D. M. Young, G. C. Zader.

During the business meeting it was announced that the Section will sponsor contests for high school students in order to stimulate further interest in mathematics.

The following papers were presented:

1. *Multilinear forms, n-ics and polynomials on a Hilbert space*, by Professor Joseph Milkman, United States Naval Academy.

Multilinear forms and n-ics on a Hilbert space were defined. Methods of finding the inner product of n-ics, polynomials, and power series were shown. The space C of all polynomials and power series of finite norms is a Hilbert space. The functionals discussed in this paper have been applied to the quantum theory of fields by K. O. Friedrichs.

2. *A new configuration associated with the osculating quadrics at a point of an analytic surface*, by Mr. P. D. Thomas, Ballistic Research Laboratories.

Certain lines in the tangent plane at a given point of an analytic surface were found as the result of certain mutual tangent plane relations between a quadric of Darboux and Bompiani's asymptotic osculating quadrics. The polar reciprocals of these lines in the tangent plane trace certain loci on a quadric of Darboux. The investigation of these loci leads to a new geometric configuration associated with a point of an analytic surface. The treatment was projective, using homogeneous coordinates.

3. *Elementary techniques in maxima and minima*, by Professor J. A. Tierney, United States Naval Academy.

The application of the first derivative in the solution of maxima minima problems is an important topic in a first course in calculus. It is interesting to note that many of the standard textbook problems of this type can be solved without the use of the calculus and require only a knowledge of the mathematics customarily taught in the secondary schools. The elementary techniques which are effective vary considerably, some being trivial in nature and others requiring a certain amount of ingenuity, whereas the calculus approach is usually straightforward. Several examples illustrating different techniques were presented.

4. *The review course in calculus at the United States Department of Agriculture Graduate School*, by Mr. Sidney Kaplan, Office, Comptroller of the Army and United States Department of Agriculture Graduate School.

The work is condensed to forty-five hours of work. Six derivatives are developed: the derivative of a constant, the derivative of a function of a function, the derivative of a sum of functions, the derivative of an inverse function, the derivative of a sine function, and the derivative of a logarithmic function. Four topics are stressed as being more essential from the applied point of view: curve tracing, differential—approximation and estimate of error, Newton-Raphson Method, and the trapezoidal and Simpson's mechanical quadrature formulae.

5. *The Stieltjes integral*, by Professor E. J. McShane, University of Virginia.

This was the invited lecture. It was pointed out that the Stieltjes integral is an appropriate device for use in many applications; for example, in investigating moments and moments of inertia of mass distributions, where the distribution may include point masses and also a continuous part; in studying the statistics associated with probability distributions; and in expressing linear functions, such as the differential effects encountered in exterior ballistics. The particular version of the Stieltjes integral presented was that due to McShane and Botts (*Duke Mathematical Journal*, vol. 19, 1952, p. 293), which has some advantages not possessed by the integral as usually defined.

C. H. FRICK, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, August 31–September 1, 1953.

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May 2, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS, Washburn Municipal University of Topeka, April 11, 1953.

KENTUCKY, University of Louisville, May 9, 1953.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA United States Naval Proving Ground, Dahlgren, Virginia, May 2, 1953.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, April 18, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, April 24, 1953.

NEBRASKA, University of Nebraska, Lincoln, May 2, 1953.

NORTHERN CALIFORNIA

OHIO, Columbus, April 18, 1953

OKLAHOMA, Oklahoma City, October, 1953.

PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

ROCKY MOUNTAIN, University of Colorado, Boulder, April 17–18, 1953.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Fort Worth, April 24–25, 1953.

UPPER NEW YORK STATE, United States Military Academy, West Point, May 9, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May 2, 1953.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.

No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN

MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 5

CONTENTS

MAY 1953

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4. Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

THE SOLUTION OF EQUATIONS BY CONTINUED FRACTIONS

J. S. FRAME, Michigan State College

1. Introduction. Continued fractions may be used to define rapidly convergent methods for solving equations. We shall study in this paper a set of continued fraction approximants P_k/Q_k for correcting a first estimate x to a required root ξ of any given equation $\phi(x)=0$, provided that $\phi(x)$ has as many continuous derivatives as may be required in a neighborhood U of the root ξ. We write

$$(1.1) \qquad y = \phi(x), \qquad y' = \phi'(x), \qquad y'' = \phi''(x), \cdots; \quad \phi(\xi) = 0.$$

$$(1.2) \qquad \nu = -y/y', \quad \beta = y''/2y', \qquad \gamma = y'''/3y'', \quad \delta = y^{\text{iv}}/4y''', \cdots;$$

and we note that the function ν is the familiar Newton-Raphson estimate for the exact correction $\epsilon=\xi-x$ that is to be added to the initial value x.*

Upon expanding the correction ϵ in a continued fraction with partial numerators a_k, denominators 1, and remainders r_k, we write

$$(1.3) \quad \epsilon = \xi - x = \cfrac{a_1}{1+\cfrac{a_2}{1+\cdots\cfrac{}{1+\cfrac{a_k}{1+r_{k+1}}}}} = \frac{P_k + r_{k+1}P_{k-1}}{Q_k + r_{k+1}Q_{k-1}}, \qquad \text{where} \quad r_k = \frac{a_k}{1+r_{k+1}}.$$

The partial numerators a_k in the continued fraction (1.3) may be determined by means of the series expansion of $\epsilon=r_1=\sum c_k\nu^k$ as a power series in ν (see 3.3) in which c_k is a certain rational function (3.2) of the first $k-1$ of the quantities $\beta, \gamma, \delta, \cdots$, evaluated at the chosen first estimate x. Since the first term in the series for r_1 is $c_1\nu=\nu$, we define $a_1=\nu$, and then by division obtain a second power series for $r_2=(a_1-r_1)/r_1=\beta\nu+\cdots$, which vanishes for $\nu=0$. In general, r_{k+1} will vanish for $\nu=0$ if we define a_k to be the first term in the series expansion for r_k. Assuming that our initial estimate x is not taken at a double root of any of the first k remainders, the ratio a_k/ν will be a rational function of the first $k-1$ quantities $\beta, \gamma, \delta, \cdots$, independent of ν. Thus we may compute successively the first few partial numerators as follows:

$$(1.4) \qquad a_1 = \nu, \quad a_2 = \nu\beta, \quad a_3 = \nu\beta - \nu\gamma, \quad a_4 = \nu\beta - \nu\gamma(\gamma-\delta)/(\beta-\gamma).$$

The procedure just outlined is complicated to apply for $k \geqq 4$, but in equation (3.8) of Theorem 2 we shall obtain concise expressions for a_k and r_k as ratios

* In case ξ is a multiple root of $\phi(x)=0$, we replace the given function $\phi(x)$ by the related function $\phi(x)/\phi'(x)$ that has a simple root at $x=\xi$. Thus the function y and the neighborhood U are chosen so that $y'\neq 0$ in U. If one or more of the higher derivatives y'', y''', *etc.*, vanish identically, an appropriate modification can be made to avoid using γ, δ, *etc.* If not, we may assume that U is chosen small enough so that the required derivatives do not vanish between x and ξ.

of certain determinants. However, the following inductive formula based on (4.3) gives a more practical procedure for actually computing consecutively the ratios a_k/ν:

$$(1.5)\qquad (k+1)\frac{a_{k+1}}{\nu} = 2\frac{a_2}{\nu} + (k-2)\frac{a_{k-1}}{\nu} - \frac{a_k}{\nu} - \frac{d}{dx}\ln\left(\frac{a_k}{\nu}\right) \qquad k \geqq 2.$$

The kth approximant to the error $\epsilon = \xi - x$ is defined to be that rational function P_k/Q_k of $a_1, \cdots, a_k$ obtained from (1.3) by replacing r_{k+1} by 0. It will be shown in Theorem 1 of Section 2 that the residual error $\epsilon_k = \epsilon - P_k/Q_k$ is of the order ν^{k+1}.

Whereas the first approximant P_1/Q_1 is the Newton-Raphson correction ν, [**2**, **3**], the second approximant

$$(1.6)\qquad \frac{P_2}{Q_2} = \frac{a_1}{1+a_2} = \frac{\nu}{1+\nu\beta} = \nu - \frac{\nu^2\beta}{1+\nu\beta}$$

defines the more accurate Halley correction [**1**] which was discovered independently by Frame [**4**, **5**], Wall [**16**], and others. The third approximant

$$(1.7)\qquad \frac{P_3}{Q_3} = \frac{\nu}{\cfrac{1+\nu\beta}{1+\nu(\beta-\gamma)}} = \nu - \frac{\nu^2\beta}{1+\nu(2\beta-\gamma)}$$

reduces to the correction proposed by J. K. Stewart [**13**], if the effect of the third derivative is neglected by setting $\gamma = 0$, but is better than Stewart's correction if $\gamma \neq 0$.

How these and higher order approximations compare in some numerical cases will be illustrated in section 2, after a simple expression R_{k+1}/Q_k has been obtained for the residual error $\epsilon - P_k/Q_k$ and somewhat similar expressions $A_k/Q_{k-1}Q_k$ and $A_k/Q_{k-1}Q_{k+1}$ for the differences of successive and alternate approximants. (Here $R_k = (-1)^{k-1}r_1r_2 \cdots r_k$, and $A_k = (-1)^{k-1}a_1a_2 \cdots a_k$).

Explicit expressions for the r_k and a_k will be obtained as ratios of determinants in Theorem 2 of Section 3.

Five derivative relations in Lemma 2 of Section 4, such as the one from which (1.5) was derived, furnish the basis for an independent proof of the formula of Tauber expressing the derivative of the iteration function $f_k(x) = x + P_k/Q_k$ in the form

$$(1.8)\qquad f_k'(x) = (x + P_k/Q_k)' = (k+1)(-1)^k a_2a_3 \cdots a_{k+1}/Q_k^2.$$

Theorem 3 also gives bounds and estimates for the residual error.

Theorem 4 of Section 5 treats the case of positive numerators $a_2, \cdots, a_{k+1}$ and shows that in this case the function $f_k(x)$ gives a better approximation to ϵ than is obtained from the kth partial sum of the corresponding power series. Here then is some solid ground for preferring the continued fraction method

among the many iteration methods that have been considered.

Several of the many papers that have discussed methods of solving equations are listed in the Index to volumes 1 to 56 of this MONTHLY under "Approximate Methods of Solution." Hamilton [8] and Householder [9] have recent papers on the subject, and Householder refers to a fundamental paper of E. Schroeder [12] in 1870 in which iterative methods of solution are discussed in general terms.

In the general iterative method for approximating a root ξ of $\phi(x)=0$, an iteration function $f(x)$ is constructed, such that if $x=x_1$ is chosen arbitrarily in some sufficiently small neighborhood U of the root ξ, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = f(x_n) \tag{1.9}$$

converges to the root ξ. If k is the largest positive number such that

$$\lim_{x\to\xi} \frac{|\xi - f(x)|}{|\xi - x|^k} < \infty, \tag{1.10}$$

then the convergence to ξ is said to be of order k.

Sufficient conditions on $f(x)$ to insure convergence in the neighborhood U were given by Schroeder [12] as follows:

(1.11a) $f(\xi) = \xi$ for some ξ in U;

(1.11b) $f(x)$ and $f'(x)$ continuous in U;

(1.11c) $|f'(x)| \leqq \theta$ for fixed $\theta < 1$ and all x in U.

Thus a region of convergence of the iteration functions

$$f_k(x) = x + P_k/Q_k \tag{1.12}$$

is indicated by the bound on $|f_k'(x)|$ in Theorem 3, and the convergence is seen to be of order $k+1$.

2. The continued fraction approximants. Having selected an arbitrary initial estimate x in a sufficiently small neighborhood U of the root ξ of the equation $\phi(x)=0$, we denote the exact difference $\epsilon=\xi-x$ by r_1, and denote the correction ν of Newton's method by a_1. Then as above we define successive remainders r_k recursively by the equations

$$r_1 = \epsilon = \xi - x, \qquad r_{k+1} = (a_k/r_k) - 1, \tag{2.1}$$

where the partial numerators a_k satisfy the conditions that a_k/a_1 is a rational function of the first $k-1$ of the quantities $\beta, \gamma, \delta, \cdots$, that approximates r_k/ν in the sense that

$$\lim_{\nu\to 0} (a_k/r_k) = 1. \tag{2.2}$$

The numerator P_k and denominator Q_k of the kth approximant P_k/Q_k in (1.3)

are polynomials in $a_1, a_2, \cdots, a_k$ without common factors, that may be defined either by the recurrence relations

$$\begin{aligned} P_0 &= 0, \quad P_1 = a_1, \qquad P_k = P_{k-1} + a_k P_{k-2} \\ Q_0 &= 1, \quad Q_1 = 1, \qquad Q_k = Q_{k-1} + a_k Q_{k-2} \end{aligned} \tag{2.3}$$

or by the equivalent matrix equation [6]:

$$\begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & a_2 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_k \\ 1 & 1 \end{pmatrix}. \tag{2.4}$$

The determinant $-A_k$ of the left member of (2.4) is the product of the k determinants $-a_j$ on the right, so by (2.3) and (2.4) we have

$$A_k = \begin{vmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{vmatrix} = \begin{vmatrix} P_{k+1} & P_{k-1} \\ Q_{k+1} & Q_{k-1} \end{vmatrix} = (-1)^{k-1} a_1 a_2 \cdots a_k; \tag{2.5}$$
$$A_0 = -1, \qquad A_1 = a_1 = \nu.$$

The corresponding products R_k of the remainders r_k are defined as follows:

(2.6) $R_k = (-1)^{k-1} r_1 r_2 \cdots r_k;\ R_0 = -1;\ R_1 = r_1 = \epsilon;\ R_2 = -r_1 r_2 = \epsilon - \nu$, *etc.*

THEOREM 1. *The residual error of the kth approximant P_k/Q_k to the continued fraction* (1.2) *is given by the equation*

$$\epsilon_k \equiv \epsilon - \frac{P_k}{Q_k} = \frac{R_{k+1}}{Q_k}. \tag{2.7}$$

It is nearly equal to A_{k+1} and is of order $k+1$ in ν when $|\nu|$ is small.

Proof: From equations (2.6) and (2.1) we have

$$R_{k+1} = r_{k+1} r_k R_{k-1} = (a_k - r_k) R_{k-1} = R_k + a_k R_{k-1}. \tag{2.8}$$

Hence R_{k+1} satisfies the same recurrence relation as do P_k and Q_k in (2.3). Thus

$$(R_{k+1} + P_k - \epsilon Q_k) = (R_k + P_{k-1} - \epsilon Q_{k-1}) + a_k(R_{k-1} + P_{k-2} - \epsilon Q_{k-2}). \tag{2.9}$$

Since by direct substitution we have

$$R_1 + P_0 - \epsilon Q_0 = \epsilon + 0 - \epsilon = 0;\ R_2 + P_1 - \epsilon Q_1 = (\epsilon - \nu) + \nu - \epsilon = 0, \tag{2.10}$$

it follows by induction that the left member of (2.9) vanishes for all $k>0$. This proves equation (2.7).

Using (2.7) and (2.5) we may express the denominator of the right member of (1.3) as follows:

$$Q_k + r_{k+1} Q_{k-1} = \frac{Q_k Q_{k-1}}{R_k}\left(\frac{R_k}{Q_{k-1}} - \frac{R_{k+1}}{Q_k}\right) = \frac{Q_k Q_{k-1}}{R_k}\left(\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}}\right) = \frac{A_k}{R_k}. \tag{2.11}$$

The ratio A_k/R_k is nearly 1 when ν is small, since it is the product of ratios

a_k/r_k which approach 1, by (2.2). Hence by (2.11) and (2.6) Q_k is near to 1, and both R_{k+1} and R_{k+1}/Q_k vanish to the order $k+1$ in ν.

The ratio A_k/Q_kQ_{k-1} (or the ratio $A_k/Q_{k+1}Q_{k-1}$) is seen from (2.5) to be the difference of successive (or alternating) approximants P_k/Q_k. Hence the exact correction ϵ may be written as a series with remainder in each of the three following ways:

$$\epsilon = \frac{A_1}{Q_1} + \frac{A_2}{Q_1Q_2} + \frac{A_3}{Q_2Q_3} + \cdots + \frac{A_k}{Q_{k-1}Q_k} + \frac{R_{k+1}}{Q_k}, \tag{2.12}$$

$$\epsilon = \frac{A_1}{Q_1} + \frac{A_2}{Q_1Q_3} + \frac{A_4}{Q_3Q_5} + \cdots + \frac{A_{2k-2}}{Q_{2k-3}Q_{2k-1}} + \frac{R_{2k}}{Q_{2k-1}}, \tag{2.13}$$

$$\epsilon = \frac{A_1}{Q_2} + \frac{A_3}{Q_2Q_4} + \frac{A_5}{Q_4Q_6} + \cdots + \frac{A_{2k-1}}{Q_{2k-2}Q_{2k}} + \frac{R_{2k+1}}{Q_{2k}}. \tag{2.14}$$

Later we shall obtain general expressions for the partial numerators a_k, but let us first see in some particular examples how much the right members of (2.12), (2.13), (2.14) improve upon Newton's correction, when the remainders R_3, R_4, and R_5 are neglected.

Consider first the equation

$$y = e^x - 1 - c = 0, \qquad |c| \leqq 1, \tag{2.15}$$

and take $x = x_1 = 0$ as the initial approximation. The solution by series would be the familiar logarithmic expansion

$$\xi = \ln(1 + c) = c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \cdots \tag{2.16}$$

which converges very slowly when $c = 1$. By the continued fraction method of solution, we have at $x = x_1 = 0$,

$$\begin{aligned} &y = -c, \quad y' = 1, \quad y'' = 1, \quad y''' = 1, \quad y^{iv} = 1, \quad \textit{etc.},\\ &\nu = c, \quad \beta = 1/2, \quad \gamma = 1/3, \quad \delta = 1/4, \cdots,\\ &a_1 = c, \quad a_2 = c/2, \quad a_3 = c/6, \quad a_4 = c/3, \text{ and in general}\\ &a_{2k} = \frac{2k}{2k-1}\,\frac{c}{4}, \qquad a_{2k+1} = \frac{2k}{2k+1}\,\frac{c}{4}. \end{aligned} \tag{2.17}$$

Thus the fourth approximant for $\ln(1+c)$ is

$$\ln(1 + c) \sim \cfrac{c}{1 + \cfrac{c/2}{1 + \cfrac{c/6}{1 + c/3}}} = \frac{c + c^2/2}{1 + c + c^2/6}; \quad \ln z \sim \frac{3(z^2 - 1)}{z^2 + 4z + 1}. \tag{2.18}$$

$$(3.4) \qquad D^*_{2k-1} = \begin{vmatrix} c_1 & c_2 & \cdots & c^*_k \\ c_2 & c_3 & \cdots & c^*_{k+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_k & c_{k+1} & \cdots & c^*_{2k-1} \end{vmatrix}; \qquad D^*_{2k} = \begin{vmatrix} c_2 & c_3 & \cdots & c^*_{k+1} \\ c_3 & c_4 & \cdots & c^*_{k+2} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k+1} & c_{k+2} & \cdots & c^*_{2k} \end{vmatrix}.$$

We modify D_k^* to define the three related determinants D_k, E_k^*, and E_k by replacing each c_j^* in the last column respectively by c_j, c^*_{j+1}, and c_{j+1}. Then since $c^*_{j+1} = (c_j^* - c_j)/\nu$ by (3.3), we obtain

$$(3.5) \qquad E^*_k = (D^*_k - D_k)/\nu, \qquad k \geqq 1.$$

It is convenient to make the special definitions

$$(3.4a) \qquad D^*_0 = D_0 = D_{-1} = -D_{-2} = 1; \qquad E^*_0 = 0.$$

Then we may prove two important identities as a lemma from which Theorem 2 will be easily obtained.

LEMMA 1. *The determinants D_k^*, D_k, E_k^*, and E_k defined in* (3.4) *and* (3.5) *satisfy the identities*:

$$(3.6) \qquad D_{k-2}D^*_{k+1} = -D_kE^*_{k-1} + D_{k-1}E^*_k, \qquad \textit{for } k > 1$$

$$(3.7) \qquad D_{k-2}D_{k+1} = -D_kE_{k-1} + D_{k-1}E_k, \qquad \textit{for } k > 1.$$

Proof: Let M, A, B, B^* be rectangular submatrices of (3.4) so chosen that

$$D_{k-2} = |\overline{M}| \qquad D_k = \begin{vmatrix} \overline{M} & \overline{B} \\ A & c_k \end{vmatrix} \qquad E^*_k = \begin{vmatrix} \overline{M} & \overline{B}^* \\ A & c^*_{k+1} \end{vmatrix}$$

$$D_{k+1} = \begin{vmatrix} 'M & B & B^* \\ 'A & c_k & c^*_{k+1} \end{vmatrix} \qquad D_{k-1} = |\,'M \quad B\,| \qquad E^*_{k-1} = |\,'M \quad B^*\,|$$

where if k is odd, $\overline{M}\overline{B}\overline{B}^* = MBB^*$ *etc.*, but if k is even, $\overline{M}\overline{B}\overline{B}^*$ is obtained from MBB^* by stripping off the top row; and where $'M = M$ and $'A = A$ if k is even, but $'M$ and $'A$ are obtained from M and A by stripping off the first column, if k is odd. Then by the Laplace expansion of determinants and some simple additions of rows and subtraction of columns we have

$$D_{k-2}D^*_{k+1} = \begin{vmatrix} 'M & B & 0 & B^* \\ 0 & 0 & \overline{M} & 0 \\ 'A & c_k & A & c^*_{k+1} \end{vmatrix} = \begin{vmatrix} 'M & B & 0 & B^* \\ '\overline{M} & \overline{B} & \overline{M} & \overline{B}^* \\ 'A & c_k & A & c^*_{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} 'M & B & 0 & B^* \\ 0 & \overline{B} & \overline{M} & \overline{B}^* \\ 0 & c_k & A & c^*_{k+1} \end{vmatrix} = \begin{vmatrix} D_{k-1} & E^*_{k-1} \\ D_k & E^*_k \end{vmatrix}.$$

This proves the identity (3.6). A simple replacement of c_j^* by c_j changes (3.6) into (3.7), and completes the proof of the lemma.

THEOREM 2. *The continued fraction partial numerators a_k and remainders r_k are expressible in terms of the determinants D_k and D_k^* of* (3.4) *by the formulas*

$$r_k = -\nu\frac{D_{k-3}D_k^*}{D_{k-2}D_{k-1}^*}, \qquad a_k = -\nu\frac{D_{k-3}D_k}{D_{k-2}D_{k-1}}. \tag{3.8}$$

Proof: When we multiply the identity (3.6) by $-\nu$ and eliminate E_{k-1}^* and E_k^* by (3.5), we obtain

$$-\nu D_{k-2}D_{k+1}^* = D_kD_{k-1}^* - D_{k-1}D_k^*. \tag{3.9}$$

Dividing (3.9) by its last term we then have

$$-\nu\frac{D_{k-2}D_{k+1}^*}{D_{k-1}D_k^*} = \frac{D_kD_{k-1}^*}{D_{k-1}D_k^*} - 1. \tag{3.10}$$

The special definitions (3.4a) were so chosen that for $k=1$ equation (3.8) becomes $r_1=\nu D_1^*=\epsilon$; $a_1=\nu D_1=\nu$. Using an induction proof, we next assume that (3.8) correctly defines $r_1, \cdots, r_k$ and $a_1, \cdots, a_k$ in accordance with the requirements of (2.1) and (2.2), and we show that (3.10) then implies the validity of (3.8) when k is replaced by $k+1$. The right member of (3.10) equals $(a_k/r_k)-1$, by (3.8). This in turn is equal to r_{k+1} by the definition (2.1). Hence the left member of (3.10) equals r_{k+1}, and the first formula of (3.8) is established for $k+1$. The formula for a_{k+1} is obtained from r_{k+1} by taking the first term in the expansion of the latter in a power series in ν. This agrees with the second formula of (3.8).

From (3.8) we next obtain simple expressions for the products R_k and A_k defined in (2.6) and (2.5). These are:

$$R_k = \frac{\nu^kD_k^*}{D_{k-2}}, \qquad A_k = \frac{\nu^kD_k}{D_{k-2}}. \tag{3.11}$$

Furthermore, from (3.8) and (3.7) we derive the formulas

$$-\frac{a_{k+1}}{a_1} = \frac{E_k}{D_k} - \frac{E_{k-1}}{D_{k-1}}, \qquad (\text{for } k \geqq 1, \text{ if we set } E_0 = 0). \tag{3.12}$$

Upon summing from 1 to k, we obtain a sum of a's (called S_k) defined by

$$S_k \equiv a_2 + a_3 + \cdots + a_{k+1} = -a_1E_k/D_k. \tag{3.13}$$

4. The derivative of the kth approximant. The error in using $f_k(x)=x+P_k/Q_k$ as an approximation to the root ξ of $\phi(x)=0$ can be estimated by means of a

surprisingly simple expression for the derivative $f_k'(x)$, which will be stated as Theorem 3. Our proof of this theorem depends on finding expressions for the derivatives with respect to x of the functions D_k, A_k, a_k, Q_k, and P_k.

LEMMA 2. *The derivatives* D_k', A_k', a_k', Q_k' *and* P_k', (*with respect to* x), *of the functions defined in* (3.4), (2.5), (3.8) *and* (2.3) *are given by the formulas*

$$-a_1D_k'/D_k = (k+1)S_k - [\tfrac{1}{2}(k+1)^2]a_2 \tag{4.1}$$

$$-a_1A_k'/A_k = k-(k-1)S_{k-2}+(k+1)S_k \tag{4.2}$$

$$-a_1a_k'/a_k = 1+a_{k-1}+a_k-(k-1)a_{k-1}+(k+1)a_{k+1}, \qquad k>1. \tag{4.3}$$

$$a_1Q_k' + S_{k-1}Q_k + (k+1)a_{k+1}a_kQ_{k-2} = 0 \tag{4.4}$$

$$a_1(P_k' + Q_k) + S_{k-1}P_k + (k+1)a_{k+1}a_kP_{k-2} = 0. \tag{4.5}$$

Proof: The elements c_{i+j-1} and c_{i+j} in the ith row and jth column of the determinants D_{2k-1} and D_{2k} were defined in (3.2). The derivative c_k' is seen to be

$$\begin{aligned} c_k' &= (y')^k y'\psi^{(k+1)}(y)/k! + k(y')^{k-1}y''\psi^{(k)}(y)/k! \\ c_k' &= (k+1)c_{k+1} - 2c_2kc_k, \qquad \text{(where } 2c_2 = -y''/y' = -2a_2/a_1). \end{aligned} \tag{4.6}$$

The derivative of a kth order determinant such as D_{2k-1} is the sum of k^2 terms obtained by multiplying the derivative d_{ij}' of each element $d_{ij}(=c_{i+j-1})$ by its cofactor C_{ij}. Each derivative c_{i+j-1}' is separated into two terms by (4.6), the first of which involves the element $c_{i+j}=d_{i,j+1}$ in the next column. Thus

$$D_{2k-1}' = \sum_{i,j=1}^{k}(i+j)c_{i+j}C_{ij} - 2c_2\sum_{i,j=1}^{k}(i+j-1)c_{i+j-1}C_{ij}. \tag{4.7a}$$

Since $C_{ji}=C_{ij}$, an interchange of i and j will replace a term with coefficient i by an equal term with coefficient j. Recalling the definition of E_{2k-1} after (3.4), we transform (4.7a) to

$$\begin{aligned} D_{2k-1}' &= 2kE_{2k-1} + \sum_{j=1}^{k-1}2j\sum_{i=1}^{k}d_{i,j+1}C_{ij} - 2c_2\sum_{j=1}^{k}(2j-1)\sum_{i=1}^{k}d_{ij}C_{ij} \\ &= 2kE_{2k-1} + \sum_{j=1}^{k-1}2j(0) \qquad - 2c_2\sum_{j=1}^{k}(2j-1)D_{2k-1}. \end{aligned} \tag{4.7b}$$

With (3.13) in mind we multiply (4.7b) by $-a_1/D_{2k-1}$ and obtain

$$-a_1D_{2k-1}'/D_{2k-1} = 2kS_{2k-1} - 2a_2\sum_{j=1}^{k}(2j-1). \tag{4.8a}$$

By a similar argument applied to the derivative of D_{2k} we obtain

$$-a_1D_{2k}'/D_{2k} = (2k+1)S_{2k} - 2a_2\sum_{j=1}^{k}(2j). \tag{4.8b}$$

Use of the square brackets in (4.1) to denote the greatest integer in $\frac{1}{2}(k+1)^2$,

enables us to express the two formulas (4.8a) and (4.8b) as a single formula (4.1).

By differentiating $\ln A_k$ as obtained from (3.11) we then have

$$(4.9)\qquad \begin{aligned} A_k'/A_k &= ky'/y - ky''/y' - D_{k-2}'/D_{k-2} + D_k'/D_k \\ -a_1A_k'/A_k &= k + 2ka_2 + a_1D_{k-2}'/D_{k-2} - a_1D_k'/D_k. \end{aligned}$$

Equation (4.2) then follows immediately from (4.9) and (4.1).

Next, since $a_k = -A_k/A_{k-1}$, we obtain (4.3) from (4.2) by subtraction as follows.

$$(4.10)\qquad \begin{aligned} -a_1a_k'/a_k &= -a_1A_k'/A_k + a_1A_{k-1}'/A_{k-1} \qquad (k > 1), \\ &= 1 - S_{k-3} + S_{k-1} - (k-1)(S_{k-2} - S_{k-3}) + (k+1)(S_k - S_{k-1}) \\ &= 1 + a_{k-1} + a_k - (k-1)a_{k-1} + (k+1)a_{k+1}. \end{aligned}$$

We prove equation (4.4) and (4.5) by induction, using the definitions (2.3) and (3.11) .We verify directly that both left members vanish for $k=1$ and $k=2$ in view of the definitions $P_{-1}=Q_0=Q_1=1$, $Q_{-1}=P_0=S_0=0$, and the values of $a_1' = P_1' = P_2'$ and $a_2' = Q_2'$ given by (4.3).

Then taking $k=n>2$, we assume that the left members of (4.4) and (4.5) both vanish for $k=n-1$ and for $k=n-2$, and we use the recurrence relations (2.3) to express Q_n' and P_n' in terms of the known quantities Q_{n-1}', Q_{n-2}' and P_{n-1}', P_{n-2}'. From (4.4) and (4.3) we obtain successively the relations

$$(4.11)\qquad \begin{aligned} 0 &= a_n[a_1Q_{n-2}' + S_{n-3}Q_{n-2} + (n-1)a_{n-1}a_{n-2}Q_{n-4}] \\ &= a_1(a_nQ_{n-2})' - a_1a_n'Q_{n-2} + S_{n-3}a_nQ_{n-2} + (n-1)a_na_{n-1}(Q_{n-2} - Q_{n-3}) \\ &= a_1(Q_n' - Q_{n-1}') - (n-1)a_na_{n-1}Q_{n-3} \\ &\qquad\qquad + a_nQ_{n-2}(-a_1a_n'/a_n + S_{n-3} + (n-1)a_{n-1}) \\ &= a_1Q_n' - a_1Q_{n-1}' - (n-1)a_na_{n-1}Q_{n-3} + a_nQ_{n-2}(1 + S_{n-1} + (n+1)a_{n+1}) \\ &= a_1Q_n' + S_{n-2}Q_{n-1} + a_na_{n-1}Q_{n-3} + a_nQ_{n-2} + a_nQ_{n-2}(S_{n-1} + (n+1)a_{n+1}) \\ &= a_1Q_n' + (S_{n-2} + a_n)Q_{n-1} + S_{n-1}a_nQ_{n-2} + (n+1)a_{n+1}a_nQ_{n-2} \\ &= a_1Q_n' + S_{n-1}Q_n + (n+1)a_{n+1}a_nQ_{n-2}. \end{aligned}$$

Hence equation (4.4) is established for $k=n$ and the induction proof is complete. By a series of steps similar to (4.11) we may prove (4.5) by induction and thus finish Lemma 2.

To find the derivative of P_k/Q_k we first multiply (4.5) by Q_k and (4.4) by P_k and subtract. This gives us the relation

$$(4.12)\qquad a_1(Q_k^2 + Q_kP_k' - P_kQ_k') - (k+1)a_{k+1}a_k(P_kQ_{k-2} - Q_kP_{k-2}) = 0.$$

When we make use of the values of A_{k-1} and A_{k+1} in (2.5), equation (4.12) becomes

$$Q_k^2 + Q_k P_k' - P_k Q_k' = (k+1)A_{k+1}/a_1. \tag{4.13}$$

Division by Q_k^2 in (4.13) gives the desired derivative of $x+P_k/Q_k$.

THEOREM 3. *The derivative of the kth iteration function $f_k(x)=x+P_k/Q_k$, defined by the kth approximant P_k/Q_k to the continued fraction* (1.3) *is given by the following formula of Tauber* [**14**]:

$$f_k'(x) = \left(x + \frac{P_k}{Q_k}\right)' = \frac{(k+1)A_{k+1}}{Q_k^2 a_1} = \frac{k+1}{Q_k^2}\frac{D_{k+1}}{D_{k-1}}\nu^k. \tag{4.14}$$

Furthermore, the ratio ϵ_k/ϵ of the residual error $\xi - f_k(x) = \epsilon_k$ to the initial error $\xi - x = \epsilon$ is less in absolute value than the maximum value of $f_k'(x)$ on an interval U containing x and ξ. Also

$$\left|\frac{\epsilon_k}{\epsilon^{k+1}}\right| < \max_{x\in U} Q_k^{-2}\prod_{s=2}^{k+1}\left|\frac{a_s}{\xi - x}\right| = \max_{x\in U} Q_k^{-2}\left(\frac{\nu}{\epsilon}\right)^k\left|\frac{D_{k+1}}{D_{k-1}}\right|. \tag{4.15}$$

Proof: Formula (4.14) follows immediately from (4.13) and (3.11). When we integrate (4.14) from x to ξ we obtain

$$\begin{aligned}\frac{\epsilon_k}{\epsilon} &= \frac{f_k(\xi) - f_k(x)}{\epsilon} \\ &= \frac{1}{\epsilon}\int_{\xi-\epsilon}^{\xi}[f_k'(t)]dt = \frac{1}{\epsilon}\int_{t=\xi-\epsilon}^{t=\xi}\left[Q_k^{-2}\prod_{s=2}^{k+1}\left(\frac{-a_s}{t-\xi}\right)\right]d(t-\xi)^{k+1}.\end{aligned} \tag{4.16}$$

By extracting from this integral the maximum value of the quantity in brackets in the interval of integration, we obtain the upper bounds for $|\epsilon_k/\epsilon^{k+1}|$ given in Theorem 3.

A good estimate of the residual error ϵ_k is $(D_{k+1}/D_{k-1})\nu^{k+1}$, since Q_k and ν/ϵ are near 1 when ϵ is small. The estimate is especially good for $k \geqq 2$ if the trial value x is chosen on that side of ξ for which y and y'' have opposite signs, so that $\beta > 0$ and $Q_2 > 1$.

5. The case of positive numerators a_k. There is a class of equations for which the kth approximant P_k/Q_k to the continued fraction (1.3) always gives a closer estimate for ϵ than the kth partial sum s_k of the corresponding power series. In this class fall the familiar quadratic equations

$$\phi(\xi) = 0, \text{ where } \phi(t) \equiv t^2 - 2At + B = (t-x)^2 + (1/\beta)(t-x) - \nu/\beta. \tag{5.1}$$

The constants ν and β may assume complex values but have the same meaning as in (1.2), and the real or complex constant x is the chosen first estimate for the root ξ. Writing $\xi - x = \epsilon$ and $\nu\beta = a$, our quadratic equation becomes

$$(\beta\epsilon)^2 + (\beta\epsilon) - a = 0. \tag{5.2}$$

The power series solution of the form (3.3) is

$$(5.3) \qquad \beta\epsilon = \tfrac{1}{2}(-1 + \sqrt{1+4a}) = a - a^2 + \frac{2\cdot 6}{2\cdot 3}a^3 - \frac{2\cdot 6\cdot 10}{2\cdot 3\cdot 4}a^4 + \cdots$$

and the corresponding continued fraction is

$$(5.4) \qquad \beta\epsilon = \cfrac{a}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \cdots}}} \qquad \text{where } a_k = a \text{ and } r_k = \beta\epsilon, \text{ for } k > 1.$$

The approximants P_k/Q_k for (5.4) are given explicitly by the formulas

$$(5.5) \qquad Q_{k-1} = (\beta/a)P_k = [(1+\beta\epsilon)^k - (-\beta\epsilon)^k]/(1+2\beta\epsilon)$$

so the residual error of the kth approximant is

$$(5.6) \qquad \epsilon_k = \epsilon - P_k/Q_k = \frac{(1/\beta) + 2\epsilon}{1 - \left(\dfrac{1+\beta\epsilon}{-\beta\epsilon}\right)^{k+1}}.$$

Thus for all complex values of x except those equally distant from the two roots of $\phi(\xi)=0$, the continued fraction (5.4) converges to the nearer root ξ [17], since $|-\epsilon|$ and $|1/\beta+\epsilon|$ are the respective distances from x to the nearer and further root of (5.1). However, if the coefficients A and B in (5.1) are fixed but x varies over the complex plane, the series (5.3) converges only if x lies inside the rectangular hyperbola with foci at the roots.

Not only does the continued function converge in a larger region than the power series, but the convergence is actually better as we see from the following theorem.

THEOREM 4. *Let the real function $\phi(x)$ and a real root ξ of $\phi(\xi)=0$ be such that the k ratios of partial numerators $b_j=a_j/a_1$ $(j=2, 3, \cdots, k+1)$ all have the same sign in a neighborhood U of ξ, and let x be chosen on that side of ξ for which $a_2, a_3, \cdots, a_{k+1}$ are all positive. Then for $k>1$ the kth approximant P_k/Q_k to the continued fraction* (1.3) *gives a closer approximation to the error $\epsilon=\xi-x$ than does the sum s_k of the first k terms $c_j\nu^j$ of the series* (3.3). *Furthermore the first $k+1$ terms of the series will then alternate in sign.*

Proof: The signs of ϵ and ν can be changed without altering $a_2, a_3, \cdots, a_{k+1}$ by replacing x by $2\xi-x$ and thus reversing the direction of the x axis. Hence it will be sufficient to prove the theorem in the case that ν and b_j are negative and a_j and r_j are positive between x and ξ, for $j=2, 3, \cdots, k+1$. From equation (2.1) or (1.3) we obtain expansions for the r_k in powers of ξ.

$$r_{m-1} = \frac{a_{m-1}}{1 + r_m} = a_{m-1} \sum_{t=0}^{\infty} (-r_m)^t = \nu b_{m-1} \sum_{t=0}^{\infty} \nu^t(-r_m/\nu)^t, \qquad m > 2. \tag{5.7}$$

The remainder r_k vanishes when $\nu = 0$ and has two negative coefficients b_k and $-b_k b_{k+1}$ in its expansion in powers of ν obtained from (5.7). Likewise r_{k-1} has three negative coefficients in its expansion, and keeps these even when r_k is replaced by a_k and ϵ by P_k/Q_k. From (5.7) it follows that the assumption that r_m has $k+2-m$ negative coefficients implies that r_{m-1} has at least one more, $(m>2)$. Hence by induction r_{k+1-j} has at least $j+1$ negative coefficients $(j=1, 2, \cdots, k-1)$. Thus $r_1=\epsilon$ has at least $k+1$ positive coefficients, since $b_1>0$. Since ν is negative, the values of the first $k+1$ terms in the power series expansion of both ϵ and P_k/Q_k must alternate in sign. The correction R_{k+1}/Q_k of (2.7) contains ν^{k+1} as a factor and has the same sign as $(-1)^k\nu$ and $c_{k+1}\nu^{k+1}$. Hence the series expansions of ϵ and P_k/Q_k agree to k terms and we have

$$| \epsilon - s_k | = | R_{k+1}/Q_k | + | P_k/Q_k - s_k | > | R_{k+1}/Q_k | = | \epsilon_k |, \quad \text{for } k > 1. \tag{5.8}$$

Thus whenever the partial numerators $a_2, \cdots, a_{k+1}$ are all positive, the continued fraction error ϵ_k is numerically less than the series remainder $\epsilon - s_k$.

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THE SEMI-ORTHOCENTRIC TETRAHEDRON

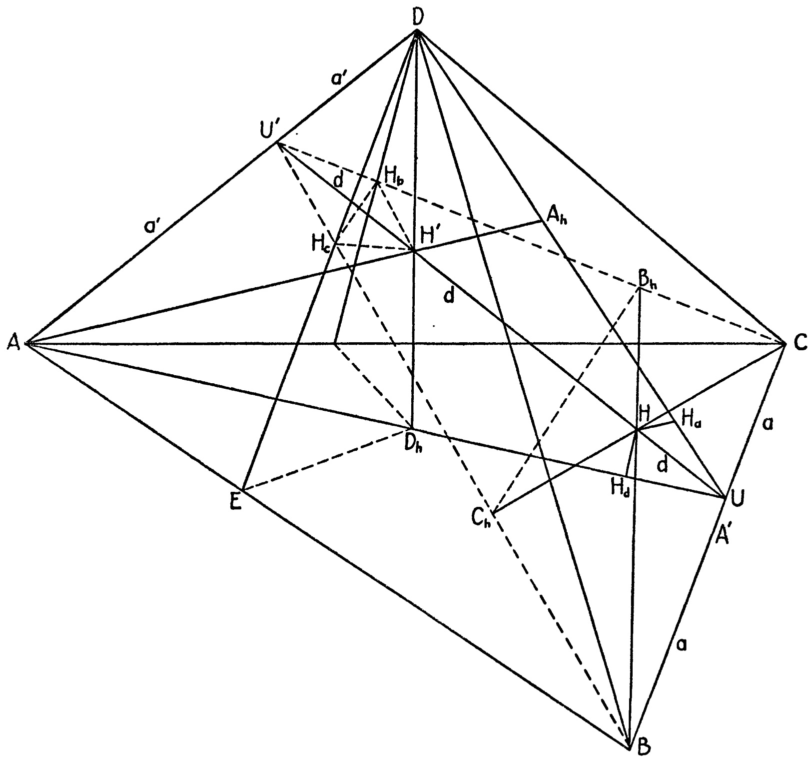
N. A. COURT, University of Oklahoma

The altitudes of a tetrahedron form, in general, a hyperbolic group of four lines.

The four altitudes have a point in common, if, and only if, each edge of the tetrahedron is perpendicular to the opposite edge. The tetrahedron is then said to be orthocentric, and the common point of the altitudes is its orthocenter.

The intermediate case of a tetrahedron with one pair of opposite orthogonal edges has so far received scant attention. Such a tetrahedron may be said to be *semi-orthocentric*. It shall be denoted by K in what follows.

The two rectangular edges of K may be referred to as its *principal* or *special* edges, and their common perpendicular as the *principal bialtitude* of K. The four remaining edges of K will be said to be *non-special*.



1. The Preliminaries. a. Let $a=BC$, $a'=DA$ be the principal edges of a semi-orthocentric tetrahedron $K=DABC$, and U, U' the feet, on a, a', respectively, of the principal bialtitude $d=UU'$.

The edges a, a' are, respectively, perpendicular to the planes $a'd=DAU$, $ad=BCU'$, and the two planes ad, $a'd$ are mutually perpendicular.

b. It is known that the two pairs of altitudes BB_h, CC_h, and DD_h, AA_h of K lie in the planes ad, $a'd$, respectively, and intersect on the principal bialtitude d of K. Conversely, if two altitudes of a tetrahedron intersect, the same holds for the other two altitudes, and the tetrahedron is semi-orthocentric [**1**; arts. 206, 204, p. 62].

c. The two points $H=(BB_h, CC_h)$, $H'=(DD_h, AA_h)$ will be referred to as the *semi-orthocenters* of K, relative to the principal edges a, a', respectively.

2. The mid-points of the non-special edges of *K*. a. The non-special edges of K form a skew quadrilateral $DBAC$ whose diagonals are the special edges DA, BC. Thus the parallelogram formed by the mid-points of the sides of this skew quadrilateral [**1**; p. 42] is a rectangle. Its diagonals are the two bimedians of K relative to the two pairs of non-special opposite edges of K, hence those two bimedians are equal, and the center of the rectangle is the centroid of K.

Moreover, the mediator (that is, the perpendicular bisecting plane) ρ of the principal bialtitude UU' being parallel to and equidistant from the two principal edges DA, BC of K, the vertices of the rectangle considered lie in the plane ρ [**1**; p. 3, art. 10]. To sum up: *In a semi-orthocentric tetrahedron K the mid-points of four non-special edges of K lie in the mediator of the principal bialtitude and are the vertices of a rectangle whose circumcenter is the centroid of K.*

b. Conversely. *If the mid-points of two pairs of opposite edges of a tetrahedron are the vertices of a rectangle, the tetrahedron is semi-orthocentric.*

3. The nine-point circles of the faces of *K*. a. The line $a=BC$ of K being perpendicular to the plane ADU of U (§1a), the lines DU, AU are the altitudes of the triangles DBC, ABC, to their common base BC. Conversely, if in a tetrahedron $DABC$ the altitudes of the triangles DBC, ABC, to their common base BC, meet that base in the same point U, the line BC is perpendicular to the plane DAU, and therefore to the line AD, that is, the tetrahedron is semi-orthocentric.

Thus in the tetrahedron $K=DABC$ the nine-point circles of the faces DBC, ABC have two points in common, namely the point U and the mid-point A' of the principal edge BC, hence: *The nine-point circles of two faces of a tetrahedron are cospherical, if, and only if, the tetrahedron is semi-orthocentric, and the common edge of the two faces is a principal element of the tetrahedron.*

b. As an immediate consequence we have: *If the nine-point circles of two faces of a tetrahedron are cospherical, the same is true of the nine-point circles of the remaining two faces of the tetrahedron.*

c. We thus associate with a semi-orthocentric tetrahedron K two spheres (G_a), (G_a') relative to the two principal elements a, a' of K.

These two spheres will be referred to as the *G-spheres* of K.

d. The sphere (G_a) containing the nine-point circles of the faces DBC, ABC, passes through the mid-points of the non-special edges DB, DC and AB, AC. The sphere (G_a') passes through the same four points, for analogous reasons.

Hence (§2a): *The radical plane of the two G-spheres of a semi-orthocentric tetrahedron K is the mediator of the principal bialtitude of K.*

The line of centers of the two G-spheres is thus parallel to the principal bialtitude of K, and, furthermore, passes through the centroid of K (§2a).

e. Given a tetrahedron $DABC$, if the perpendiculars to the faces DBC, ABC at their nine-point centers meet in a point G_a, the sphere having G_a for center and passing through the midpoint A' of the edge BC will pass through the nine-point circles of those two faces, hence (§3a): *The perpendiculars erected to two faces of a tetrahedron as their nine-point centers are coplanar, if, and only if, the tetrahedron is semi-orthocentric, and the common edge of the two faces considered is a principal edge.*

4. The orthocenters of the faces of K. a. If from the foot D_h of the altitude DD_h of K a perpendicular D_hE is drawn to the line AB, the line DE is an altitude of the triangle DAB.

The plane DAB intersects the plane $ad=BCU'$ along the line BU'. Now the special edge $a'=DA$ is perpendicular to the plane ad (§1a), hence DA is perpendicular to $U'B$. Thus the point $H_c=(BU', DE)$ is the orthocenter of the triangle DAB.

The point H_c belongs to each of the planes DD_hE, $ad=BCU'$, and so does the semi-orthocenter H' of K (§§1b, 1c), hence $H'H_c$ is the line of intersection of those two planes. Now the plane DAB passes through the lines BA, DA which are, respectively, perpendicular to the planes DD_aE, $ad=BCU'$, hence the line $H'H_c$ is the perpendicular to the face DAB at its orthocenter H_c. Similarly for the orthocenters H_a, H_b, H_d of the faces BCD, CDA, ABC. of K. Thus: *In a semi-orthocentric tetrahedron K the semi-orthocenter relative to a given principal edge of K is the point of intersection of the perpendiculars erected to the two faces of K passing through that special edge, at the orthocenters of those faces.*

b. Conversely. *If the perpendiculars to two faces of a tetrahedron at their respective orthocenters intersect, the tetrahedron is semi-orthocentric.*

If in a tetrahedron $DABC$ the perpendiculars to the faces BCA, BCD at their orthocenters H_d, H_a intersect, in a point H, the plane HH_aH_d is perpendicular to each of the planes BCA, BCD, and therefore to their line of intersection BC, at a point U. The line UH_d is thus the altitude of the triangle ABC relative to the side BC and therefore passes through its vertex A. The line UH_a passes through the vertex D, for analogous reasons.

Thus the edge DA of the tetrahedron lies in the plane HH_aH_d perpendicular to BC, hence the two edges DA, BC are perpendicular, which proves the proposition (1a).

5. The polar circles of the faces of K. a. Let (H) be the sphere having the point H for center and orthogonal to the sphere (BC) having the principal edge BC for diameter.

The plane ABC cuts the sphere (BC) along a great circle (bc) having BC for diameter, and the sphere (H) along a small circle η orthogonal to (bc) [1; p. 176,

lies on the principal bialtitude and bisects the segment determined by the semi-orthocenters of K.

8. The orthocentric tetrahedron. An orthocentric tetrahedron may be considered as a semi-orthocentric tetrahedron which has three pairs of "principal" edges (§1). The basic properties of an orthocentric tetrahedron may be derived from the properties of a semi-orthocentric tetrahedron established in the preceding paragraphs. Here is an example.

In an orthocentric tetrahedron (L) the three edges of any face are "principal" edges, hence the perpendicular to any face of (L) at the nine-point center of that face meets the analogous perpendiculars relative to the remaining three faces of (L). Now these four perpendiculars cannot lie in one plane (for that would imply that the four faces of (L) are perpendicular to that plane), hence they have a point in common.

Thus the three pairs of G-spheres (§3) relative to the three pairs of opposite edges of (L) have a common center. This center is therefore the centroid of (L) and the six spheres coincide in one sphere which passes through the nine-point circles of the four faces of (L) [1; pp. 261, 262].

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OPTIMAL HORSE RACE BETS

RUFUS ISAACS, The Rand Corporation

1. Introduction. The problem of placing straight win bets on a horse race so as to maximize the expected value of the profit is intellectually lucrative, although fiscally academic unless the bettor is extraordinarily opulent, has an excellent dopester, and can do elaborate computations on very short notice.

The crowd bets on the various horses. The total sum, after a fixed percentage being deducted as the track's profit, is allocated to the bettors on the winning horse in proportion to their bets. We assume that we know the true probability of each horse's winning.

There is nothing unsound or new in our basic wagering principle. It is known, in track parlance, as "overlays." It consists in taking advantage of the collective error of the crowd in appraising the probabilities as registered by the amounts they bet. What we believe is new is the exact solution.

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Let us take an example. The third column is the amounts bet by the crowd, drastically scaled down from race track reality.

Horse	Probability of Winning	Amount Bet by Crowd
A	.4	\$1000
B	.2	350
C	.2	300
D	.1	250
E	.1	100
	1.0	\$2000

If we neglect, for the moment, the track's deduction, it is clear that E is the soundest investment; for he returns twenty times the stake and has a win probability of 0.1. But if our bet on him is large, the increment to the \$100 will affect the entire picture and possibly even cause another horse to be the most favorable. We now can see that to maximize our expected profit we should place bets on a certain subset (possibly null) of horses.

Since this bagatelle first resulted from a conversation with a turf-minded friend, we have learned of its wider interpretations. It is akin to the current field of non-linear programming whose advocates have suggested alternative techniques. But in such fields, solutions by such definite means as algorithm and formula, as here obtained, are uncommon and we have been told our work should hold some exoteric interest. We acknowledge, with admiration, the suggestions of the referee who elegantly abridged our derivation.

2. The problem. Let the win probability of the jth horse be p_j, the amount wagered on him by others than ourself s_j, and our wager on him be x_j. Let the total sum wagered be multiplied by Q $(0<Q<1)$ prior to distribution to the winning bettors. Our problem then is:

What values of

$$x_j \; (x_j \geqq 0, j = 1, \cdots, n)$$

render the expected value of our profit,

$$F(x_1, \cdots, x_n) = Q\left[\sum_{j=1}^{n} (x_j + s_j)\right] \sum_{j=1}^{n} \frac{p_j x_j}{x_j + s_j} - \sum_{j=1}^{n} x_j \tag{1}$$

a maximum, the value of the maximal F to be positive? Here Q and p_j, s_j are constants subject to

$$0 \leqq p_j \leqq 1, \qquad \sum_{j=1}^{n} p_j = 1,$$

$$s_j > 0,^* \qquad 0 < Q < 1,$$

and, of course, $n > 1$.

3. The solution. Let c_j $(j = 1, \cdots, n)$ be such that

$$c_j \geqq 0, \qquad \sum_{j=1}^{n} c_j = 1.$$

If we put $x_j = c_j u$ in (1), F becomes a function of u denoted by $f(u)$. It is easy to verify.

$$f(0) = 0 \tag{2}$$

$$\lim_{u \to \infty} \frac{f(u)}{u} \leqq Q - 1 < 0 \tag{3}$$

$$f'(0) = \sum_{j=1}^{n} c_j \left[Q \frac{p_j}{s_j} \sum_{i=1}^{n} s_i - 1 \right]. \tag{4}$$

Let u range from 0 to ∞. From (2) and (3) we see that f begins as 0 but ultimately becomes and remains negative. If at least one of the brackets of (4) is positive, the c_j can be chosen so that $f'(0) > 0$. Thus we state:

LEMMA 1. *There is always a non-negative maximum of F; if*

$$\max_{1 \leqq j \leqq n} \frac{p_j}{s_j} > \frac{1}{Q \sum_i s_i} \tag{5}$$

then this maximum is positive.

Let $\bar{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ furnish a maximum. If $\bar{x}_i > 0$, then

$$\left(\frac{\partial F}{\partial x_i}\right)_{X = \bar{X}} = Q \sum_{j=1}^{n} \frac{p_j \bar{x}_j}{\bar{x}_j + s_j} + \frac{p_i s_i}{(\bar{x}_i + s_i)^2} Q \sum_{j=1}^{n} (\bar{x}_j + s_j) - 1 = 0. \tag{6}$$

Then $p_i s_i / (\bar{x}_i + s_i)^2$ has the same value for all i such that $\bar{x}_i > 0$ and we call it $1/\lambda^2$. Thus for such i

$$\bar{x}_i = \lambda \sqrt{p_i s_i} - s_i. \tag{7}$$

LEMMA 2. *The maximal F never occurs when all the $\bar{x}_i$ are positive*; or: *Don't bet on all the horses.*

Substituting from (7) for *every* i into (6), the central member of the latter becomes $Q - 1$ which is negative.

* We will show later that there is not necessarily a solution if some s_j are zero.

Recognizing that the higher p_j is, and the lower s_j is, the better bet Horse j is, it is clear that the ratio $\rho_j = p_j/s_j$ provides a heuristic measure of the soundness of a bet on Horse j. Let us renumber the horses, if necessary, in order of increasing value by this criterion, *i.e.*, so that $\rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_n$. The next proposition tells us that, in maximizing our expectation, if we bet on a certain horse, then we bet on all horses which are as good or better bets in this ordering. It also asserts that the optimal amounts of wager are uniquely determined by the set of horses on which we bet.

LEMMA 3. *The only positive maxima of F are achieved by* $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ *of the form*:

$$\bar{x}_1 = \cdots = \bar{x}_{t-1} = 0, \qquad \bar{x}_t > 0, \cdots, \bar{x}_n > 0 \tag{8}$$

where $\overline{X}$ *is uniquely determined by t as follows*:

$$\bar{x}_i = \lambda_t \sqrt{p_i s_i} - s_i \qquad \textit{for } i = t, \cdots, n. \tag{7}$$

Also

$$\rho_i \leqq \frac{1}{\lambda_t^2} \textit{ for } i = 1, \cdots, t-1 \tag{9}$$

where

$$\lambda_t^2 = Q \sum_{j=1}^{t-1} s_j \Big/ \left(1 - Q \sum_{j=t}^{n} p_j\right). \tag{10}$$

Remark: Lemma 2 assures us that $t > 1$, so that (10) is guaranteed a meaning.

Proof: From (6)

$$\frac{1}{\lambda^2} = \frac{1 - Q \displaystyle\sum_{j=1}^{n} \frac{p_j \bar{x}_j}{\bar{x}_j + s_j}}{Q \displaystyle\sum_{j=1}^{n} (\bar{x}_j + s_j)}. \tag{11}$$

Remark that $\bar{x}_i > 0$ implies

$$\rho_i > \frac{p_i s_i}{(\bar{x}_i + s_i)^2} = \frac{1}{\lambda^2} \tag{12}$$

while $\bar{x}_i = 0$ implies

$$\left(\frac{\partial F}{\partial x_i}\right)_{X=\overline{X}} \leqq 0$$

and (6) becomes

$$Q\sum_{j=1}^{n}\frac{p_j\bar{x}_j}{\bar{x}_j+s_j}+\rho_i Q\sum_{j=1}^{n}(\bar{x}_j+s_j)-1\leqq 0.$$

A glance at (11) shows that now

$$\rho_i \leqq \frac{1}{\lambda^2}. \tag{9}$$

Hence (8) is the only possible form for a solution.

Suppose (8) to hold for some t, we calculate λ by substituting from (7) for $i=1, \cdots, t-1$ into (6). The resulting λ depends on t; it is denoted by λ_t and given by (10).

LEMMA 4. *If*

$$\rho_t > \frac{1}{\lambda_t^2},$$

then

$$\rho_t > \frac{1}{\lambda_{t+1}^2} \qquad \textit{for } t = 2, \cdots, n-1.$$

For

$$\frac{1}{\lambda_{t+1}^2}=\left[\left(1-Q\sum_{j=t}^{n}p_j\right)+Qp_t\right]\Big/Q\sum_{j=1}^{t}s_j$$

and

$$1-Q\sum_{j=t}^{n}p_j=Q\sum_{j=1}^{t-1}s_j\Big/\lambda_t^2<Q\rho_t\sum_{j=1}^{t}s_j,$$

so that

$$\frac{1}{\lambda_{t+1}^2}<\left[\rho_t\sum_{j=1}^{t-1}s_j+\rho_t s_t\right]\Big/\sum_{j=1}^{t}s_j=\rho_t.$$

Let us write the inequalities

$$\rho_v > 1/\lambda_v^2 = \left[q+\sum_{j=1}^{v-1}p_j\right]\Big/\sum_{j=1}^{v-1}s_j \tag{K_v}$$

for $v=2, \cdots, n$. Here $q=(1-Q)/Q$, the new form of λ_v being convenient for computation.

LEMMA 5. *If any of the (K_v) hold, F has a positive maximum.*

Proof: If (K_v) holds with $v<n$, then, from Lemma 4, $\rho_{v+1}\geqq\rho_v>1/\lambda_{v+1}^2$ so that (K_{v+1}) holds. Then (K_n) holds and it is

$$\rho_n > [1-Qp_n]/Q\sum_{j=1}^{n-1}s_j=\left[\frac{1}{Q}-\rho_n s_n\right]\Big/\left[\sum_{j=1}^{n}s_j-s_n\right]$$

or

$$1/Q \sum_{j=1}^{n} s_j < \rho_n = \max \frac{p_j}{s_j},$$

and we apply Lemma 1.

THEOREM. *If all of (K_v) are false, the $\bar{x}_j$ are all zero. Otherwise the t of Lemma 3 is the smallest v such that (K_v) is true.*

Proof: Suppose all the (K_v) false. If the $\bar{x}_j$ were not all zero, we would have the situation described in Lemma 3 with a $t>1$. Then (7) for $i=t$ would imply the truth of (K_t).

If any (K_v) hold, Lemma 5 assures us that t exists. Again (7) tells us that (K_t) is true. Let $t>2$ and suppose (K_{t-1}) true. Then $\rho_{t-1}>1/\lambda_{t-1}^2$ and, from Lemma 4, $\rho_{t-1}>1/\lambda_t^2$, contradicting (9) with $i=t-1$.

The calculatory procedure is now clear: We successively test the inequalities (K_v). The first true one (if any) furnishes us t and λ_t. The $\bar{x}_j$ are computed from (7).

A Remark: We show now that if some of the s_j are zero a solution need not exist. Let us take $n=2$, $s_1>0$, $s_2=0$, and $p_2>0$. Then a brief calculation shows that

$$\sup F(x_1, x_2) = F(0, 0+) = Qp_2s_1 > 0$$

while

$$F(0, 0) = 0.$$

An example: Applying our method to the above example and taking Q as 0.9, we find we should wager

$28.40 on *B*
50.33 on *C*
43.02 on *E*.

4. Some ramifications. R. M. Thrall* has developed some interesting variants. The present mathematical scheme fits a problem of merchandising economics; here the restriction $Q<1$ may be dropped. Then the bigger the x_j, the bigger F and a maximum no longer exists unless the sum of the x_j be bounded.

Reverting to the original setting, he observes that after a bettor has made an optimal wager, it may still be possible for a second bettor to do likewise with a positive expected profit. It will be small and the first bettor's profit will be severely diminished. Thrall's problem is: What is our optimal bet if we know that a succeeding bettor is to wager optimally?

* R. M. Thrall, Some results in non-linear programming. To be published.

MATHEMATICAL NOTES

Edited by F. A. Ficken, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

THE MATRICES *AB* AND *BA*

W. V. Parker, Alabama Polytechnic Institute

There has been considerable interest recently in the pair of matrices AB and BA. It is well known that, for square matrices A and B, the matrices AB and BA have the same characteristic equation [1]. The proof given here may be of interest because it is short and elementary.

If A is non-singular, we have

$$A^{I}(AB)A = BA$$

and hence the matrices have the same characteristic equation since they are similar. If both A and B are singular Roth [2] has pointed out that AB and BA need not be similar. Flanders [3] has recently shown that AB and BA have the same elementary divisors except for those corresponding to the characteristic root zero even when A and B are not square matrices.

Now write

$$F(x, y) = |Ix - (A - Iy)B| = x^n + f_1(y)x^{n-1} + \cdots + f_n(y) \tag{1}$$

and

$$G(x, y) = |Ix - B(A - Iy)| = x^n + g_1(y)x^{n-1} + \cdots + g_n(y). \tag{2}$$

Since $F(x, y)$ and $G(x, y)$ are polynomials of degree n in x and $y, f_i(y)$ and $g_i(y)$ are polynomials of degrees not greater than n. Also from the statement above $f_i(y) = g_i(y)$ whenever $|A - Iy| \neq 0$. Since these polynomials are equal for an infinite number of values of y it follows that $f_i(y) \equiv g_i(y)$ and $F(x, y) \equiv G(x, y)$. Setting $y = 0$ we have

$$|Ix - AB| \equiv |Ix - BA|.$$

The argument here is based on a field with an infinite number of elements. The referee has suggested that this is not necessary for the proof to be valid. It would suffice to have a field with more than $2n$ elements.

References

1. H. S. Thurston, On the characteristic equation of products of square matrices, this Monthly, vol. 38, 1931, pp. 322–324.

2. W. E. Roth, A theorem on matrices, this Monthly, vol. 44, 1937, p. 95.

3. Harley Flanders, Elementary divisors of AB and BA, Proceedings of the Amer. Math. Soc. vol. 2, 1951, pp. 871–874.

TWO EXAMPLES IN REAL VARIABLES

ALBERT WILANSKY, Lehigh University

Every Riemann integrable function is Lebesgue integrable (we speak only of bounded functions on bounded sets). Furthermore, a Riemann integrable function can be made non-integrable by redefining its values at a set of measure zero; for example, the characteristic function of the rationals is obtained in this way from the identically zero function. Our first example shows that not every Lebesgue integrable function can be obtained in this way; even though such a function can be approximated uniformly, except on a set of arbitrarily small measure, by a continuous (*a fortiori* Riemann integrable) function.

The set of Lebesgue measurable functions includes, as another subclass, the functions of Baire class I. Our second example shows that this class does not include the class of Riemann integrable functions. That it is not included in this latter class follows from example 1 and the fact that every measurable function can be obtained by redefining, on a set of measure 0, the values of a function of Baire class I. In fact, the function constructed for example 1 is itself of Baire class I.

Our examples are as follows:

1. *A bounded Lebesgue integrable function f such that there is no Riemann integrable function g with $f=g$ almost everywhere.*

2. *A Riemann integrable function which is not of Baire class* I.

1. Let E be an open set which includes all the rational numbers in the unit interval, such that $m(E)<\frac{1}{2}$, where $m(E)$ is the Lebesgue measure of E. Let f be the characteristic function of E. Clearly f is lower semi-continuous, thus of Baire class I, thus Lebesgue integrable. Suppose g is a function with $g=f$ almost everywhere. Then there is a set H of measure 0 with $g(x)=1$ for x in $E-H$, $g(x)=0$ for x in $C(E)-H$, where $C(E)$ is the complement of E.

Since E includes a dense set, E is itself dense, as is $E-H$. Thus $g(x)=1$ on a dense set, and so g is discontinuous at each point x where $g(x)\neq 1$, in particular for x in $C(E)-H$. But the measure of $C(E)-H$ is greater than $\frac{1}{2}$ and so g is not Riemann integrable.

2. Let K be the Cantor nowhere-dense perfect set. Let $\{x_n\}$ be a sequence of elements of K which is dense in K, for example the set of endpoints of the "removed intervals." $f(x)=0$ if x is not in K, $f(x)=1$ if $x=x_n$ for some n, $f(x)=2$ if x is in K but $x\neq x_n$ for all n.

Then, f is continuous on a set of measure one and so is Riemann integrable. However, there is a perfect set K such that f, considered as a function on K alone, has no point of continuity. Thus f is not of Baire class I.

Example 1 is implicit in a paper of Oxtoby, Bulletin of the American Mathematical Society, vol. 43, 1937, Th. 3, p. 247. Examples similar to Example 2 occur in the early issues of *Fundamenta Mathematicae.*

where U_1 and U_2 are functions of u^1 and u^2 respectively. Then we have (5) reducible to

$$K_{,2}\, g_{11}\left(\frac{\overset{2}{U_1}U_2 - 1}{U_1^2 U_2}\right) = 0, \qquad K_{,1}\, g_{22}\left(\frac{\overset{2}{U_2}U_1 - 1}{U_2^2 U_1}\right) = 0,$$

which imply that $K_{,1}=0$ and $K_{,2}=0$ in the orthogonal coordinate system. Since $K_{,\alpha}$ is a tensor which vanishes in one coordinate system, it will vanish in all coordinate systems. Hence K is a constant. Therefore the surfaces S and $\bar{S}$ are applicable.

References

1. W. C. Graustein, Differential Geometry, The Macmillan Co., 1947.
2. L. P. Eisenhart, An Introduction to Differential Geometry with Use of the Tensor Calculus, Princeton University Press, 1947.

ON BOSE NUMBERS

B. MISRA, Ravenshaw College, Cuttack, India

1. R. C. Das has established* a method for writing down the repetend of the recurring decimal for $1/n$, n being a positive integer prime to 10. In this connection, two numbers b (Bose number) and e (end number) were defined. These are the least positive integers that satisfy the relation

$$n\cdot e = 10\cdot b - 1. \tag{A}$$

Let $1/n = .\dot{a}_1 a_2 \cdots \dot{a}_k$, so that

$$(10^k - 1)/n = [a_1 a_2 \cdots a_k] \tag{B}$$

where $[a_1a_2 \cdots a_k]$, with a_i integers and $0 \leqq a_i \leqq 9$ $(i=1, \cdots, k)$, stands for the expression $10^{k-1}a_1 + 10^{k-2}a_2 + \cdots + a_k$. Das has shown that from a knowledge of b and e only, the repetend $[a_1a_2 \cdots a_k]$ can be written down. In the present note it is shown that b and e can also be used in a different manner to write down the repetend.

2. The present method consists of dividing the end number e by b, giving rise to a quotient e_1 and a remainder r_1. Then e_1 is placed after r_1, and the number $10r_1 + e_1$ so formed is again divided by b, giving rise to a quotient e_2 and a remainder r_2. The process is repeated. Our object is to prove the

THEOREM (i) *Continuation after k steps of the above process will cause the quotients $e_1, e_2, \cdots, e_k$ $(0 \leqq e_i \leqq 9)$ to repeat in the same order; and* (ii) *$[e_1e_2 \cdots e_k]$ is the repetend of the recurring decimal for $1/n$, i.e. $[e_1e_2 \cdots e_k] = [a_1a_2 \cdots a_k]$.*

Proof: From the method of formation of the quotients $e_1, e_2, \cdots, e_k$ and the remainders $r_1, r_2, \cdots, r_k$, we can immediately write down the following equations

* This MONTHLY, vol. 56, No. 2, 1949, pp. 87–89.

where U_1 and U_2 are functions of u^1 and u^2 respectively. Then we have (5) reducible to

$$K_{,2}\, g_{11}\left(\frac{\overset{2}{U_1}U_2 - 1}{U_1^2 U_2}\right) = 0, \qquad K_{,1}\, g_{22}\left(\frac{\overset{2}{U_2}U_1 - 1}{U_2^2 U_1}\right) = 0,$$

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$$\text{(A)} \qquad n\cdot e = 10\cdot b - 1.$$

Let $1/n = .\dot{a}_1 a_2 \cdots \dot{a}_k$, so that

$$\text{(B)} \qquad (10^k - 1)/n = [a_1 a_2 \cdots a_k]$$

where $[a_1a_2 \cdots a_k]$, with a_i integers and $0 \leqq a_i \leqq 9$ $(i=1, \cdots, k)$, stands for the expression $10^{k-1}a_1 + 10^{k-2}a_2 + \cdots + a_k$. Das has shown that from a knowledge of b and e only, the repetend $[a_1a_2 \cdots a_k]$ can be written down. In the present note it is shown that b and e can also be used in a different manner to write down the repetend.

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Proof: From the method of formation of the quotients $e_1, e_2, \cdots, e_k$ and the remainders $r_1, r_2, \cdots, r_k$, we can immediately write down the following equations

* This Monthly, vol. 56, No. 2, 1949, pp. 87–89.

$$\text{(C)} \qquad \begin{cases} e = be_1 + r_1 \\ 10r_1 + e_1 = be_2 + r_2 \\ \cdots\cdots\cdots\cdots \\ 10r_{k-1} + e_{k-1} = be_k + r_k. \end{cases}$$

Multiplying the k equations (C) by $10^k, 10^{k-1}, \cdots, 10$ successively and adding, we get the equation (D), after simplifying with the help of relation (A);

$$\text{(D)} \qquad (10^k - 1)e - ne[e_1e_2 \cdots e_k] = (10r_k + e_k) - e.$$

The equation (D) shows that e divides $10r_k+e_k$. Let $pe=10r_k+e_k$. Now dividing the equation (D) throughout by ne and using the relation (B) we get

$$\text{(E)} \qquad [a_1a_2 \cdots a_k] - [e_1e_2 \cdots e_k] = (p - 1)/n.$$

It can be easily seen that $10r_k+e_k \leqq 10b-1$, which implies $p \leqq n$.* Relation (E) therefore is an equality between an integer and a proper fraction, which is absurd unless $p=1$ and, consequently, $[e_1e_2 \cdots e_k] = [a_1a_2 \cdots a_k]$. The latter relation establishes the second part of the proposition and $p=1$ implies that $r_k=0$ and $e_k=e$. As a consequence of these values of r_k and e_k, it is obvious that the process repeats itself after the kth step. Hence the proposition is established.

Example: Take $n=123$. Then $b=37$ and $e=3$. Dividing 3 by 37 and continuing the process as indicated above, we have

```
37) 3     ( 0 0 8 1 3
    0
    ---
    30
     0
    ---
    300
    296
    ---
      48
      37
      ---
      111
      111
      ---
       03 (The process repeats after this step.)
```

Hence

$$\frac{1}{123} = .\dot{0}081\dot{3}.$$

* From the inequalities $1 \leqq e \leqq 9$, $1 \leqq b \leqq n-1$, and $0 \leqq r_i \leqq b-1$ (all i) it is not hard to show first that $e_1 \leqq e \leqq 9$, and then that if $e_i \leqq 9$ then $10r_i+e_i=ne_{i+1}+r_{i+1} \leqq 10b-1$. It thus follows by induction that $0 \leqq e_i \leqq 9$ and hence moreover, for $i=k$, that $pe=10r_k+e_k \leqq 10b-1=ne$. (Referee.)

(ii) Take $$m/n = 41/123 \quad (=1/3).$$

Dividing $b=37$ into $me=123$, we get,

$$\begin{array}{l} 37)\ 123\ \ (\ 3 \\ \quad\ \ \underline{111} \\ \quad\ \ \ 123 \text{ (The process repeats after this step.).} \end{array}$$

Hence the repetend is 3. So $41/123=.\dot{3}$.

A NOTE ON NUMBER THEORY*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. Legendre [1] mentions a "very ingenious" proof by Sophie Germain that the Fermat equation

$$a^m + b^m + c^m = 0$$

has no integral solution in which $2m+1$ is prime and divides none of the numbers a, b, and c. He also considered the case in which m divides at least one of the numbers a, b, c whereas Germain [2] had assumed that none of the three is a multiple of m. It is very easy to remove the restriction on whether m divides one or more of the numbers a, b, c in the special case which follows.

2. THEOREM: *If $2m+1(m>2)$ is a prime, there exists no integral solution of the equation*

$$a^m + b^m + c^m = 0$$

in which $2m+1$ fails to divide one of the integers a, b, c.

If a is relatively prime to $2m+1$, Fermat's theorem states that

$$a^{2m} - 1 \equiv (a^m - 1)(a^m + 1) \equiv 0 \pmod{2m+1}$$

since $2m+1$ is prime by hypothesis. Moreover, either a^m-1 or a^m+1, but not both, is divisible by $2m+1$ since their greatest common divisor cannot exceed 2. Hence, using analogous arguments in the case of b and c,

$$(1) \qquad a^m \equiv \pm 1, \qquad b^m \equiv \pm 1, \qquad c^m \equiv \pm 1 \pmod{2m+1},$$

thus proving the theorem.

COROLLARY: *If $4m+1$ is a prime, there exists no integral solution of the equation*

$$a^m + b^m + c^m = 0$$

in which $4m+1$ fails to divide one of the integers a, b, c, $a^{2m}+1$, $b^{2m}+1$, $c^{2m}+1$.

* Translated from the French by R. L. Wilson.

If $a \not\equiv 0 \pmod{4m+1}$, $a^{4m}-1 \equiv (a^{2m}-1)(a^{2m}+1) \equiv 0 \pmod{4m+1}$. Since $a^{2m}+1 \not\equiv 0 \pmod{4m+1}$, $a^{2m}-1 \equiv (a^m-1)(a^m+1) \equiv 0 \pmod{4m+1}$. Thus $a^m \equiv \pm 1 \pmod{4m+1}$ and the proof proceeds as before.

3. THEOREM: *If $2m+1$ is a prime and k is a positive integer, there exists no integral solution of the equation*

$$[k(2m+1)-1]a^m = b^m + c^m \tag{2}$$

in which $2m+1$ fails to divide one of the integers a, b, c.

Assume that none of the numbers a, b, c is divisible by $2m+1$. By Theorem 1, $a^m \equiv \pm 1 \pmod{2m+1}$ and hence $[k(2m+1)-1]a^m \equiv \pm 1 \pmod{2m+1}$. On the other hand $b^m \equiv \pm 1$, $c^m \equiv \pm 1 \pmod{2m+1}$, whence either $b^m+c^m \equiv 0$ or $b^m+c^m \equiv \pm 2 \pmod{2m+1}$. Therefore, at least one of the integers a, b, c must be a multiple of $2m+1$.

Example. If $m=3$, $k=2$, equation (2) becomes

$$13a^3 = b^3 + c^3.$$

This possesses at least one integral solution, namely $a=3$, $b=7$, $c=2$.

References

1. Legendre, *Essai sur la Theorie des Nombres*, (Second Supplément), *Sphinx-Oedipe*, 1909, pp. 97–128.
2. Cf. also Bachmann, *Niedere Zahlentheorie*, II, 1910, p. 475.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

TRANSFORMATIONS OF THE LAPLACIAN

R. P. AGNEW, Cornell University

It is the purpose of this note to make some simple remarks about the familiar tedious procedure involved in using a transformation of coordinates to convert the Laplacian operator from rectangular to spherical coordinates. Let the rectangular, cylindrical, and spherical coordinates of a point be, respectively, (x, y, z), (ρ, ϕ, z), and (r, ϕ, θ). The problem is to start with the Laplacian Δu defined by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{1}$$

If $a \not\equiv 0 \pmod{4m+1}$, $a^{4m}-1 \equiv (a^{2m}-1)(a^{2m}+1) \equiv 0 \pmod{4m+1}$. Since $a^{2m}+1 \not\equiv 0 \pmod{4m+1}$, $a^{2m}-1 \equiv (a^m-1)(a^m+1) \equiv 0 \pmod{4m+1}$. Thus $a^m \equiv \pm 1 \pmod{4m+1}$ and the proof proceeds as before.

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$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{1}$$

been adequately publicized. In the first place, (2) is very often derived by tedious use of (3). In the second place, there have been times when (5) was derived by use of (4), but the derivation of (2) was omitted on the ground that the derivation was too tedious.

Finally, we may ask whether there are other occasions in which one can make significant use of the fact that, except for the names of the variables involved, the transformations (4) and (6) are identical.

THE BERNOULLI DIFFERENTIAL EQUATION

A. Buckley, Huddersfield Technical College

In an interesting article on the differential equation $(dy/dx)+Py=Qy^n$, under Classroom Notes (this Monthly vol. 59, 1952, pp. 632–633), M. R. Spiegel discusses solutions in the particular cases $n=0$ and 1. The same conclusions are evident if the equation is solved by the following method.

Multiply throughout the equation by an integrating factor I of the left-hand side, to give

$$I\left(\frac{dy}{dx}+Py\right)=Qy^n\frac{I^n}{I^{n-1}},$$

which reduces to

$$\frac{d(Iy)}{(Iy)^n}=\frac{Q}{I^{n-1}}\,dx.$$

The advantage of this form is that it holds for $n=0$ and 1 and only in the latter case does the easily derived solution differ from that obtained for all other values of n.

This method of solution was first published by Professor Neville in the Mathematical Gazette (vol. XVIII, 1934, p. 321) and was referred to by the present writer in a note to the same journal (vol. XXXIV, 1950, p. 304).

A MODIFICATION OF THE SIEVE OF ERATOSTHENES

V. C. Harris, San Diego State College

The following modification of the method known as the sieve of Eratosthenes may be of interest to classes studying elementary number theory.

Write the integers $1, 2, \cdots, n$ in natural order. Cross out every integer. Cross out every second integer, beginning with 2. Cross out every third integer, beginning with 3. Continue similarly for $4, 5, \cdots, n$. We find

(a) each unit has been crossed out once (there is just one unit, namely, 1),

(b) each prime has been crossed out twice, and the number of these is $\pi(n)$, the number of primes $\leqq n$,

(c) each composite number m, $1\leqq m\leqq n$, has been crossed out a number of times equal to $\tau(m)$, the number of divisors of m.

Centering our attention on a particular integer, say m, $1 \leqq m \leqq n$, we can write down the integer begun with each time that m is crossed out. Then

(d) the sum of these is $\sigma(m)$, the sum of the divisors of m,

(e) of these, the integers crossed out twice are the distinct prime divisors of m.

If, however, the process as described is carried out not for all of $1, 2, \cdots, n$ but only for the distinct prime divisors of n, then

(f) the number of integers not crossed out is $\phi(n)$, the number of (positive) integers $\leqq n$ and relatively prime to n.

THE RATE OF INTEREST IN INSTALLMENT PAYMENT PLANS

H. E. Stelson, Michigan State College

The purpose of this paper is to develop by a new method the formulas which are now in current use, to compare them to compound interest, and to present some new formulas.

1. Historical approach. The formulas which are currently presented in texts are known as the Constant Ratio, Series of Payments and Interest at End formulas. The formulas have been developed independently from different assumptions.

The Constant Ratio formula is derived on the assumption that each payment is composed of a principal repayment and an interest repayment in the same ratio that the original unpaid balance is to the interest. The basic assumption for the Series of Payments formula is that the sum of the series of payments is the outstanding debt at the beginning of the installment term. The Interest at End or Residuary formula* is derived on the assumption that the payments should be used to repay the principal first, then after the principal has been repaid, to pay the interest.

Formulas for the three methods are obtained by considering the principal outstanding during each period, finding the sum of all these principals (an arithmetic progression) and substituting this total principal, considered to be in use for one payment period, in the simple interest formula.

2. Alternate derivation—new method. The values of different obligations can be compared only if they are accumulated or discounted to the same date. That is, the value of the debt on a certain date must equal the value of the payments. The date chosen for comparison is called the comparison or focal date.

We now make the following assumption, which will be used to derive the three formulas already mentioned. For all payments (or debts) made before the focal date, the accumulation factor $(1+nr)$ is used, while for all payments made after the focal date, the discount factor $(1-nr)$ is used.

The following symbols are used:

* This assumption is equivalent to the Merchant's Rule.

Centering our attention on a particular integer, say m, $1 \leqq m \leqq n$, we can write down the integer begun with each time that m is crossed out. Then

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The following symbols are used:

* This assumption is equivalent to the Merchant's Rule.

R = periodic payment,
r = periodic rate,
n = number of periodic payments (not counting the down payment),
B = unpaid balance at the beginning of the credit period (cash price less the down payment, if any),
I = total carrying charge or cost of the loan, or
$I = Rn - B$.

As an aid in considering the focal date, we present a line diagram for an installment payment plan.

$$\begin{array}{lccccc} \text{Payments} & & R & R & \cdots & R \\ & | & \text{———}|\text{———} & | & \text{———} & | \\ \text{Time} & 0 & 1 & 2 & \cdots & n. \end{array}$$

(A) If the focal date is taken at n in the line diagram (the end of the payment period), we use the Merchant's rule to obtain the equation,

$$B(1 + nr) = R[1 + (n - 1)r] + R[1 + (n - 2)r] + \cdots + R(1 + r) + R.$$

Solving this equation for r gives the Interest at End (or the Residuary) formula,

$$r_I = \frac{2I}{n(B + R - I)} = \frac{2I}{B(n + 1) - I(n - 1)}.$$

(B) If the focal date is taken at the average time, $(n+1)/2$, we obtain the Constant Ratio formula,

$$r_c = \frac{2I}{B(n + 1)}.$$

(C) If the focal date is taken at 0 (the beginning of the payment period), we have the Series of Payments formula,

$$r_s = \frac{2I}{Rn(n + 1)} = \frac{2I}{B(n + 1) + I(n + 1)}.$$

It can be shown by using the second formulas for r_I and r_s that the denominator for r is increased by $2I$ for each period the focal date is moved to the left in the line diagram. Hence

$$r_s < r_c < r_I.$$

3. A new formula—interest charged uniformly. In order to get a closer approximation to r, the compound interest rate, we move the focal date to $n/2$. This gives the very simple formula

$$r_B = \frac{2I}{n(B + R)}.$$

It can readily be proved that r_B is the harmonic mean of r_s and r_I.

In the derivation of the formulas for Constant Ratio, Series of Payments, and Interest at End, it is assumed that the charge for the loan, $I=Rn-B$, is made at the end of the period. It is more accurate to assume that the charge is paid uniformly throughout the payment interval or that it is paid at the middle of the payment interval. In the Constant Ratio method, the focal date is taken at time $(n+1)/2$. If the interest charge is paid at time $n/2$ then the debtor gets to hold the interest for $1/2$ period and the debt should be increased by $rI/2$. Hence using the focal date $(n+1)/2$ we get

$$B\left(1+\frac{n+1}{2}r\right)+\frac{Ir}{2}=R\left(1+\frac{n-1}{2}r\right)+\cdots+R\left(1+\frac{r}{2}\right)$$
$$+R\left(1-\frac{r}{2}\right)+R\left(1-\frac{3r}{2}\right)+\cdots+R\left(1-\frac{n-1}{2}r\right)$$

or

$$r=\frac{2I}{n(B+R)}=r_B.$$

If a similar correction is made in the same manner to the formulas for Interest at End and Series of Payments, they also reduce to the formula for r_B.

In order to compare the accuracy of r_B with r, we expand the formula for compound interest,

$$B = Ra_n \text{ in the series, } (Rn = B + I)$$

$$B=\frac{2I}{r(n+1)}\left[1-\frac{(n-1)r}{6}+\frac{(n-1)(n-2)r^2}{36}+\cdots\right]. \tag{1}$$

Also

$$B=\frac{2I}{r_B(n+1)}\left[1-\frac{r_B}{2}\right]. \tag{2}$$

Dividing (1) and (2) and solving for r_B-r we obtain the error,

$$r_B-r=\frac{n-4}{6}r^2-\frac{n-2}{4}r^3\cdots.$$

4. The direct ratio formula—Interest charged throughout the payment period in proportion to the loan outstanding. The direct ratio formula assumes that the total interest charge is distributed to each installment in proportion to the outstanding balance at that time. Under this assumption, we find the average time for the total loan. Since the principal outstanding for the first period is B, for the second period is $B-(B/n)$, *etc.*, the average time is

$$\frac{B\cdot 1+\left(B-\frac{B}{n}\right)\cdot 2+\left(B-\frac{2B}{n}\right)\cdot 3\cdots+\left[B-\frac{(n-1)B}{n}\right]n}{B\quad+\left(B-\frac{B}{n}\right)\quad+\left(B-\frac{2B}{n}\right)\quad\cdots+\left[B-\frac{(n-1)B}{n}\right]}=\frac{n+2}{3}.$$

Placing the focal date at $(n+2)/3$ and using the line diagram, we obtain in the same way that the formulas for r_s, r_c, and r_I were derived, a formula for the rate

$$r_a=\frac{6I}{3B(n+1)+I(n-1)}.$$

This formula gives results remarkably close to the compound interest rate.

The error is given by

$$r-r_a<\frac{(n-1)(n+2)r^3}{36}.$$

The formula for r_a assumes that all of the payments are equal. However, many installment plans have an irregular last payment. Again, we assume the focal date at $(n+2)/3$, and using the line diagram in the same manner as in the derivation of r_a we obtain the formula

$$r_a'=\frac{2I}{n(B+Z)+(I/3)(n-4)}$$

where Z is the last irregular payment.

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4. Rider and Fischer, Mathematics of Investment, Rinehart & Co., 1951, pp. 19–22.

MORE ON TAYLOR'S THEOREM IN A FIRST COURSE

C. P. NICHOLAS, U.S. Military Academy

This MONTHLY (vol. 58, 1951, pp. 559–562) carried a derivation of Taylor's Theorem which I offered as suitable for simplifying the subject in a first course. The derivation can be further simplified, as shown below. Like the earlier derivation, the new one presupposes that the student can evaluate a successive integral; and also that he has been led into the problem sufficiently to know that we wish to find an approximate value of $f(x)$, based on known values of $f(x)$ and its first n derivatives at a neighboring point $x=a$.

E 1070. *Proposed by A. W. Walker, University of Toronto*

Find the lowest order differential equation satisfied by all conics with a given eccentricity e.

SOLUTIONS

Jumping Off an Asteroid

E 1036 [1952, 633] *Proposed by W. R. Ransom, Tufts College*

How large an asteroid could a man jump clear off of?

Solution by W. M. Stone, Boeing Airplane Co., Seattle, Wash. Using the subscript e to refer to earth magnitudes, the acceleration due to gravity at the surface of any spherical asteroid of radius R and mass M would be

$$g = g_e M R_e^2 / M_e R^2.$$

To jump a vertical height h on the earth's surface a man must acquire a velocity $v_e = (2g_e h)^{1/2}$. We equate this velocity to the velocity of escape from the asteroid, $v = (2gR)^{1/2}$:

$$2g_e h = 2g_e M R_e^2 / M_e R = 2g_e C R^2 / C_e R_e,$$

or

$$R^2 = (C_e/C) R_e h,$$

where C and C_e are the specific densities of the asteroid and earth respectively.

Also solved by Ray Aronson and H. S. Wilf (jointly), R. H. Boyer, Julian Braun, C. W. Bruce, A. M. Glicksman, H. W. Hickey, P. F. Hultquist, P. B. Johnson, M. S. Klamkin, C. S. Ogilvy, Azriel Rosenfeld, C. M. Sandwick, Sr., C. Swanson, and the proposer.

Glicksman considered also the interesting possibility where the man jumps horizontally, that is, tangentially. The man could then continue to move around the asteroid, like a satellite, just above its surface, and he could be said to have jumped "clear" of the asteroid. In this case one finds

$$R^2 = 2(C_e/C) R_e h,$$

whence "a man can jump completely *around* an asteroid whose radius is $\sqrt{2}$ times as large as that of the biggest asteroid he could jump completely *away* from."

Johnson emphasized the fact that h is the height a man can lift his center of gravity, and must not be confused with his high jump mark. In a good high jump the jumper's center of gravity starts off several feet above the ground and actually passes under the bar as the jumper goes over. It takes a good jumper to have an h of two feet.

Criterion for a Real Cubic to Have Pure Imaginary Roots

E 1037 [1952, 633]. *Proposed by A. G. Anderson, Oberlin College*

Show that a necessary and sufficient condition that a real cubic equation $ax^3+bx^2+cx+d=0$ have one real and two pure imaginary roots is that $ba=ad$ and $ac>0$.

Solution by Sidney Kravitz, Great Notch, N. J. The equation whose roots are A, $+Bi$, and $-Bi$, where A and $B\neq 0$ are real, is $x^3-Ax^2+B^2x-AB^2=0$, thus satisfying the conditions $bc=ad$ and $ac>0$. Conversely, if $bc=ad$ and $ac>0$, then the real cubic equation $ax^3+bx^2+cx+d=0$ has the real root $-b/a$, and the two pure imaginary roots $+\sqrt{c/a}\,i$ and $-\sqrt{c/a}\,i$.

Also solved by J. M. Anderson, J. W. Baldwin, A. P. Boblétt, Julian Braun, W. E. Briggs, C. S. Carlson, F. E. Cothran, Marian E. Daniells, I. A. Dodes, David Ellis, H. Emich, A. L. Epstein, H. M. Feldman, Calvin Foreman, Arthur Gregory, Bernard Greenspan, Emil Grosswald, S. W. Hahn, B. A. Hausmann, Vern Hoggatt, Douglas Holdridge, Willard James, W. C. Janes, John Jones, Jr., Ray Jurgensen, Aida Kalish, J. M. Kingston, M. S. Klamkin, J. D. E. Konhauser, A. E. Livingston, D. C. B. Marsh, B. Martin, Morris Morduchow, Leo Moser, Roberto Muguercia, C. S. Ogilvy, L. L. Pennisi, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., John Sawyer, A. Sisk, William Small, P. Somanadham and K. Subba Rao (jointly), C. R. Sparks, R. H. Sprague, W. M. Stone, C. Swanson, W. R. Talbot, W. C. Taylor, Jr., J. A. Tierney, Peter Treuenfels, R. Z. Vause, R. R. Williams, Jr., David Zeitlin, and the proposer. Late solutions by S. Parameswaran and Louisa S. Grenstein.

Pennisi pointed out that if the condition $bc=ad$ and $ac>0$ be changed to $bc=ad$ and $ac<0$ we get a necessary and sufficient condition that all three roots be real with two of them equal but of opposite sign.

James noted that since $bc=ad$ and $ac>0$ imply that a and c must have the same sign and that b and d must have the same sign we have an easy and quick check on the possibility of a real cubic having pure imaginary roots.

Stone called attention to von Karman and Biot, *Mathematical Methods in Engineering*, McGraw-Hill, New York (1940), p. 242.

Divisibility by 27

E 1038 [1952, 633]. *Proposed by S. B. Townes, University of Hawaii*

Show that if a number of $3n$ digits is divisible by 27, then any number with the same digits cyclically permuted will also be divisible by 27.

Solution by Leo Moser, University of Alberta. We shall prove the following more general result:

If N has sn digits when expressed in base r, then $t\mid N$, $t\mid(r^s-1)$, and $(r, t)=1$ imply t divides any number obtained from N by a cyclic permutation of its digits (in base r).

Proof. Let the last (on the right) digit of N be a. It clearly suffices to consider

the effect of putting this digit first. If the resulting number is M then

$$M = (N - a)/r + ar^{sn-1} = [N + a(r^{sn} - 1)]/r.$$

Now $t|(r^s-1)$ and $(r^s-1)|(r^{sn}-1)$, so that $t|(r^{sn}-1)$. Further, $t|N$ and $(r, t)=1$. It follows that $t|M$, and the proof is complete.

The given problem is the case $r=10$, $s=3$, $t=27$. The case $r=10$, $s=3$, $t=37$ shows that 27 may be replaced by 37 in the given problem.

Also solved by Norman Anning, J. W. Baldwin, R. H. Boyer, Julian Braun, Monte Dernham, I. A. Dodes, A. L. Epstein, Lloyd Fulk, Bernard Greenspan, Arthur Gregory, S. W. Hahn, B. A. Hausmann, Vern Hoggatt, Douglas Holdridge, J. E. Householder, H. K. Humphrey, H. I. James, M. S. Klamkin, T. C. Littlejohn, A. E. Livingston, D. C. B. Marsh, L. A. Ringenberg, C. M. Sandwick, Sr., P. Somanadham and K. Subba Rao (jointly), R. H. Sprague, J. A. Tierney, R. Z. Vause, Leroy Warren, and the proposer. Late solutions by J. P. Jackson, Herb Lechner, Hilbert Levitz, R. J. Painter, S. Parameswaran, and R. H. Spencer.

Minimum Product of Perpendicular Diameters of an Ellipse

E 1039 [1952, 633]. *Proposed by I. W. Burr, Purdue University*

Minimize the product of two perpendicular central chords of a given ellipse.

Solution by Julian Braun, Washington, D. C. Employing usual notation, the polar equation of an ellipse with pole at the center is

$$r^2 = b^2/(1 - e^2 \cos^2 \theta).$$

Let

$$r_1 = b/(1 - e^2 \cos^2 \theta_1)^{1/2}.$$

Then the length of the semi-central chord perpendicular to r_1 is

$$r_2 = b/(1 - e^2 \sin^2 \theta_1)^{1/2}.$$

The product of the two perpendicular central chords is

$$4r_1r_2 = 4b^2/[1 - e^2 + (e^4/4) \sin^2 2\theta_1]^{1/2},$$

which is clearly a minimum for $\theta_1=45°$. Thus the required central chords make angles of 45° with the axes of the ellipse and their product is $4b^2/(1-e^2/2)$ or $8a^2b^2/(a^2+b^2)$.

Also solved by Leon Bankoff, A. P. Boblétt, W. E. Briggs, R. C. Clelland, I. A. Dodes, Bernard Greenspan, Arthur Gregory, Vern Hoggatt, Douglas Holdridge, R. Huck, Ray Jurgensen, M. S. Klamkin, D. M. Mandelbaum, Jerome Manheim, D. C. B. Marsh, George Millman, C. N. Mills, R. K. Morley, C. S. Ogilvy, L. A. Ringenberg, P. Somanadham and K. Subba Rao (jointly),

C. R. Sparks, C. Swanson, W. R. Talbot, J. A. Tierney, Peter Treuenfels, H. S. Wilf, and the proposer. Late solutions by H. H. Berry and A. E. Livingston.

Sparks pointed out the interesting fact that the minimum product of two perpendicular central chords of a given ellipse is the *harmonic* mean, whereas the maximum product is the *geometric* mean, of the squares of the axes of the ellipse.

Comparison of Merchant's Rule and Residuary Method

E 1040 [1952, 633]. *Proposed by H. E. Stelson, Michigan State College*

On page 20 of Rider and Fischer, *Mathematics of Investment*, is the statement, "It does not appear to have been generally recognized that the assumptions underlying the Residuary method are equivalent to those of the Merchant's Rule and hence the two methods yield identical results."

Prove that the Merchant's Rule and the Residuary (Interest at End) method give the same results for the case where the payments are all equal except the last and where the interest payment amounts to more than a single payment.

Solution by Julian Braun, Washington, D. C. Let A = initial unpaid balance, K = carrying charge, m = total number of payments per year, n = total number of payments in the contract, i = simple rate of interest, P = amount of each payment (except the last).

In the Residuary method the amount on which interest is computed for the jth period is $A-(j-1)P$, so that, according to this method,

$$K = (i/m)\sum_{j=1}^{n} [A - (j - 1)P],$$

whence

$$i = 2mK/[2An - nP(n - 1)].$$

In the Merchant's Rule the principal earns interest to the final date. The jth payment ($j<n$) earns interest for $n-j$ periods, and the last payment is $(A+K)-(n-1)P$. Thus, for this method,

$$A(1 + ni/m) = P\sum_{j=1}^{n-1} [1 + (n - j)i/m] + (A + K) - (n - 1)P.$$

Solving this for i yields the same result as above.

Also solved by the proposer.

C. R. Sparks, C. Swanson, W. R. Talbot, J. A. Tierney, Peter Treuenfels, H. S. Wilf, and the proposer. Late solutions by H. H. Berry and A. E. Livingston.

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Solving this for i yields the same result as above.

Also solved by the proposer.

Let $\mu(j)$ be the Möbius function.

Let $\lambda(j)=\alpha_1+\alpha_2+\cdots$ for $j=p_1^{\alpha_1}p_2^{\alpha_2}\cdots$, (the decomposition of j into prime powers).

Prove that, for all integers j and t such that $t>\lambda(j)$,

$$\mu(j) = -\sum_{k=1}^{t}(-1)^k\binom{t}{k}d_{k-2}(j).$$

4542. *Proposed by C. D. Olds, San Jose State College, Calif*

The n points (x_i, y_i), $i=1, 2, \cdots, n$, should lie on a circle but they fail to do so. What circle shall we take which most nearly fulfills the condition that $\sum d_i^2$ is a minimum where

$$d_i = [(h-x_i)^2+(k-y_i)^2]^{1/2} - r?$$

The origin of coördinates is at the centroid of the given points; and (h, k) is the center, r the radius, of the circle to be found. If necessary, assume that d_i is small compared with r.

SOLUTIONS

Area of the Morley Triangle of a Triangle

4477 [1952, 110]. *Proposed by C. E. Springer, University of Oklahoma*

Given a triangle with angles α, β, γ. Show that the ratio of the area of the Morley triangle of the given triangle to the area of the given triangle is

$$\frac{\left(\prod \sin^2\dfrac{\alpha}{3}\right)\left(1-4\sum\cos^2\dfrac{\alpha}{3}+16\prod\cos\dfrac{\alpha}{3}\right)}{\left(\prod\sin\alpha\right)\left(\prod\sin\dfrac{\pi-\alpha}{3}\right)}.$$

(If the angles of a triangle be trisected, the intersections of the pair of trisectors adjacent to each side determine an equilateral triangle called the Morley triangle of the given triangle. See also [1943, 552].)

Solution by Roscoe Woods, The State University of Iowa. Let the given triangle ABC be chosen as the reference triangle of a system of trilinear normal coordinates. Then it is known that the area K of any triangle $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ is given by the formula $2K=(R/S)(x_1y_2z_3)$, where R and S are respectively the circumradius and area of the triangle ABC, and where the symbol $(x_1y_2z_3)$ is a familiar determinant of the third order with the elements x_1, y_2, z_3 along its principal diagonal. In the application of this formula the coördinates x_i, y_i, z_i must be the actual coördinates of the points $P_i(i=1, 2, 3)$.

Let P_1 be the vertex of the Morley triangle formed by the intersections of the trisectors adjacent to the side a of the triangle ABC. The actual coördinates of the vertex P_1 are then readily found by the use of trigonometry to be

$$\left\{ \frac{a \sin \frac{\beta}{3} \sin \frac{\gamma}{3}}{\sin\left(\frac{\beta+\gamma}{3}\right)}, \quad \frac{2a \sin \frac{\beta}{3} \sin \frac{\gamma}{3} \cos \frac{\gamma}{3}}{\sin\left(\frac{\beta+\gamma}{3}\right)}, \quad \frac{2a \sin \frac{\beta}{3} \sin \frac{\gamma}{3} \cos \frac{\beta}{3}}{\sin\left(\frac{\beta+\gamma}{3}\right)} \right\}.$$

The coördinates of the other two vertices P_2 and P_3 of the Morley triangle may be written down by permuting the symbols a, b, c and α, β, γ. The area K of the Morley triangle is then readily found by the above formula to be

$$(1) \qquad 2K = \frac{Rabc\left(\prod \sin^2 \frac{\alpha}{3}\right)\left(1 - 4\sum \cos^2 \frac{\alpha}{3} + 16 \prod \cos \frac{\alpha}{3}\right)}{S\left(\prod \sin \frac{\pi - \alpha}{3}\right)}.$$

From the relations $4RS = abc$ and $S = 2R^2 \sin \alpha \sin \beta \sin \gamma$, equation (1) reduces at once to the proposed form.

Also solved by Mrs. R. A. Hudson and the Proposer.

Basis in a Banach Space

4478 [1952, 186]. *Proposed by Albert Wilansky, Lehigh University*

A basis in a Banach space B is a set of elements $\{x_n\}$ such that every element of the space is a unique, finite or infinite, linear combination of elements of the set.

(1) Prove that the x_n are isolated.

(2) If $\sum x_n$ is an element of B, show that there is a divergent sequence $\{s_n\}$ of numbers such that $\sum s_n x_n$ is an element of B.

Solution by G. G. Lorentz, University of Toronto, Canada. (1) If x_p is not isolated, then

$$x_p = \lim x_{n_k}, \qquad p < n_1 < n_2 < \cdots.$$

In this case,

$$x_p = x_p,$$

$$x_p = x_{n_1} + (x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \cdots + (x_{n_k} - x_{n_{k-1}}) + \cdots$$

are two different representations of x_p, which is a contradiction.

(2) The hypothesis that the x_n form a basis is not necessary here. Let $\sum x_n$ converge. Then $\|x_n\| \to 0$ and there is a sequence $n_1 < n_2 < \cdots$ such that $\|x_{n_k}\| \leq k^{-3}$. Putting $s_{n_k} = k$, $k = 1, 2, \cdots$, $s_n = 1$ for $n \neq n_k$, we see that $\sum s_n x_n$ is convergent, while s_n diverges.

Also solved by Abraham Charnes, R. C. James, Solomon Leader, Karl Zeller, and the Proposer.

An Infinite Set a_n such that $\sum a_n^p = 0$ for all p

4479 [1952, 187]. *Proposed by D. J. Newman, Harvard University*

Suppose that all the series

$$\sum_{n=0}^{\infty} \alpha_n, \quad \sum_{n=0}^{\infty} \alpha_n^2, \quad \sum_{n=0}^{\infty} \alpha_n^3, \cdots$$

converge to zero, where the α_n are complex numbers. Does it follow that all the $\alpha_n = 0$?

Solution by Karl Zeller, Tübingen, Germany. Put $a = \{\alpha_k\}$, $k = 1, 2, \cdots$; $\theta a = \{\theta \alpha_k\}$ for any constant θ; $\phi_p(a) = \sum_{k=0}^{\infty} \alpha_k^p$, whence $\phi_p(\theta a) = \theta^p \cdot \phi_p(a)$. If $b = \{\beta_0, \cdots, \beta_q \neq 0, 0, 0, \cdots\}$ and $c = \{\gamma_0, \cdots, \gamma_j \neq 0, 0, 0, \cdots\}$, let $b \circ c = \{\beta_0, \cdots, \beta_q, \gamma_0, \cdots, \gamma_j, 0, 0, \cdots\}$.

A sequence $a \neq \{0, 0, \cdots\}$ can be exhibited which has the property

$$\phi_p(a) = 0, \qquad p = 1, 2, \cdots.$$

Let a be the sequence whose terms are the non-zero terms of the sequences a_n determined inductively as follows:

$$a_1 = \{1, -1, 0, 0, \cdots\},$$

$$a_n = a_{n-1} \circ \frac{\rho_n}{n} a_{n-1} \circ \cdots \circ \frac{\rho_n}{n} a_{n-1},$$

in which $|\rho_n| = 1$, $\rho_n^n = -1$, and the set $(\rho_n/n)a_{n-1}$ occurs n^n times. It is then true that

$$\phi_n(a_n) = \phi_n(a_{n-1}) + n^n \frac{\rho_n^n}{n^n} \phi_n(a_{n-1}) = 0.$$

Thus, for every $p = 1, 2, \cdots$, we have a relation of the form

$$a = a_p \circ \lambda_2 a_p \circ \lambda_3 a_p \circ \cdots$$

with $\lambda_j \to 0$, from which the desired result follows.

Also solved by Robert Frucht, Fritz Herzog, E. C. Straus, and the Proposer, all of whom note that if $\sum \alpha_n^p$ is absolutely convergent for all p, then all $\alpha_n = 0$.

A Three Dimensional Cantor Set

4480 [1952, 187]. *Proposed by Rufus Isaacs, the Rand Corporation, Santa Monica, Calif.*

Given a closed solid polyhedron with faces which are equilateral triangles, we can construct another by the following process. Join the mid-edge points of each face dividing it into four triangles. On each face place externally a regular tetrahedron having the central triangle for a base.

If we start with a regular tetrahedron and repeat this process, show that the limiting figure is a cube less a certain point set which is non-enumerable yet of measure zero.

Solution by E. M. Zaustinsky, University of Southern Calif. Let the length of an edge of the given tetrahedron be 1. If at the nth step of the construction the figure has M faces each with edge s, at the $(n+1)$th step it will have $6M$ faces each with edge $s/2$. Now the $(n+1)$th step adds the volume of M tetrahedrons with edge $s/2$. We note that the vertices of the original tetrahedron and the new vertices produced by the first step are the eight vertices of a cube of edge $\sqrt{2}/2$. Since the volume of a regular tetrahedron with edge s is $V = s^3\sqrt{2}/12$, the sum of the volumes of the tetrahedrons up to the nth step is

$$V_n = \frac{\sqrt{2}}{12}\left\{1 + \frac{4}{8} + \frac{4\cdot 6}{8^2} + \cdots + \frac{4\cdot 6^{n-1}}{8^n}\right\}.$$

The volume of the cube is $\sqrt{2}/4$. We regard the tetrahedrons as being subtracted from the cube during the construction. The complement, then, of the sum of the tetrahedrons with respect to the cube is a Borel set, is measurable, and has measure at the nth step equal to

$$\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12}\left\{1 + \frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{6}{8}\right)^i\right\},$$

whence the measure of the residual set when $n \to \infty$ is 0.

Now let us consider that subset of the residual set which lies in one edge of the cube. At the nth step of the construction we removed only a finite number of points from this subset and therefore only a countable number have been removed from the limiting set. The points in the residual set along the edge are therefore uncountable and, *a fortiori*, so are the points of the entire residual set.

Also solved by the Proposer.

Cubes with Same Unit's Digit

4481 [1952, 187]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration with base B, there are n one-digit numbers less than B whose cubes have $B-1$ as the unit's digit. Determine the relation between B and n.

Solution by Emil Grosswald, Institute for Advanced Study. Let

$$B = 3^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} q_1^{b_1} \cdots q_t^{b_t}$$

where p_i, q_j are primes, $p_i \equiv 1 \pmod 3$, $q_j \equiv 2 \pmod 3$. We will show that

If we start with a regular tetrahedron and repeat this process, show that the limiting figure is a cube less a certain point set which is non-enumerable yet of measure zero.

Solution by E. M. Zaustinsky, University of Southern Calif. Let the length of an edge of the given tetrahedron be 1. If at the nth step of the construction the figure has M faces each with edge s, at the $(n+1)$th step it will have $6M$ faces each with edge $s/2$. Now the $(n+1)$th step adds the volume of M tetrahedrons with edge $s/2$. We note that the vertices of the original tetrahedron and the new vertices produced by the first step are the eight vertices of a cube of edge $\sqrt{2}/2$. Since the volume of a regular tetrahedron with edge s is $V = s^3\sqrt{2}/12$, the sum of the volumes of the tetrahedrons up to the nth step is

$$V_n = \frac{\sqrt{2}}{12}\left\{1 + \frac{4}{8} + \frac{4\cdot 6}{8^2} + \cdots + \frac{4\cdot 6^{n-1}}{8^n}\right\}.$$

The volume of the cube is $\sqrt{2}/4$. We regard the tetrahedrons as being subtracted from the cube during the construction. The complement, then, of the sum of the tetrahedrons with respect to the cube is a Borel set, is measurable, and has measure at the nth step equal to

$$\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12}\left\{1 + \frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{6}{8}\right)^i\right\},$$

whence the measure of the residual set when $n \to \infty$ is 0.

Now let us consider that subset of the residual set which lies in one edge of the cube. At the nth step of the construction we removed only a finite number of points from this subset and therefore only a countable number have been removed from the limiting set. The points in the residual set along the edge are therefore uncountable and, *a fortiori*, so are the points of the entire residual set.

Also solved by the Proposer.

Cubes with Same Unit's Digit

4481 [1952, 187]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration with base B, there are n one-digit numbers less than B whose cubes have $B-1$ as the unit's digit. Determine the relation between B and n.

Solution by Emil Grosswald, Institute for Advanced Study. Let

$$B = 3^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} q_1^{b_1} \cdots q_t^{b_t}$$

where p_i, q_j are primes, $p_i \equiv 1 \pmod 3$, $q_j \equiv 2 \pmod 3$. We will show that

that $n_2=3$ if $P=3^a$, $a\geqq 2$ or $P=p$, it follows that $n=n_1$ or $n=3n_1$, respectively. Using the induction assumption for n_1, the validity of (1) follows.

A Measure-preserving Transformation

4482 [1952, 187]. *Proposed by H. S. Shapiro, Chatham, New Jersey.*

Let $n_1<n_2<\cdots$ be any sequence of positive integers (finite or infinite) and let $T_{n_1n_2\cdots}x$, where x $(0<x<1)$ is expanded in dyadic notation, denote a transformation which alters precisely the $n_1, n_2, \cdots$ digits in x. Then $T_{n_1n_2\cdots}$ is measure-preserving on subsets of $(0, 1)$.

Solution by the Proposer. First, for any integer n and any measurable subset A of $(0, 1)$ we have

$$|T_nA| = |A|, \tag{1}$$

where $|A|$ denotes measure of A. Indeed, T_nx translates x by a distance 2^{-n}, to the left or right according as the nth digit of x is odd or even, and (1) follows at once.

Next, for any interval I with dyadic rational end-points (i.e., having terminating dyadic expansion) and any $T=T_{n_1n_2\cdots}$,

$$|TI| = |I|. \tag{2}$$

Indeed, for k large enough $T_{n_{k+1}n_{k+2}\cdots}\ I=I$, hence

$$|T_{n_1n_2\cdots}I| = |T_{n_1\cdots n_k}I| = |I|.$$

This last statement follows from the fact that $T_{n_1\cdots n_k}=T_{n_1}\cdots T_{n_k}$, and from (1).

Finally, let A be an arbitrary measurable set; cover it with intervals $I_1, I_2, \cdots$ each of which has dyadic rational endpoints, and such that $\sum|I_n|<|A|+\epsilon$. Write $T_{n_1n_2\cdots}=T$. Now $A\subseteq\sum I_n$ whence $TA\subseteq T\sum I_n\subseteq\sum TI_n$. Therefore

$$|TA| \leqq |\sum TI_n| \leqq \sum |TI_n| = \sum |I_n| < |A| + \epsilon.$$

Since ϵ is arbitrary, $|TA|\leqq A$. Applying the same reasoning with A replaced by TA. and noting $TTA=A$, we have $|A|\leqq|TA|$, and so $|A|=|TA|$.

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The chapter headings after III are: IV. Projective Geometry of Two Dimensions, V. Conic Loci and Conic Envelopes, VI. Further Properties of Conics, VII. Linear Systems of Conics, VIII. Higher Correspondences, Apolarity, and the Theory of Invariants, IX. Transformations of the Plane, X. Projective Geometry of Three Dimensions, XI. The Quadric, XII. The Twisted Cubic Curve and Cubic Surfaces, XIII. Linear Systems of Quadrics, XIV. Linear Transformations of Space, XV. Line Geometry, XVI. Projective Geometry of n Dimensions.

There is a wealth of excellent material in this text book. At the end of each chapter there is a large list of both illustrative and supplementary exercises. This text is a definite contribution to the field of algebraic projective geometry of one, two and three dimensions and lays a good foundation for further development. The text does not attain the standard of rigor sought by some of our modern writers in the field of algebraic geometry. For example, in their definition of a projective space of n dimensions they state that "The projective properties of $S_n(K)$ are those properties of which the expression in every allowable coordinate system is the same." It is not clear from this definition what is meant by the word "expression." Also on page 105 some explanation of what is meant by a "general point" or a "general figure" is offered but even so a lot of detail is left to the reader. However "in general" the authors appear to have accomplished their main purpose in writing this text.

W. F. Atchison
University of Illinois

Intermediate College Mechanics. By D. E. Christie, McGraw-Hill Book Co. Inc., New York, 1952. 16+454 pages. $7.00.

Chapters 1–11 of this book contain an excellent treatment of the mechanics of a particle, of a system of particles, and of a rigid body. The remaining six chapters contain brief introductions to a number of independent topics of a more advanced nature. The treatment is three-dimensional, and extensive use is made of vector methods. The arrangement of topics is quite good; explanations are clear, systematic, and detailed; and the text contains many examples worked out carefully and completely. The problem exercises seem quite adequate.

This book is an excellent one for classroom use. The only prerequisite seems to be calculus, though a fuller appreciation of portions of the last six chapters would be realized if the reader has some knowledge of differential equations and advanced calculus.

Chapters 1 and 2 deal with vector algebra. There is some emphasis on geometric proofs. The differentiation and integration of vectors is deferred until Chapter 5. Chapters 3 and 4 contain Newton's laws, and a treatment of the properties of force systems. Also, there appears here the statics of a particle and of a rigid body, including the funicular polygon and the statics of beams. Chapters 5 and 6 deal with the kinematics of a particle and of a rigid body. Chapters 7 and 8 deal with the dynamics of rigid body. The motion of a particle

relative to a moving frame of reference is included, but the motion of a rigid body relative to a moving frame of reference appears only as one of a set of problems at the end of Chapter 11. Chapters 9 and 10 contain some simple methods of solving special problems. Included here are the principles of work-energy, conservation of energy, and virtual work. Chapter 11 deals with impulses and problems involving impacts.

In Chapter 12 there is a treatment of the simple one-dimensional oscillator both with and without damping and an exciting force. Chapter 13 deals with conservative fields of force, and includes a discussion of planetary motions and the two-body problem. Chapters 14 and 15 contain very brief introductions to some of the concepts and equations of the mathematical theories of elasticity and hydrodynamics. Included here are the notions of divergence and curl of a vector, as well as the theorems of Green and Stokes. In Chapter 16, wave motion in one dimension is considered. The partial differential equation governing three elementary vibrating systems is deduced in each case. The general solution of this equation is assumed, and various phenomena such as beats and the Doppler effect are discussed. Chapter 17 contains an introduction to the kinetic theory of a monatomic gas, in which each molecule is treated as a perfectly elastic sphere.

The reviewer can make only one criticism of this book. In his opinion, a more extensive consideration of the motion of a rigid body relative to a moving frame of reference, with applications to the top and gyroscope, would round out nicely an otherwise rather complete book.

G. E. Hay
University of Michigan

Methods of Applied Mathematics. By F. B. Hildebrand, Prentice-Hall, Inc., New York, 1952, 523 pages. $7.75.

"The principal aim of this volume is to place at the disposal of the engineer or physicist the basis of an intelligent working knowledge of a number of facts and techniques relevant to four fields of mathematics which usually are not treated in courses of the 'Advanced Calculus' type, but which are useful in various fields of application."

In the opinion of the reviewer, Professor Hildebrand has achieved his noteworthy aim. His book should be a most useful text and reference volume for seniors and first year graduate students.

It would appear that we are now entering a second phase of American higher education in which greater efforts are being spent on raising the level of instruction in the sciences (including engineering). Professor Hildebrand's book certainly is a large step forward in this direction through his treatment of the four basic topics: Chapter 1, Matrices, Determinants and Linear Equations; Chapter 2, Calculus of Variations and Applications; Chapter 3, Difference Equations; Chapter 4, Integral Equations.

Throughout the text complete derivations of mathematical equations from

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Throughout the text complete derivations of mathematical equations from

physical principles are followed by detailed analyses of the mathematical and the numerical methods that are appropriate. In this way a well rounded and thorough understanding of the basic techniques is transmitted to the reader. Many of the exercises (which are large in number, excellent in quality and supplied with answers or hints) were devised to extend the theory. Of course, Professor Hildebrand's book was not intended as a treatise and it therefore contains careful statements, discussions and references for those topics which could not be completely treated.

Chapter 1 "deals principally with linear algebraic equations, quadratic and Hermitian forms, and operations with vectors and matrices, with special emphasis on the concept of characteristic values. A brief summary of corresponding results in function space is included." The author stops short of deriving the Jordan canonical form for matrices, but describes it with care. His rule for evaluating a determinant is a fine idea for classroom discussion, but should probably not be given to the exclusion of the customary one involving subscripts.

Chapter 2 carefully introduces the variational notation and the Euler equations. Natural boundary conditions, generalized coordinates, Hamilton's principle, and Lagrange's equations are then treated. Normal coordinates, minimal principles for elasticity, *etc.*, and some of the direct methods of the calculus of variations complete the scope of the second chapter.

"The third chapter combines the presentation of available methods for solving the simpler types of difference equations with a description of the application of finite-difference methods to the approximate solution of problems governed by partial differential equations and includes consideration of the troublesome problems of convergence and stability." In his efforts to present in integrated form some of the modern developments of numerical analysis, the author has done a great service. (Even though his discussion of stability and convergence is not as clean-cut or inclusive as it can be made today.)

"The concluding chapter deals with the formulation and theory of linear integral equations, and with exact and approximate methods for obtaining their solutions, particular emphasis being placed on the several equivalent interpretations of the relevant Green's function."

The author has succeeded in writing four self-contained fundamental chapters, and has managed in addition to treat many of those aspects which are common to the various fields. This makes his volume a remarkably well coordinated unit.

EUGENE ISAACSON
New York University

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

MAGIC CIRCLES

S. W. MCINNIS, University of Florida

This paper has for its purpose the compiling of the sketchy material on magic circles, enlarging that information, and bringing together the various illustrations found in the literature.

A magic circle consists of $R+1$ concentric circles divided into sections by n diameters or N radii; an arbitrary number C is placed in the central circle and a set of consecutive numbers from a to b are placed in the other sections so that the sum along each diameter is constant. We classify magic circles into three classes: simple, compound, and perfect. Illustrations and properties of these different classes follow.

Let S_r be the sum along any radius, S_d be the sum along any diameter, and S_o be the sum in any ring (excluding the central circle). A property common to all magic circles is

$$S_{d_1} = S_{d_2} = \cdots = S_{d_n}. \tag{1}$$

If (1) is the only property a magic circle has, it is called *simple*. Figures 1 and 2 are illustrations of this class.

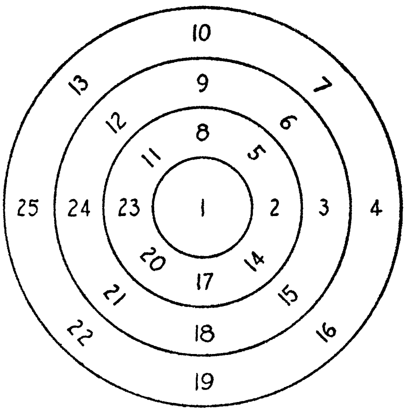

FIG. 1

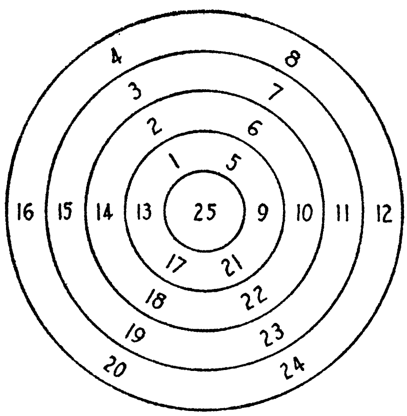

FIG. 2

If simultaneously

$$S_{r_1} = S_{r_2} = \cdots = S_{r_N} \tag{2}$$

and

(3) $$S_{o_1} = S_{o_2} = \cdots = S_{o_R},$$

the magic circle is *compound*. In this event, it follows easily from (2) and (3) that

(4) $$S_r = \tfrac{1}{2}(a + b)R + C,$$

and

(5) $$S_o = \frac{N}{R}(S_r - C).$$

Figures 3 and 4 are examples of this class. If $N=R$, the magic circle is *perfect*. From (5) we have $S_o = S_r - C$. Figure 5 is an example of a perfect magic circle.

Figure 3 is an example of a compound magic circle formed from the numbers 1 to 320 inclusive. Its designer writes:*

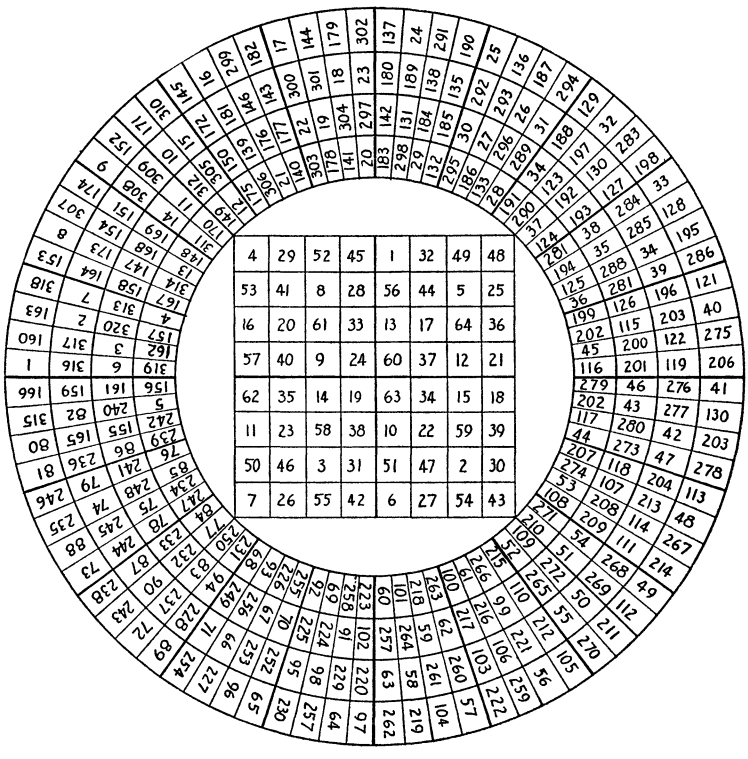

FIG. 3

"The rings of this circular diagram contain the first 320 numbers arranged in twenty subdivisions each of which forms a magic square. Each row, column and diagonal totals 642.

"In the entire ring every right and left diagonal totals 642. Four extra magic

* R. V. Heath, Scripta Mathematica, vol. 3, 1935, p. 340.

squares are formed where the following numbers appear on the outer edge:

267, 124, 49, 112,
227, 254, 84, 72,
207, 174, 9, 152,
187, 294, 129, 32.

"In the center there is a perfect 8 by 8 magic square composed of the numbers 1–64. Each quarter section forms a magic square. Adding any number in a quarter section with the number in like position in the other sections we always get 130.

"The upper tier of the square forms a pan-diagonal rectangle, so does the lower tier."

Here $a=1$, $b=320$, $R=4$, and $N=80$, whence by (4) and (5), $S_r-C=\frac{1}{2}(1+320)\cdot 4=642$ and $S_o=20(642)=12{,}840$.

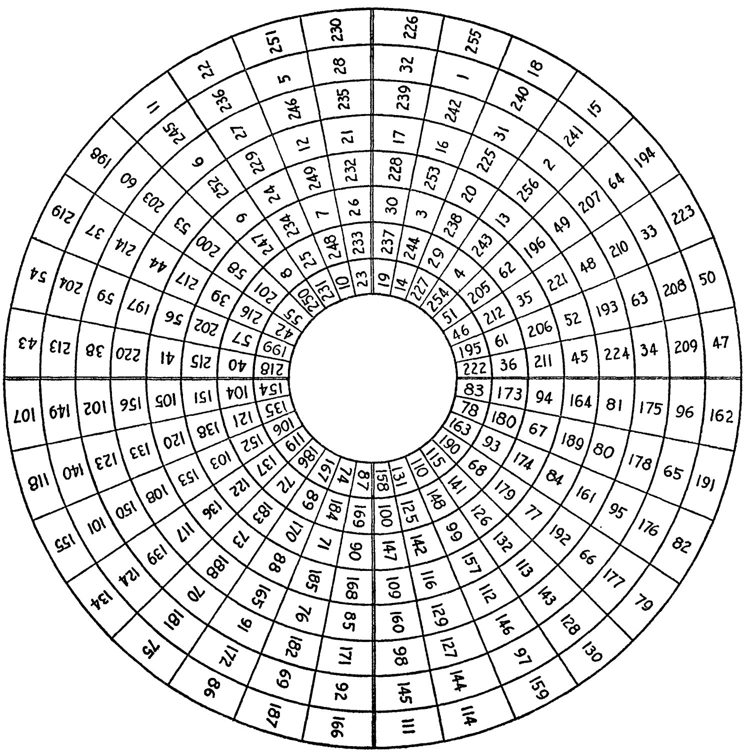

FIG. 4

Figure 4 is another compound magic circle formed from the numbers 1 to 256 inclusive arranged in four subdivisions or quarter sections of the circle such that each quarter forms an 8 by 8 magic square. Here $a=1$, $b=256$, $C=0$, $R=8$, and $N=32$, so that by (4) and (5), $S_r=\frac{1}{2}(1+256)\cdot 8=1028$ and $S_o=4(1028)=4{,}112$.

added to each ring.

Two interesting questions arise. Is the number of perfect magic circles finite? If finite, can an example of each different set be illustrated?

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

MEETING OF MATHEMATICS DIVISION OF A.S.E.E.

The Mathematics Division of the American Society for Engineering Education will meet on June 24–26, 1953 at the University of Florida, Gainesville, Florida. In addition to contributed papers the program will include the following: Two lectures by Dr. J. H. Curtiss, Chief of the National Applied Mathematics Laboratories, National Bureau of Standards, on "Programming for Electronic Digital Computers"; a luncheon address by Professor C. O. Oakley, Haverford College, entitled "Mathematics for Engineers Who Will Never Use It"; a symposium on "Instruction in Mathematics for Undergraduate Engineering Students," under the chairmanship of Professor F. W. Kokomoor, University of Florida. For further information about the meeting write to Chairman of the Division, Dr. C. V. Newsom, Associate Commissioner for Higher Education, State University of New York, Albany 1, New York.

CONGRESS AND SEMINAR OF THE CANADIAN MATHEMATICAL CONGRESS

The biennial Seminar of the Canadian Mathematical Congress will be held at Queen's University and the Royal Military College, Kingston, Ontario, August 10 to September 4, 1953. The topics are Topology and Geometry. Research lectures will be given by Professors L. E. J. Brouwer, Henri Cartan, M. H. A. Newman and Beniamino Segre. Seminars will be conducted by Professors H. S. M. Coxeter and H. Zassenhaus. Professors G. deB. Robinson, Peter Scherk, A. W. Tucker and Max Wyman will give instructional lectures.

The Congress will meet from August 31 to September 4 in conjunction with meetings of the Mathematical Association of America and the American Mathematical Society. For further information write to: Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, Quebec, Canada.

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MATHEMATICS WORKSHOP AT UNIVERSITY OF ARKANSAS

The University of Arkansas, with the cooperation of the Arkansas Council of Teachers of Mathematics, will conduct a Workshop in Mathematics on July 13–15, 1953. Consultants will be Miss Martha Hildebrandt, Proviso Township High School, Maywood, Illinois, past president of the National Council of Teachers of Mathematics; Professor Maurice L. Hartung, Department of Education, The University of Chicago; Miss Christine Poindexter, Senior High School, Little Rock, Arkansas; Mrs. Virginia Sue Wilson, Supervisor of Elementary Education, College of Education, The University of Arkansas. There will be general meetings and discussion groups for secondary teachers and for elementary teachers.

Participants in the Workshop may obtain accommodations in the facilities of the University. Inquiries regarding the Workshop should be addressed to Professor D. P. Richardson, Department of Mathematics, University of Arkansas, Fayetteville, Arkansas.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1953:

Boston University. June 1 to July 11: Professor Johanson, vector analysis; Professor Scheid, infinite series. July 13 to August 22: Dr. Browder, the theory of games; Professor Noether, introduction to mathematical statistics.

Columbia University. July 6 to August 14: Dr. Brown, introduction to higher algebra; Dr. Finn, differential equations; Professor Cohen, foundations of mathematics, general topology; Professor Levi, theory of functions of a real variable; Professor Lorch, the calculus of finite differences; Professor Murray, probability, Lebesgue measure and integrals; Dr. Taylor, theory of fields, topics in the modern theory of partial differential equations.

Duke University. June 10 to July 18: Professor Carlitz, theory of equations, thesis seminar; Professor Gergen, statistics, thesis seminar; Professor Thomas, plane geometry and trigonometry from the advanced standpoint, thesis seminar. July 21 to August 26: Professor Dressel, probability, thesis seminar; Professor Roberts, thesis seminar. July 20 to July 24: Professor Dressel, Director for Mathematics, Laboratory Conference for Teachers of Science and Mathematics.

Stanford University. June 22 to September 1: Professor Mordell of St. Johns College, Cambridge University, Diophantine equations. July 28 to August 31: Professor Salem of Massachusetts Institute of Technology, trigonometric series and random distribution.

University of Delaware. June 22 to July 31: Professor Barrett, Fourier series; Professor Remage, fundamentals of geometry.

University of Illinois. June 19 to August 15: Professor Day, group theory, functions of real variables.

University of Minnesota, Department of Mathematics, Institute of Technology. June 15 to July 18: Professor Milgram, differential equations and functions of

a complex variable; Professor Munro, intermediate calculus and vector analysis. July 20 to August 22: Professor Polansky, differential equations and advanced calculus.

University of Texas, Department of Mathematics. June 3 to July 15: Professor Ettlinger, differential equations and applications, research in differential equations; Professor Moore, introduction to foundations of geometry, theory of sets; Professor Lane, introduction to the applications of continued fractions, smoothing experimental data; Professors Ettlinger and Moore, thesis for the master's degree, thesis for the degree of Doctor of Philosophy. July 16 to August 26: Mr. Mohat, differential equations and applications, curve fitting; Professor Lubben, topics in modern algebra, introduction to modern projective geometry; Professor Wall, functions of a complex variable, infinite processes.

University of Texas, Department of Applied Mathematics and Astronomy. June 3 to July 15: Professor Craig, vector and tensor analysis, applications of tensor analysis; Professor Greenwood, graphical and numerical computation, mathematical theory of strategy; Mr. Weaver, introduction to the theory of integers; Staff, advanced calculus, differential equations. July 16 to August 26: Professor Cooper, theory of functions of a complex variable, group theory of differential equations; Professor Guy, modern operational methods; Staff, advanced calculus, differential equations.

University of Wisconsin. June 29 to August 21: Professor McCoy of Smith College, determinants and matrices, rings and fields; Professor MacDuffee, survey of the foundations of algebra, tensor analysis; Professor Kleene, elementary plane topology; Professor Fullerton, Fourier analysis; Professor Korevaar, mathematical applications; Dr. Immel, introduction to the theory of probability.

University of Wyoming. June 15 to July 17: Professor Barr, seminar in geometry; Professor S. R. Smith, ordinary differential equations; Professor Steen, college geometry; Professor Varineau, theory of equations, theory of numbers, fundamental concepts of mathematics. July 20 to August 21: Professor Neubauer, history of mathematics; Professor Schwid, partial differential equations; Professor W. N. Smith, mathematical theory of probability.

PERSONAL ITEMS

Professor O. J. Ramler of the Catholic University of America was appointed to represent the Association at the inauguration of President H. R. Anderson of the American University on February 24, 1953.

Dr. Warren Weaver, director for the natural sciences of the Rockefeller Foundation, is President-elect of the American Association for the Advancement of Science; Professor Tibor Rado of Ohio State University has been elected Vice-President and Chairman of the Mathematics Section of the Association; Dean M. H. Ingraham of the University of Wisconsin has been re-elected a member of the Board of Directors for the period 1953–56.

Brown University announces the formation of a physical sciences council

under the chairmanship of Professor William Prager; Professor E. H. Lee replaces Professor Prager as chairman of the Graduate Division of Applied Mathematics.

Iona College reports the following: Mr. F. J. Lane, Brother E. I. Duggan, and Brother S. A. Ryan have been appointed to instructorships; Brother J. G. McKenna has been promoted to the position of Dean of the College; Brother C. A. Lynam has retired.

New York University announces the formation of an Institute of Mathematical Sciences with Professor Richard Courant as scientific director; the Institute consists of three divisions: Mathematics and Mechanics, Mathematics Research, and Computing Services.

Oregon State College announces the following: Professor A. T. Lonseth was granted the Carter Award for inspiring teaching by the School of Science during the fall term of 1952; Mr. R. V. Jamison has been appointed to an instructorship.

Tennessee Polytechnic Institute announces the following appointments: Mr. J. H. Hoelzer of Williams College to an associate professorship; Mr. F. J. Witt, previously head of the Department of Mathematics of Roanoke Rapids High School, North Carolina, to an assistant professorship.

University of Denver makes the following announcements: Assistant Professor Mary E. Waller of Central State College, Edmund, Oklahoma, has been appointed to an instructorship to replace Assistant Professor T. J. Bartlett, who is on leave of absence and is engaged in graduate study; Assistant Professor Olna H. Fant has retired with the title of Professor Emeritus.

At the University of Detroit: Assistant Professor E. D. McCarthy has been promoted to an associate professorship; Instructor E. M. Steinbach has been promoted to an assistant professorship; Mr. S. F. Dice, formerly instructor in mathematics and physics at West Liberty State College, and Mr. J. G. Sowul, previously analytical engineer, Dynamic Analysis Group, Chance Vought Aircraft, Dallas, Texas, have been appointed to instructorships.

University of Florida announces: Mr. H. D. Sprinkle has been appointed to an interim instructorship; Instructor E. J. Lytle, Jr. has returned to the University after two years of military service; Assistant Professor M. E. McCarty has retired.

University of Massachusetts reports the following: Mr. Oscar Litoff of Illinois Institute of Technology has been appointed to an assistant professorship; Mr. A. G. Davis, formerly a mathematics consultant at the Naval Research Laboratory, Mr. Valdemars Punga, and Mr. N. T. Watson have been appointed to instructorships; Mr. Edward Halpern and Mr. W. E. Mientka are on leave of absence during 1952–53.

Institute of Technology, University of Minnesota announces: Professor A. N. Milgram of New York University has been appointed Associate Professor of Mathematics; Associate Professor H. L. Turrittin has been promoted to a professorship.

University of Rochester announces the following: Dr. Walter Rudin, formerly C. L. E. Moore instructor at Massachusetts Institute of Technology, has been appointed to an assistant professorship; Mr. Hewitt Kenyon, previously part-time instructor at the University of California at Berkeley, and Mr. Ralph Raimi, formerly part-time instructor at the University of Michigan, have been appointed to instructorships; Mr. D. C. Barton and Mr. A. E. Danese have been promoted to instructorships; Professor Wladimir Seidel is on sabbatical leave and is at the Institute for Advanced Study.

Reverend H. B. Albiser, formerly at St. Michael's College, Vermont, is teaching now at Cardinal Mindszenty High School, Dunkirk, New York.

Dr. Mabel S. Barnes of Occidental College has been promoted to an assistant professorship.

Mr. D. Y. Barrer of Northwestern University has accepted a position as a staff member with the Operations Evaluation Group, Office of the Chief of Naval Operations, Washington, D. C.

Associate Professor Iacopo Barsotti of the University of Pittsburgh has been promoted to a professorship.

Assistant Professor J. K. Baumgart of Elmhurst College is now in military service.

Mr. F. S. Beckman, formerly an instructor at Pratt Institute, has a position as Senior Mathematician with the International Business Machines Corporation, New York City.

Mr. R. J. Beeber, previously a graduate student at Columbia University, has been appointed to an instructorship at St. Peter's College.

Mr. B. E. Beeman of Texas Agricultural and Mechanical College has been appointed to an instructorship at North Texas State College.

Mr. Joseph Blum, previously of the Armed Forces Security Agency, Washington, D. C., is now an analyst with the Defense Department.

Dr. S. G. Bourne of the University of Connecticut has been appointed to an instructorship at Temple University.

Mr. W. G. Brady, previously a graduate assistant at the University of Pittsburgh, has been appointed to an assistant professorship at Washington and Jefferson College.

Associate Professor J. R. Britton of the University of Colorado has been promoted to a professorship.

Dr. A. H. Brown has been appointed to an instructorship at Rice Institute.

Mr. G. H. Butcher of Howard University has been promoted to an assistant professorship.

Dr. H. H. Campaigne, formerly a mathematician in the Navy Department, Washington, D. C., has been appointed Chief, Armed Forces Security Agency, Defense Department, Washington, D. C.

Assistant Professor W. B. Caton of DePaul University has been promoted to an associate professorship.

Dr. T. Y. Chow, who has been a teaching fellow at Cornell University, has

been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Miss Helen E. Clarkson of Creighton University is now with the Equitable Life Assurance Society of the United States, New York City.

Dr. Mary D. Clement has been appointed Lecturer at the University of Chicago.

Dr. A. H. Clifford has been appointed to an associate professorship at Johns Hopkins University.

Mr. T. S. Dean, formerly president of Design Associates, Sherman, Texas, has a position as Designer-Surveyor with Dean and Alexander, Dallas, Texas.

Assistant Professor R. D. Depew is on leave of absence from Florence State Teachers College, Alabama, and is engaged in graduate study at Vanderbilt University.

Associate Professor R. P. Dilworth of California Institute of Technology has been promoted to a professorship.

Miss Flora Dinkines of the University of Chicago has been appointed to an assistant professorship at the University of Illinois, Chicago, Illinois.

Dr. Mary P. Dolciani of Vassar College has been promoted to an assistant professorship.

Assistant Professor Jim Douglas, Jr. of the University of Alabama has accepted a position as an assistant research engineer with Humble Oil and Refining Company, Houston, Texas.

Dr. Joanne Elliott of Swarthmore College has been appointed to an assistant professorship at Mount Holyoke College.

Mr. Norbert Ellmann of Marquette University is now with the Bell Aircraft Corporation, Niagara Falls, New York.

Dr. Paul Erdös, formerly a visiting lecturer of the University of Aberdeen, Scotland, has accepted a position as a mathematician at the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, California.

Assistant Professor Trevor Evans is on leave of absence from Emory University and is spending the year at the Institute for Advanced Study.

Mr. J. E. Faulkner, previously a graduate assistant at Kansas State College, has been appointed to an instructorship at Utah State Agricultural College.

Mr. W. R. Ferrante has been appointed to an instructorship at Lafayette College.

Professor A. H. Fox of Union College is on leave of absence and has been appointed a consultant at the Oak Ridge National Laboratory.

Miss Joyce B. Friedman has a position as a mathematician with the Defense Department, Washington, D. C.

Dr. J. W. Gaddum, formerly with the National Bureau of Standards, Los Angeles, California, has accepted a position as a mathematician with the United States Air Force, Washington, D. C.

Dr. L. D. Gates, Jr. of Iowa State College of Agriculture and Mechanic Arts has a position as a mathematician with the Department of Defense, Washington, D. C.

Professor F. C. Gentry of Arizona State College, Tempe, has been appointed to an associate professorship at the University of New Mexico.

Assistant Professor J. B. Giever, formerly of Boston University, is now at the Instrumentation Laboratory, Massachusetts Institute of Technology.

Mr. H. W. Godderz, formerly with the Cessna Aircraft Company, Wichita, Kansas, has been appointed to an assistant professorship at Ohio Northern University.

Mr. M. L. Goldwater is an electronics engineer with Hughes Aircraft Company, Culver City, California.

Mr. G. B. Hedrick of Stanford University has a position as a training specialist with Northrop Aircraft, Hawthorne, California.

Assistant Professor D. M. Hester of Baker University is teaching now at Liberty High School, Texas.

Mr. Joseph Hilsenrath of National Bureau of Standards has been promoted to the position of Physical Science Administrator.

Dr. M. P. Jarnagin of the University of Maryland has accepted a position as a mathematician with the Naval Aviation Ordnance Test Station, Chincoteague, Virginia.

Mr. P. W. M. John, formerly of the University of Oklahoma, is teaching now at Casady School, Oklahoma City, Oklahoma.

Dr. V. L. Klee, who has been at the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Virginia.

Mr. Mark Leum of the University of Iowa has been appointed a mathematician at the Woodward Governor Company, Rockford, Illinois.

Dr. Werner Leutert of the Ballistic Research Laboratories, Aberdeen Proving Ground, has been promoted to Chief, Computing Laboratory.

Mr. R. D. Lowe, previously a graduate assistant at Northwestern University, has accepted a position as a mathematician at International Business Machines, Chicago, Illinois.

Dr. G. H. Lundberg of Vanderbilt University has been appointed to an associate professorship.

Professor A. C. Maddox has retired from Northern State College of Louisiana and is now a member of the staff of Southern State College, Magnolia, Arkansas.

Dr. Jean Maranda has been appointed to an assistant professorship at the University of Montreal.

Assistant Professor William Marcaccio of Xavier University has been promoted to an associate professorship.

Mr. B. W. Marks is teaching at San Jacinto Junior High School, Midland, Texas.

Captain G. T. McCready is now with the Navy Department, Washington, D. C.

Mr. J. E. McKeehan, formerly head of the Department of Mathematics of Skagit Valley Junior College, Mt. Vernon, Washington, is teaching at Lake

Washington High School, Kirkland, Washington.

Assistant Professor K. F. McLaughlin of the United States Naval Academy has been appointed to the position of Associate Professor and Technical Assistant in the Test Service Bureau at Florida State University.

Mr. R. L. McNeal has retired from his position at General Motors Proving Ground, Milford, Michigan.

Professor Karl Menger of Illinois Institute of Technology conducted two programs during March, 1953, in the Institute's new educational TV series.

Dr. W. M. Miller has been appointed to an assistant professorship at Washington and Lee University.

Mr. P. D. Minton, formerly a graduate student at the University of North Carolina, has been appointed to an assistant professorship at Southern Methodist University.

Dr. E. R. Mullins, Jr. has been appointed to an instructorship at Swarthmore College.

Dr. Rufus Oldenburger, chief mathematician of the Woodward Governor Company, Rockford, Illinois, gave a series of lectures recently at the University of Paris on mathematical engineering analysis.

Dr. Daniel Orloff, previously at Bell Aircraft Corporation, Niagara Falls, New York, has accepted a position as a mathematician at Cornell Aeronautical Laboratories, Buffalo, New York.

Associate Professor Margaret Owchar of Southwest Missouri State College has a position as a research worker and statistician with the Manitoba Cancer Relief and Research Institute, Winnipeg, Manitoba, Canada.

Mr. D. H. Pilgrim, who has been a graduate student at State University of Iowa, has accepted a position as a statistician with the United States Rubber Company, Passaic, New Jersey.

Mr. Costas Plithides, previously a graduate student at Columbia University, has been appointed to an instructorship at Newark College of Engineering.

Mr. H. E. Reinhardt, who has been a statistician at the General Electric Company, Richland, Washington, has been appointed to an instructorship at State College of Washington.

Dr. Helene Reschovsky of the University of Connecticut has been promoted to an assistant professorship.

Mr. E. C. Rice, formerly a graduate student at George Peabody College for Teachers, has been appointed to an assistant professorship at Monticello Agricultural and Mechanical College.

Assistant Professor L. G. Riggs of Ohio University has been appointed to an assistant professorship at San Diego State College.

Dr. L. V. Robinson has accepted a position as a mathematician with Wright-Patterson Air Force Base, Dayton, Ohio.

Dr. R. M. Robinson of Iowa State College of Agriculture and Mechanic Arts has been appointed Director of Placement at the University of Arizona.

Dr. Charles Roth of the United States Military Academy has accepted a

position as a guidance counselor at City College of New York City.

Mr. J. A. Ryan of Gonzaga University has accepted a position with Lockheed Aircraft Company, Burbank, California.

Mr. Yomei Sawanobori of Princeton University has a position as a junior mathematician with the Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. R. E. Shear, who has been an assistant at Harpur College, has a position as a mathematician at Aberdeen Proving Ground.

Mr. Milton Siegel, previously at the United States Naval Proving Ground, Dahlgren, Virginia, has accepted a position as a mathematical statistician with the Bureau of the Census, Suitland, Maryland.

Professor Jack Silber of Roosevelt College has returned from a tour of duty as an operations analyst with the Fifth Air Force in Korea.

Mr. D. W. Stoddard of Utah State Agricultural College is now with the Sandia Corporation, Albuquerque, New Mexico.

Miss Dorothy J. Stodola has been appointed to an instructorship at Marquette University.

Mr. Irwin Stoner, who was an assistant scientist at Rosemount Research Center, University of Minnesota, has a position now as a dynamics engineer at Bell Aircraft Corporation, Niagara Falls, New York.

Mr. J. R. Sullivan of Clemson Agricultural College has been promoted to an assistant professorship.

Mr. Peter Terwey, Jr., formerly a part-time instructor at the University of North Carolina, has been appointed to an assistant professorship at Davidson College.

Mr. A. E. Ventriglia of Manhattan College has been promoted to an assistant professorship.

Assistant Professor L. F. Walton of University of California at Santa Barbara has been promoted to an associate professorship.

Mr. W. J. Wells, who was a graduate student at the University of Minnesota, has accepted a position as a mathematician with the United Aircraft Corporation, East Hartford, Connecticut.

Mr. C. S. Williams, Jr. has a position as an electronics engineer with Sandia Corporation, Albuquerque, New Mexico.

President Emeritus H. N. Davis of the Stevens Institute of Technology died on November 3, 1952.

Professor M. W. Dehn of Black Mountain College died on June 27, 1952.

Assistant Professor J. D. Newburgh of Tulane University died on January 3, 1953.

Professor Leigh Page of Yale University died on September 14, 1952.

Professor Emeritus W. P. Russell of Pomona College died on January 10, 1953; he was a charter member of the Association.

Professor Emeritus W. S. Schlauch of New York University died on January 27, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE JANUARY MEETING OF THE SOUTHWESTERN SECTION

The January meeting of the Southwestern Section of the Mathematical Association of America was held at the New Mexico College of Agriculture and Mechanic Arts at State College, New Mexico, on January 2 and 3, 1953. Professor J. H. Butchart, Chairman of the Section, presided at the sessions.

Fifty persons attended the meetings including the following thirty members of the Association:

O. B. Ader, J. W. Beach, A. I. Benson, J. W. Branson, C. E. Buell, J. H. Butchart, J. T. Clausen, Jr., G. A. Culpepper, D. G. Duncan, R. S. Fouch, Gordon Fuller, F. C. Gentry, A. S. Gregory, E. A. Hazlewood, W. P. Heinzman, M. S. Hendrickson, R. C. Hildner, Verba M. Iturralde, Max Kramer, W. W. Mitchell, Jr., H. E. Pickett, Irene Price, E. J. Purcell, W. L. Shepherd, R. B. Stiles, Deonisie Trifan, R. S. Underwood, Earl Walden, D. L. Webb, J. W. T. Youngs.

The following were elected officers for the year 1953: Chairman, Professor M. S. Hendrickson, University of New Mexico; Vice Chairman, Professor D. L. Webb, Arizona University; Lecturer for 1953, Professor Max Kramer, New Mexico College of Agriculture and Mechanic Arts.

An invited address entitled "The Matter of Topology" was given, following the banquet on Friday evening, by Professor J. W. T. Youngs of Indiana University. Dr. Youngs also read a paper during the day sessions.

The following papers were presented during the two day sessions:

1. *New properties of the quadrilateral*, by Professor J. H. Butchart, Arizona State College, Flagstaff, Arizona.

Using primary methods of inversive geometry, the author showed that the circumcenters of the four triangles of any quadrilateral are the vertices of four triangles similar to the given triangles, and that lines through these circumcenters parallel to the corresponding lines of the given quadrilateral form a homothetic figure, the ratio being -1. A theorem of H. C. Gossard to the effect that the Euler lines of the three triangles formed by the Euler line of a given triangle with its sides constitute a triangle congruent to the given one was proven anew.

2. *A construction in* [4] *for a Cremona involution in* [3], by Professor E. J. Purcell, University of Arizona.

3. *Extended analytic geometry applied to simultaneous equations*, by Professor R. S. Underwood, Texas Technological College.

In this paper a coordinate system is considered which provides readily obtainable loci on a plane for equations in three or more variables. Successive positive half-axes, numbered in counterclockwise sequence, are 90° apart, so that the fifth and sixth axes, for example, coincide respectively with the first and second. Since the simple lattice points of plane analytic geometry are retained, the system is well adapted for finding integral as well as real solutions. The discussion deals chiefly with systems of linear equations in general and with sets of quadratic equations in three or four unknowns.

4. *Remarks on surface area*, by Professor J. W. T. Youngs, Indiana University.

11. *Finite projective planes and their associated Latin squares*, by Professor Charles Wexler and Professor L. J. Paige, Arizona State College, Tempe, Arizona. Paper presented by title.

The following papers having to do with teaching problems and organization of courses were presented on Saturday morning, January 3.

12. *The mathematical preparation of college freshmen from New Mexico high schools*, by Professor F. C. Gentry, University of New Mexico.

13. *Some experiments in the organization and administration of freshman mathematics*, by Professor E. A. Hazlewood, Texas Technological College.

14. *Elimination of variables from systems of equations as an aid in unifying freshman mathematics*, by Professor W. L. Shepherd, Texas Western College.

15. *Contemporary psychology and an experiment in the teaching of mathematics*, by Professor R. S. Fouch, Arizona State College, Tempe, Arizona.

16. *Multi-sensory aids in the teaching of mathematics*, by Mrs. Verba M. Iturralde, Bowie High School, El Paso, Texas.

17. *Report of committee on high school cooperation*, by Professor Max Kramer, New Mexico College of Agriculture and Mechanic Arts.

The committee reported that it is planning to hold statewide mathematical contests in New Mexico this year. Official Association awards are to be given. The program was planned with the cooperation of the Mathematics Section of the New Mexico Education Association.

R. L. WESTHAFER, *Secretary*

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirtieth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Millsaps College, Jackson, Mississippi, on February 13 and 14, 1953. Professor F. A. Rickey, Chairman of the Section, presided at the Friday afternoon session. Professor T. L. Reynolds, Mississippi Vice-Chairman, presided at the Saturday morning session.

There were ninety-seven persons present including the following fifty members of the Association:

T. A. Bickerstaff, Elsie T. Church, W. H. Cleveland, G. J. Corley, Margaret R. Davis, M. P. Dossey, W. L. Duren, Jr., D. O. Etter, L. R. Ford, L. M. Garrison, A. L. Gilmore, Jr., A. C. Grimes, J. A. Hardin, R. H. Hopkins, L. H. Kanter, H. T. Karnes, C. G. Killen, R. J. Koch, Margaret M. LaSalle, Mrs. Helen W. Lindley, Z. L. Loflin, J. W. McClimans, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, Benjamin Evans Mitchell, S. B. Murray, M. M. Ohmer, Arthur Ollivier, R. L. O'Quinn, W. V. Parker, B. J. Pettis, T. J. Pignani, P. K. Rees, T. L. Reynolds, F. A. Rickey, A. A. Ritchie, D. R. Scholz, H. F. Schroeder, Fariebee P. Self, S. W. Shelton, P. K. Smith, W. H. Spragens, Jr., V. B. Temple, W. B. Temple, B. B. Townsend, G. J. Trammell, Jr., B. O. Van Hook, Eleanor B. Walters, Dale Woods.

The following officers were elected for the coming year: Chairman, Professor M. E. Gillis, Blue Mountain College; Louisiana Vice-Chairman, Professor M. M. Ohmer, Southwestern Louisiana Institute; Mississippi Vice-Chairman, Professor B. O. Van Hook, Mississippi Southern College; Secretary-Treasurer,

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Professor Z. L. Loflin, Southwestern Louisiana Institute.

The following motion was passed at the business session: "The meetings of the Section shall consist of those programs and only those programs open to all members of the Mathematical Association of America and the secretary is instructed to send a notice of the Section meetings to all colleges in Louisiana and Mississippi."

The invited speaker for the meeting was Professor W. L. Duren, Jr., of Tulane University. His address on Friday evening was entitled "The Reform of the Mathematical Curriculum," and at the Saturday morning session, "Elementary Scientific Measurement and the Real Numbers."

The following papers were presented:

1. *Transformations obtained from a certain cubic transformation*, by Professor Elsie T. Church, Northwestern State College.

From results obtained in *A Certain Cubic Transformation*,* the three quadratic transformations A, B, C were derived, where

$$A:\ \begin{aligned} x_1' &= x_2x_3 \\ x_2' &= x_1x_2 \\ x_3' &= x_3x_1 \end{aligned} \qquad B:\ \begin{aligned} x_1' &= \omega x_2x_3 \\ x_2' &= x_1x_2 \\ x_3' &= x_3x_1 \end{aligned} \qquad C:\ \begin{aligned} x_1' &= \omega^2 x_2x_3 \\ x_2' &= x_1x_2 \\ x_3' &= x_3x_1. \end{aligned}$$

The product of any two of these transformations A, B, C, can be reduced to one of the three linear transformations (α), (β), (γ) where

$$(\alpha):\ \begin{aligned} x_1' &= \omega x_1 \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned} \qquad (\beta):\ \begin{aligned} x_1' &= \omega^2 x_1 \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned} \qquad (\gamma):\ \begin{aligned} x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= x_3. \end{aligned}$$

The six transformations A, B, C, (α), (β), (γ) form a group H of order six and (α), (β), (γ) form a subgroup of H of order three.

2. *A set of cyclicly related functional equations*, by Professor T. J. Pignani, Loyola University.

In a paper, of this same title, published by W. M. Whyburn in *Bulletin of the American Mathematical Society*, December, 1930, the following system was considered:

$$Y_i' = \sum_{k=1}^{m} A_k(x) Y_{i+hk+m}, \qquad i = 1, 2, \cdots n\,,\ Y_{j+n} \equiv Y_j,$$

where n is a positive integer, n and h are integers or zero, and $A_k(x)$ are L-integrable functions on an interval of definition X. In this work Whyburn obtained explicit solutions for this system of equations.

A more general extension of this system is as follows:

$$(1) \qquad Y_i' = \sum_{k=1}^{m} A_k(x) Y_{bi+hk+m}, \qquad i = 1, 2, \cdots ;n,\ Y_{j+n} \equiv Y_j,$$

under the same hypothesis and where b is a positive integer.

* See this MONTHLY, vol. 59, 1952, pp. 314–315.

In particular, consider the case where $(h, n) = 1$ and $b = n$. Then (1) becomes

$$Y_i' = \sum_{s=1}^{m} B_s(x) Y_s, \qquad i = 1, 2, \cdots, n. \tag{2}$$

Solutions satisfying (2) are readily obtained, and they are:

$$Y_j = \exp \int_a^x P(v)dv \left[\int_a^x Q(v) \exp - \int_a^v P(t)dt + K \right] \tag{3}$$

where K is an arbitrary constant, $P(v) = \sum_{s=1}^{m} B_s(v)$, $Q(v) = \sum_{s=1, s\neq j}^{m} c_s B_s(v)$, and $1 \leq j \leq n$,
(4) $Y_i = Y_j + c_i$, $i \neq j$, where c_i is an arbitrary constant, $i = 1, 2, \cdots, n$.

Direct substitution of (3) and (4) verify that they satisfy (2).

3. *Quaternions as rotations*, by Miss Margaret M. LaSalle, Southwestern Louisiana Institute.

Quaternions with real coefficients are considered as matrices, with left and right multiplication noted. The general rotation

$$(x, y, 0, 0) + (0, 0, z, t)$$

is expressed as a matrix, M, the angles of rotation being θ and ϕ respectively. The product of two quaternion matrices of the same form does not yield general M; however, the product of two, one from left and the other from right multiplication, gives M, provided the coefficients are expressed in terms of sines and cosines of A and B and $|A+B| = \theta$, $|A-B| = \phi$.

By analogy with two- and three-space, the sum of the squares of rows (and columns) is one, the sum of cross products of rows (and columns) is zero and distances are preserved. It is shown that every vector is changed in length only, that the plane is transformed into itself and that M represents a general rotation.

4. *Mathematics and modern philosophy*, by Professor N. B. Fleming, Millsaps College, introduced by the Secretary.

Modern science and modern philosophy really began when the methods and criteria of mathematics were applied to man's experience and also to his investigation of nature. This development began when the number-mysticism of Pythagoras and the mathematical metaphysics of Plato became more empirical for Kepler and Descartes.

5. *Orthic and oblique hyperbolas*, by Professor V. B. Temple, Louisiana College.

Take non-parallel lines L_1 and L_2, not passing through the origin O. Draw two lines through T, one making a chosen angle w with the vertical through S in P_1, the other making w with the horizontal through S in P_2. Each point traces an orthic or oblique hyperbola as OA rotates about O, according as w is or is not 90°; w is the angle of obliquity and is equal to the angle between the asymptotes.

If e and e' are conjugate eccentricities we have

$$e^2 = \frac{2}{1 + \cos w}, \quad e'^2 = \frac{2}{1 - \cos w}, \quad \text{and} \quad e^2 + e'^2 = e^2 e'^2 = 4 \csc^2 w.$$

In particular if L_1 and L_2 are respectively parallel to the y and x axes and $w = 90°$, P_2 becomes a fixed point, and this point with its symmetric point about O constitute a two-point hyperbola.

6. *The group of automorphisms of cyclic groups*, by Miss Allean McKnight, Sunflower Junior College, Mississippi, introduced by the Secretary.

In order to exhibit an automorphism of a cyclic group a correspondence is set up between

which is endless, dense, and Dedekind complete. The theory of the stream includes much of the basic theory of functions of a real variable. With additional assumptions of uniformity and of order-related algebraic operations the stream specializes into the real numbers but there are intermediate systems which have interesting relationships to elementary scientific measurement. In fact, it is seldom true that measurements of physical quantities have a natural algebra which includes all of the properties of the real numbers as an ordered field.

12. *The Madison symposium on the training of mathematics teachers*, by Professor Z. L. Loflin, Southwestern Louisiana Institute.

A summary and a critical analysis of the sessions and discussion groups of the Madison Symposium was given.

Z. L. LOFLIN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, Canada, August 31–September 1, 1953.

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May 2, 1953.
ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.
INDIANA, Ball State Teachers College, Muncie, May 2, 1953.
IOWA
KANSAS
KENTUCKY, University of Louisville, May 9, 1953.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Proving Ground, Dahlgren, Virginia, May 2, 1953.
METROPOLITAN NEW YORK
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NEBRASKA, University of Nebraska, Lincoln, May 2, 1953.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City, October, 1953.
PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.
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SOUTHEASTERN
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TEXAS
UPPER NEW YORK STATE, United States Military Academy, West Point, May 9, 1953.
WISCONSIN, Mount Mary College, Milwaukee, May 2, 1953.

which is endless, dense, and Dedekind complete. The theory of the stream includes much of the basic theory of functions of a real variable. With additional assumptions of uniformity and of order-related algebraic operations the stream specializes into the real numbers but there are intermediate systems which have interesting relationships to elementary scientific measurement. In fact, it is seldom true that measurements of physical quantities have a natural algebra which includes all of the properties of the real numbers as an ordered field.

12. *The Madison symposium on the training of mathematics teachers*, by Professor Z. L. Loflin, Southwestern Louisiana Institute.

A summary and a critical analysis of the sessions and discussion groups of the Madison Symposium was given.

Z. L. LOFLIN, *Secretary*

CALENDAR OF FUTURE MEETINGS

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GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 6

CONTENTS

JUNE–JULY 1953

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

TEACHER EDUCATION IN ALGEBRA*

C. C. MACDUFFEE, University of Wisconsin

For many years we at the University of Wisconsin have been experimenting with our curriculum in algebra. Both the preparation of the students and the development of the subject itself have changed so rapidly that it has not been possible to develop a curriculum which would remain satisfactory for more than a few years. It is therefore with a feeling of little confidence that I approach this subject. I can only relate some of our experiences and experiments under the supposition that other colleges are having similar experiences, and state a few of our conclusions for whatever they may be worth.

The very spotty preparation in algebra of most of our freshmen is a major item among our troubles. Even among those students who enter the engineering college, the second year of algebra in high school is rapidly disappearing. Even more serious is the lack of competence in arithmetic and algebra of those who claim a year's preparation. Until fashions in Education swing back, there seems to be nothing effective that we can do to stem this trend. To entering engineering students we give an entrance test on elementary skills, and those students who are completely incompetent are required to spend a preliminary semester in a no-credit course where they study high school algebra. In the College of Letters and Science we give an entrance test but have no non-credit course.

Let us admit, then, that the first semester of college mathematics is largely remedial and that very little is actually learned beyond the basic skills. This course is primarily aimed at providing the tools for analytic geometry and the calculus and a minute examination of its contents does not seem to be important.

It is in the courses in analytic geometry and the calculus that the student becomes adept in algebraic manipulations. The grossly incompetent students are no longer present, and imperceptibly the problems have involved more and more complex algebraic operations so that at long last the student has acquired some skill in algebraic manipulation.

The question which we have come here to discuss is the proper content of a course or courses in algebra for a student who has just completed elementary calculus. There are many possibilities, of which I list three, because they seem to represent most common procedures:

1. A problem-solving course in permutations and combinations, probability *etc.* with advanced skills as the prime objective.

2. A development of some of the more elegant topics in classical algebra, a course in which skills and theory are kept in balance.

3. An introduction to abstract algebra.

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* A lecture delivered at the Symposium on Teacher Education in Mathematics, August 27. 1952, in Madison, Wisconsin.

Before deciding which of these procedures to follow, one must examine carefully the objectives of the course. Some students at this level are aiming to take graduate work in mathematics, but more of them are not. In fact, we have graduate and undergraduate students from almost all scientific departments in the university and we cannot afford to ignore their demands for algebraic skills. It is this diversity of objectives which makes the problem difficult and calls for a compromise answer.

Since physics has become such an important science, both undergraduate and graduate students in physics have reached a level where they can be treated with mathematics majors. Most of them now enter into the spirit of a mathematics course and enjoy the theory as well as the applications. They present no particular problem.

It is frequently otherwise with advanced undergraduate and graduate students from other departments. I have some superior students from chemistry and others who are superior only in their attitude toward the Mathematics Department. They object to learning anything whose immediate application to chemistry is not evident. It is difficult to say how far our obligation to such students goes.

A few engineering students elect this course, but by and large these students keep pretty closely to their differential equations and mechanics. There is a scattering of students from agriculture, biology and the other sciences but these students are not numerous and they are inclined to be not very good. Their principal interest in algebra is as a prelude to statistics.

Approximately half of our students in junior algebra are mathematics majors. They fall sharply into two groups, graduate students who enter with a deficiency in algebra, and undergraduates working for a teacher's certificate. The latter sometimes find the competition a little rough.

To this list we might add a few students who are interested in the actuarial examinations, and a group which bids fair to increase in the next few years, namely students who are interested in computing machines.

If the University were large enough so that each of these groups could be segregated and an algebra course tailored to the individual needs of each group, the problem of the curriculum would be a small one. However, this course usually runs in at most two sections so that this cannot be done.

Probably very few mathematicians would be willing to go as far as some of our applied brothers seem to desire and omit all theory. In fact, if such an experiment were tried, I am sure that the students would become so confused that it would have to be abandoned. On the other hand, a straight course in mathematical theory is out of place with a group such as we have in our classes. A skillful teacher will know how to keep these features in balance.

With such a diversified clientele it seems undesirable to deviate too greatly from the traditional course, which I assume to be the theory of equations. There are certain definite advantages in this course. It enables one to teach skills in computing that are of definite value to students in the sciences and at the same

time develop elegant processes of reasoning that were not employed in the calculus.

While I do not believe that the student should be plunged into a course in abstract algebra at this stage in his development, I do believe that the shadow of abstract algebra falls upon this course and indicates certain changes in its organization and development which should be made.

The course which I have been giving at Wisconsin for the last couple of years is still entitled the Theory of Equations, but might more properly be called the Theory of Polynomials. This approach seems to unify the somewhat scattered topics in the theory of equations, and to give a deeper insight into the subject which is particularly valuable to those who go on in algebra and to those who contemplate teaching algebra.

You may not agree with me in bringing in about a week of the theory of numbers, but I have found it desirable, and after all the only way to check a pedagogical theory is to try it out. Only a little of the theory of numbers is required, the definitions of primes and units, scales of notation, the greatest common divisor algorithm, and the unique factorization theorem. The ideas are here presented in their simplest form free of computational difficulties. Later the corresponding concepts and theorems must be proved for polynomials. Just as a matter of experience I have found that students have trouble with this topic if it is first presented to them by way of polynomials. The introduction of this bit of number theory seems necessary in order to teach the polynomial theory regardless of its intrinsic interest.

After this bit of number theory it is easy to attack the problem of finding the integral solutions of an equation having integral coefficients, and the rational solutions of an equation having rational coefficients. Let us consider for a moment the theorem that if an equation with integral coefficients has a rational solution, when this solution is expressed in lowest terms the numerator is a divisor of the constant term of the equation. The proof depends upon the theorem in number theory that if a number divides a product and is relatively prime to one of the factors, it must divide the other factor. This students are ordinarily asked to accept as obvious, but I have seen some able students quite disturbed by it. It is proved in our bit of number theory and students having had this week of number theory really seem to understand the proof.

The plan which I have been following is to start with polynomials over the rational field, later take up polynomials over the real field, and finally polynomials over the complex field. It is well to call attention to the properties of a field, but it is not necessary to call them postulates or to introduce the notion of an abstract field. Other fields, if introduced at all, are postponed until late in the course.

The high points in the theory of polynomials are the Euclid algorithm, the unique factorization theorem, the representation of one polynomial as a polynomial in powers of a second, and the properties of the derivative in regard to multiple zeros. All of these results hold for every coefficient field but can first be

introduced for polynomials over the rational field. At this point the decomposition of a rational function into a sum of partial fractions can be rigorously established. The students are familiar with the process from integral calculus and some of them are astonished when you point out that the universality of the method had not been proved to them.

Most books on the theory of equations scrupulously avoid assuming any knowledge of the calculus on the part of the student. There may be some historical reason for this assumption, but it is no longer valid in American colleges. It is very unusual for any student to elect the theory of equations before he has had his first semester of calculus, and when it *is* done, it should not be. I am accustomed to use the derivative freely and I thus avoid the awkward situation of having to explain why the limit process is out of bounds in an algebra course.

I have always maintained that an intuitive approach should precede an abstract approach, and I feel that this is particularly true in the introduction of the real numbers. All the mystery can be taken out of the real numbers if one can show the student that the existence of an approximation process with arbitrarily small error establishes the existence of the real number. This is no place to go into the intricacies of real variable theory, but an intuitive grasp of the meaning of the real numbers is not beyond the legitimate objectives of a course in the theory of equations.

I would leave a consideration of the complex field until the last instead of introducing it at the beginning, as has lately become fashionable. To deflate the complex field from its paramount position in mathematics is one of the objectives of abstract algebra.

One of the oddities of modern convention is the introduction by many books of the complex numbers by means of Hamilton's number pairs. This is certainly in the spirit of modern abstract algebra, but it is such an isolated bit of abstraction that it seems decidedly out of place in a book that is otherwise purely intuitive. The reason why the complex numbers are frequently so introduced, and not the rational numbers nor the real numbers, is simply a matter of history. Hamilton introduced the complex numbers in this manner in 1835, but the work of Steinitz in 1910 is still probably too new for similar incorporation. In fact, I prefer to do neither, but to keep the presentation on an intuitive level at this point. The complex numbers can be nicely introduced by means of their correspondence with the points of the plane.

I would like to put in a plea here for a few days devoted to symmetric functions of the roots of an equation. Graduate students nearly always tell me that it was omitted from their course in the theory of equations on the grounds that it is of no practical value. This is a point which I am unwilling to concede. Not only is it essential for more advanced work in algebra, but it shows with great clarity how the coefficients of an equation determine the roots without showing favoritism to any one root or group of roots.

The rest of the course is pretty conventional, featuring the factorization of polynomials with real and complex coefficients. This leads to the solutions of the

cubic and quartic equations in terms of radicals. Since determinants are not a part of this course, some time is devoted to the solution of linear systems of equations by applying elementary transformations to them. This treatment of systems of equations is fast gaining in popularity. It is never more laborious than to use determinants, and in the irregular cases it is quicker and more satisfactory. A systematic treatment of systems of higher degree is also included.

It is possible to make this course more intuitive and less formal than is sometimes done. One can teach methods rather than formulas, as in the instance of finding bounds for the roots of an equation, and methods stay with a student better than a formula.

In the seventeen years that I have been at Wisconsin, this junior algebra course has vacillated between a one semester course and a two semester course several times. Now I think it has permanently become a two semester course, the second semester being devoted to matrices and determinants. Until recently it has been difficult to find a text well suited to such a course, but the publishers now seem to be erupting with many such books. This seems to indicate that other institutions than Wisconsin are planning courses along these same lines.

The recent growth of the theory of matrices in public esteem has been remarkable, but not at all surprising. Matrices are now employed in statistics, differential equations, and in computational mathematics of various kinds, to mention only a few applications. Students from other departments are coming to us with requests for work in matrices and we are having about as many students in this second semester of algebra as in the first. Probably well over half of our mathematics majors in Education take it after they have completed the Theory of Equations course.

A course in matrices at the junior level must be given with some restraint. This is not the point for an instructor or text book writer to unload all he knows. The concept of matrix must be motivated at the beginning until the student has got some idea of what it is all about.

As I have usually given the course, the first few weeks are given over to the treatment of determinants of matrices. This treatment differs very little from the old-fashioned development of determinants except in the statement of the theorems. A determinant is a number and does not have rows or columns. The matrix has the rows and columns, but a matrix is not equal to an ordinary number. Thus the accurate statement of a well-known theorem is that if two rows of a matrix are proportional, the determinant of the matrix is zero.

It seems quite natural to most students to set up the detached coefficients of a linear system of equations, and to call this array a matrix. Elementary operations on the equations, which replace the given system of equations by an equivalent system, correspond to elementary row operations on the matrix which replace it by a row-equivalent matrix. By a simple sequence of row operations every linear system may be replaced by one whose matrix is in Hermite form, whereupon the system of equations is solved. The process is applicable regardless of the number of equations or of unknowns, of whether the system is con-

sistent or inconsistent, of whether there is one solution or infinitely many.

Without doubt the next most important topic to be treated is quadratic form theory. This has familiar geometric motivations, and important applications to statistics, relativity and other fields. There are some computational difficulties, so that the first problems to be worked should be chosen so as to minimize these difficulties. The problem should be formulated in matric notation as well as in the notation of quadratic forms, and pretty soon the student will be dealing with matrices as abstract entities.

This is a most excellent place to point out the significance of the concept of field, and the fundamental differences between the rational, real and complex fields. The concept of invariant is well illustrated by the rank and signature of a quadratic form.

Custom now calls for a treatment of orthogonal matrices, and the orthogonal reduction of a real symmetric matrix to diagonal form. For third order matrices this is the principal-axis transformation of solid analytic geometry. To students who have clawed rather ineffectually at this problem in a geometry class, the complete solution of the problem is revealing.

Such then, is the content of our course **115** which most candidates for the teacher's certificate with major in mathematics take at Wisconsin. It is a fairly substantial course, but I believe I speak for my colleagues when I say that we are never entirely satisfied with it. There are so many things that it does not contain that it would be good for a teacher of mathematics to know. It is true that a few of the concepts of abstract algebra are worked into it, such as field, ring and group, but so much is omitted that a modern student of mathematics should know.

You are probably about to ask the question, why do we not put the prospective teachers into a separate section and give them abstract algebra. We have tried it without much success. Most of our Education students are not in the genius class and they absorb the abstract point of view very slowly. Also they do not have the basic algebraic facility nor the fund of illustrative examples to make the subject meaningful. It looks as if we shall have to be content to keep our junior algebra course at a somewhat old-fashioned level and surreptitiously to work in as many abstract concepts as the material allows.

But to my mind the situation indicates pretty strongly that a well qualified teacher of mathematics in high school should have a master's degree or at least a little summer work beyond the bachelor's degree. This is not a revolutionary suggestion, but the really radical and communistic suggestion that I am going to make is that the master's work should include at least one course in the subject which the candidate teaches. Every summer we have thousands of teachers in our Summer School, a dozen or so of whom ever go inside North Hall.

We have for about eight years offered courses in the Summer Session expressly for high school teachers of mathematics. They are entitled Foundations of Arithmetic and Foundations of Algebra, and are given in alternate years. We had believed that as time went on these courses would attract an increasing

clientele, but the classes remain small while hundreds of teachers go next door for courses in the Principles of Education.

The course in Foundations of Arithmetic exposes the student to some modern mathematics. The natural numbers are introduced by means of the Peano postulates, the rational integers as pairs of natural numbers, the rational numbers as pairs of rational integers, the real numbers as sequences of rational numbers, the complex numbers as pairs of real numbers, and finally the real quaternions as tetrads of real numbers. This is Hamilton's idea carried to a logical conclusion. Within this framework there is opportunity to take up scales of notation other than the decimal scale, proof of the rule of casting out nines, and a critical examination of all of the elementary operations of arithmetic including cube root. The student is treated to proofs of all the fundamental laws of arithmetic including the fact that zero times one is zero. It is true that the proof is not one that the teacher can show to an elementary student, but at least the teacher knows that it is true.

The course has been very successful with those whom we have been able to persuade to take it. The point of view has been rather lofty but we have not required a very high level of attainment on the part of the student teacher. We have felt that it was sufficient to get across the general idea without requiring the memorization of too many details. In fact, there was frequently marked enthusiasm by the class when we finally proved some point which had previously bothered these teachers when they had attempted to present the subject to their own elementary classes.

The other course, given in alternate summers with the Foundations of Arithmetic, was aimed at strengthening the foundations of elementary algebra. In my own opinion it was not as good a course as the first. Almost without exception the students registering for the course had not had the theory of equations, so that we were up against the situation described earlier · · · no technical competence. As a result this course has overlapped considerably with the theory of equations.

I am afraid that in this course I have spent some time presenting antidotes to some of the ideas which the students have picked up in methods courses—not at the University of Wisconsin, I hasten to add. I shall take just one example. In a methods course, there are various distinct methods for solving a quadratic equation: factoring, completing the square, and using the formula. But from the point of view of a person who is interested in fundamental principles, there is just one: a product is zero if and only if one of its factors is zero.

I wonder if it is really a good idea to break down a subject such as algebra into a large number of steps, methods and processes. Perhaps it is the only way with very unintelligent students. But with students of medium intelligence or above, I think it is a step in the wrong direction. The number of fundamental principles in algebra is small and if these principles can be kept to the forefront, and the student made to realize that he can rely on common sense, the whole subject should appear vastly simplified. I have often said that if I were teaching

And a more delightful summer vacation may be had by by-passing subject matter courses.

Possibly some good might be done if the books on How to Teach Mathematics were of a little higher calibre. An Educational expert who does not know any mathematics beyond elementary algebra is not in a position to write such a book. It is perhaps also true that a research mathematician is not able to restrain himself sufficiently to keep the book within the comprehension of his audience, in which case his book will simply not be read. What we really need is another J. W. Young and an American David Hilbert who will write elegant but elementary expositions for the benefit of those who are not very experienced in the field, men who are not afraid to tell less than they know about a subject in fear that some book reviewer will misjudge the depth of their scholarship.

ON FACTORIZATION OF POLYNOMIALS

J. B. KELLY, Michigan State College

1. Introduction. The problem of determining, in a finite number of steps, the factorization over the rational field of a given numerical polynomial has been solved by Kronecker. His method is based on the fact that, when the polynomials $f(x)$ and $g(x)$ have integral coefficients, $f(x)$ can be divisible by $g(x)$ only if $f(n)$ is divisible by $g(n)$ for any integer n. Even when one takes advantage of the improvements in Kronecker's method which have been developed by various writers, its application usually involves lengthy and cumbersome calculations. In this paper we present an alternative procedure which focuses attention on the coefficients of the polynomial, rather than on its values at the integers. We believe that our method requires less effort than Kronecker's.

2. The ∇-functions.
Let

$$f(x) = x^n + \sum_{j=1}^{n} (-1)^j s_j x^{n-j}$$

be a polynomial with rational integral coefficients and leading coefficient one. s_j is the elementary symmetric function of order j of the roots $r_1, r_2, \cdots, r_n$ of $f(x)$. We define the integer $\nabla_{j,k} f(x)$ by means of the equation,

$$\nabla_{j,k} f(x) = \prod \sigma_j(r_1', r_2', \cdots, r_k'),$$

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$$\nabla_{j,k}f(x) = \prod \sigma_j(r_1', r_2', \cdots, r_k'),$$

where $r_1', r_2', \cdots, r_k'$ is a particular combination of k of the n roots of $f(x)$, $\sigma_j(r_1', r_2', \cdots, r_k')$ is the elementary symmetric function of order j of these k roots, and the product is taken over all combinations of the roots of $f(x)$ taken k at a time. The symmetric function theorem tells us that $\nabla_{j,k}f(x)$ may be written as a polynomial in $s_1, s_2, \cdots, s_n$ with rational integral coefficients. The calculation of this polynomial is rather tedious, but it is something that needs to be done only once for particular values of j, k and n. We give here a table of $\nabla_{j,k}$ for $n=4$ and $n=5$.

$$n = 4 \qquad f(x) = x^4 - s_1x^3 + s_2x^2 - s_3x + s_4$$

$$\begin{aligned}
\nabla_{1,1}f(x) &= s_4\\
\nabla_{1,2}f(x) &= s_1s_2s_3 - s_3^2 - s_1^2s_4\\
\nabla_{1,3}f(x) &= s_1^2s_2 - s_1s_3 + s_4\\
\nabla_{2,3}f(x) &= s_2s_3^2 - s_1s_3s_4 + s_4^2
\end{aligned}$$

$$n = 5 \qquad f(x) = x^5 - s_1x^4 + s_2x^3 - s_3x^2 + s_4x - s_5$$

$$\begin{aligned}
\nabla_{1,1}f(x) &= s_5\\
\nabla_{1,2}f(x) &= s_1s_2s_3s_4 - s_3^2s_4 - s_1^2s_4^2 - s_5^2 + 2s_1s_4s_5 + s_2s_3s_5 - s_1s_2^2s_5\\
\nabla_{1,3}f(x) &= (s_1^2s_2 - s_1s_3 + s_4)(s_1s_2s_3 - s_3^2 - s_1^2s_4) - s_5^2 - 2s_1s_4s_5 + s_2s_3s_5\\
&\quad + 3s_1^2s_3s_5 - s_1^3s_2s_5 + s_1^5s_5\\
\nabla_{1,4}f(x) &= s_1^3s_2 - s_1^2s_3 + s_1s_4 - s_5\\
\nabla_{2,3}f(x) &= (s_3s_4^2 - s_2s_4s_5 + s_1s_5^2)(s_2s_3s_4 - s_2^2s_5 - s_1s_4^2) - s_5^4\\
&\quad - 2s_1s_4s_5^3 + s_2s_3s_5^3 + 3s_2s_4^2s_5^2 - s_3s_4^3s_5 + s_4^5\\
\nabla_{2,4}f(x) &= s_2^2s_3^2 - s_1s_2s_3s_4 + s_2s_4^2 + s_5^2 - s_1s_4s_5 - 2s_2s_3s_5 + s_1^2s_3s_5\\
\nabla_{3,4}f(x) &= s_3s_4^3 - s_2s_4^2s_5 + s_1s_4s_5^2 - s_5^3
\end{aligned}$$

Let $g(x) = x^k + \sum_{j=1}^{k}(-1)^j t_j x^{k-j}$ be a divisor of $f(x)$ with rational integral coefficients. Then t_j is a divisor of $\nabla_{j,k}f(x)$. For the quotient $\nabla_{j,k}f(x)/t_j$ is on the one hand rational, since both numerator and denominator are, and on the other hand, an algebraic integer, since it is the product of algebraic integers.

3. The method. We wish to determine whether or not $f(x)$ has any rational factors of degree k. There is no loss of generality in supposing that $f(z)$ has integral coefficients with leading coefficient unity, for a simple substitution will always put $f(x)$ in this form. Suppose that

$$f(x) = g(x)h(x)$$

where

$$h(x) = x^{n-k} + \sum_{j=1}^{n-k} (-1)^j t_j' x^{n-k-j}$$

and $f(x)$ and $g(x)$ are as before. Then t_1 is a divisor of $\nabla_{1,k}f(x)$, t_1' is a divisor of $\nabla_{1,n-k}f(x)$ and

$$t_1 + t_1' = s_1. \tag{1}$$

If no pairs of divisors of $\nabla_{1,k}$ and $\nabla_{1,n-k}$ satisfy (1), then $f(x)$ has no factor of degree k. If several such pairs (τ_1, τ_1') exist, it is possible to eliminate some of them by the following process. Compute $f(x+a)$ for some suitable integer a. To the pair of divisors (τ_1, τ_1') of $\nabla_{1,k}f(x)$ and $\nabla_{1,n-k}f(x)$ must correspond the pair of divisors $(\tau_1+ka, \tau_1'+(n-k)a)$ of $\nabla_{1,k}f(x+a)$ and $\nabla_{1,n-k}f(x+a)$. This is readily seen by replacing x by $x+a$ in $g(x)$ and $h(x)$. If either τ_1+ka does not divide $\nabla_{1,k}f(x+a)$ or $\tau_1'+(n-k)a$ does not divide $\nabla_{1,n-k}f(x+a)$, then the original pair (τ_1, τ_1') may be removed from consideration. It may happen that $f(x)$ has a factor of degree k in some extension of the rational field and that the first coefficient, τ_1, of this factor is rational. In this case, no choice of a would eliminate the corresponding pair (τ_1, τ_1'). As an example, consider the polynomial

$$x^4 + 2x^3 + x^2 + 1 = (x^2 + x + i)(x^2 + x - i).$$

This polynomial is actually irreducible over the rational field, as would become apparent after the application of the next step of our method.

If $M = \max |s_i|$, then it is easy to show that $|r_i| \leqq M+1$, $i=1, 2, \cdots, n$. Hence $|t_1| \leqq k(M+1)$ and $|t_1'| \leqq (n-k)(M+1)$. It is possible to eliminate some of the pairs (τ_1, τ_1') by means of these inequalities. Similar inequalities may be derived for the subsequent coefficients.

If, after a few choices of a, one or more pairs (τ_1, τ_1') are not eliminated, it is probable that $f(x)$ has several factors of degree k. Take a particular pair (τ_1, τ_1') and call it (t_1, t_1'). (In cases where $f(x)$ has just one factor of degree k, (t_1, t_1') will usually be determined uniquely.) We then proceed to the determination of t_2 and t_2'. Now t_2 is a divisor of $\nabla_{2,k}f(x)$, t_2' is a divisor of $\nabla_{2,n-k}f(x)$ and

$$t_2 + t_2' = s_2 - t_1t_1'. \tag{2}$$

We search for pairs of divisors of $\nabla_{2,k}f(x)$ and $\nabla_{2,n-k}f(x)$ which satisfy (2). Again, if a number of such pairs (τ_2, τ_2') exist, many can be eliminated by considering $f(x+a)$. Then, as one sees by replacing x by $x+a$ in $f(x)$ and $g(x)$, $\tau_2 - t_1(k-1)a + \frac{1}{2}k(k-1)a^2$ must divide $\nabla_{2,k}f(x+a)$ and $\tau_2' - t_1'(n-k-1)a + \frac{1}{2}(n-k)(n-k-1)a^2$ must divide $\nabla_{2,n-k}f(x+a)$.

After t_2 and t_2' have been found, one continues in an exactly similar fashion and determines the remaining coefficients. Our method may then be applied to the factors $g(x)$ and $h(x)$ until finally $f(x)$ is broken down into its irreducible factors. Ordinarily it is not necessary to use our method for the calculation of all the coefficients of $g(x)$ and $h(x)$. Frequently simpler devices suggest themselves.

4. Examples. We illustrate the foregoing with two examples.

Example 1. $f(x)=x^4-2x^3+6x^2+5x+6$.

Our method could be used to test for linear factors, ($k=1$), but it does not involve any great saving of labor; in fact, in most cases the traditional procedure, to which ours is similar, is better. It is easily seen that in this instance $f(x)$ has no linear factors.

We now look for quadratic factors. Using the table, we see that $\nabla_{1,2}f(x)=-109$. The only possible values of t_1 and t_1', are -1 and -1. But $f(x-1)=x^4-6x^3+18x^2-17x+10$ and $\nabla_{1,2}f(x-1)=1187$, which is not divisible by -3. Hence $f(x)$ has no rational quadratic factors and is consequently irreducible. These latter calculations might have been somewhat simplified by computing $f(x-1)$ and $\nabla_{1,2}f(x-1)$ modulo 3.

A better way of eliminating the pair $(-1, -1)$ is the following. If $f(x)$ has a pair of quadratic factors, (x^2-x+t_2) and (x^2-x+t_2'), then examination of the last two coefficients shows that we must have $(-x+t_2)(-x+t_2')=x^2+5x+6$, whence $t_2=-3$ and $t_2'=-2$. But $(x^2-x-3)(x^2-x-2)\neq f(x)$, since the coefficients of x^2 do not agree.

Example 2. $f(x)=x^5-3x^4-x^3+x^2+5x+15$.

As before, there are no linear factors. We seek quadratic factors. From the table we have

$$\nabla_{1,2}f(x) = -\ 860 = -\ 2^2\cdot 5\cdot 43,$$
$$\nabla_{1,3}f(x) = -\ 3392 = -\ 2^6\cdot 53.$$

The only pairs of divisors of $\nabla_{1,2}f(x)$ and $\nabla_{1,3}f(x)$ which satisfy the condition $\tau_1+\tau_1'=-3$ are $(1, -4)$, $(5, -8)$, $(-1, -2)$, $(-2, -1)$, $(-4, 1)$ and $(-5, 2)$. Now

$$f(x+1) = x^5+2x^4-3x^3-10x^2-3x+18$$

and

$$\nabla_{1,2}f(x+1) = -\ 240 = -\ 2^4\cdot 3\cdot 5,$$
$$\nabla_{1,3}f(x+1) = -\ 860 = -\ 2^2\cdot 5\cdot 43.$$

This eliminates the pair $(5, -8)$, since $\nabla_{1,2}f(x+1)$ is not divisible by $5+2=7$, and the pair $(-2, -1)$, since $\nabla_{1,2}f(x+1)\neq 0$. We try

$$f(x-1) = x^5-8x^4+21x^3-24x^2+17x+8.$$

Here

$$\nabla_{1,2}f(x-1) = 2^8\cdot 3^5$$

while

$$\nabla_{1,3}f(x-1) = 2^6\cdot 5\cdot 79\cdot 101.$$

This eliminates $(-5, 2)$ since $\nabla_{1,2}f(x-1)$ is not divisible by $-5+2(-1)=-7$ and $(1, -4)$ since $\nabla_{1,3}f(x-1)$ is not divisible by $-4+3(-1)=-7$. There remain only $(1, -2)$ and $(4, -1)$; the latter may be eliminated by considering $f(x+2)$ and observing that $\nabla_{1,2}f(x+2)\neq 0$.

We investigate t_2 and t_2'. Notice that, from (2), $t_2+t_2'=-3$. We find that $\nabla_{2,3}f(x)=0$. Simple field-theoretic considerations show that, if $f(x)$ is reducible, we must have $t_2'=0$. Then we are forced to put $t_2=-3$, $t_3'=-5$ and $f(x)=(x^2-x-3)(x^3-2x^2-5)$. It is readily seen that this is a factorization of $f(x)$. Since neither factor has any rational roots, $f(x)$ can be factored no further.

5. Related methods. A method closely related to ours was introduced by Sebastião e Silva.* He observes that if $f(x)$ has a factor, $g(x)$, of degree k, then the equations

$$\prod (t - \sigma_j(r_1', r_2', \cdots, r_k')) = 0, \qquad i = 1, 2, \cdots, k,$$

where the products are taken over all combinations of the roots of $f(x)$ taken k at a time, must have rational roots. These roots, if they exist, may be found by the traditional method. One notices that our $\nabla_{j,k}f(x)$ are simply the constant terms of these equations. Our method does not require the computation of the remaining coefficients. On the other hand, Sebastião e Silva's method does not require the replacement of x by $x+a$ and a repetition of the process, as ours frequently does.

Ore† has remarked that if $f(x)$ has a quadratic factor $g(x)=x^2-t_1x+t_2$, then the discriminant of $f(x)$ must be divisible by $t_1^2-4t_2$, the discriminant of $g(x)$. He uses this fact to restrict the possible values of t_1. This method does not seem as direct as ours, and ordinarily will require the calculation and factorization of larger numbers. However, in many investigations, one has to determine the factors of the discriminant of $f(x)$ for other purposes.

* Jose Sebastião e Silva, Problems concerning rational functions of the roots of an algebraic equation. Portugaliae Math. 2, 1941, pp. 20–35.

† Oystein Ore, A note on factorization of polynomials—Revista Ci. Lima 41, 1939, pp. 587–592.

MATHEMATICS IN SCHOOL AND COLLEGE*

Editorial Note: The following article is an extract from the book *General Education in School and College*, Harvard University Press, 1952, pp. 52–57. This book is the outcome of a study designed to improve the integration between school and college undertaken by members of the faculties of Andover, Exeter, Lawrenceville, Harvard, Princeton, and Yale. The portion of the report dealing with mathematics was based, in part, on a survey conducted by Richard S. Pieters (Andover) and on a panel discussion in which the committee was joined by A. W. Tucker (Princeton), E. G. Begle (Yale), Peter J. Kiernan (Lawrenceville), Ransom Lynch (Exeter), Winfield M. Sides and Richard S. Pieters (Andover).

A reading of the entire report reveals that the recommendations below were framed with special reference to the better-than-average student. Also it should be noted that the conclusions are intended to apply to the relationships between the six institutions represented and are not necessarily valid in other situations. Your editor believes, however, that the basic ideas expressed can be of great value to mathematicians in other schools and universities who may be interested in incorporating suitable modifications of these suggestions into their own curricula.

No subject is more properly a major part of secondary education than mathematics. None has a more distinguished history or a finer tradition of teaching. Perhaps the very excellence of the topic has helped, in recent decades, to keep the content and order of its teaching largely unexamined. One of the most remarkable of our sessions was the one in which we consulted with a group of first-rate school and college teachers of mathematics and discovered, as the evening progressed, a very high degree of consensus on the view that school offerings in mathematics are ready for drastic alteration and improvement.

At present the basic four-year course of mathematics in the schools we have studied covers two years of algebra, one of plane geometry, and one of solid geometry and trigonometry (the latter year elective). A small group of boys (less than one in five) are advanced more rapidly in two of the schools so that in their last year they are introduced to the calculus in a course roughly comparable to first-year college calculus. This basic four-year curriculum has the sanction of the ages, and there can be no doubt that every subject in it has a value of its own. But the Committee was still more impressed by the value of what is usually omitted, and by the fact that in some cases the words of principle seem to be obscured by the trees of constant repetition and problem-solving.

Mathematics is the rigorous application of notions that are vaguely apparent even to the nonmathematical; its wonder lies in its demonstration of the extraordinary power which comes from thinking closely and connectedly. It is not remarkable that an apple drops with growing speed from tree to ground, but what Newton did to this idea is one of the heroic advances of the human mind. The means by which he did it are not easily understood; the power of abstract thinking is matched by its difficulty. This is true for all the basic ideas of mathematics, yet it is just these basic ideas that are of the greatest value to the student. As the Harvard Report has noted, "It is unfortunately true that those

aspects of algebra and geometry that are of greatest interest in general education are also more difficult to teach, and are much harder for the student to grasp, than are the technical skills of mathematical manipulation."* From every point of view—that of the college mathematics teacher as well as that of general education—we are persuaded that the greatest business of the school's mathematics curriculum should be to communicate as many of these central concepts as it can; we think the better student can do much more of this than he has been doing, and we are certain that he should try.

But from the point of view of a crowded curriculum we are convinced that each branch of mathematics operates under a law of rapidly diminishing returns. Once the basic notions are solidly understood—and this will require drill as well as thought—we think it is unwise to linger in loving elaboration of a set of ideas grown familiar. Of course it is possible to design problems of bewildering complexity in every subject from long division to trigonometry; it is also a waste of time. We do not press the paradox, but we suggest that it is almost true that the better the student, the fewer problems he should be asked to solve.

The impact of these notions upon the present mathematical curriculum is heavy, for large parts of it turn out to be relatively unhelpful elaborations of principles which are better taught in other ways. The greatest single offender in this sense is solid geometry. It is a beautiful subject, but in the strictly mathematical sense it is an elaboration of plane geometry, and elaboration is not the point of mathematics. The real value of solid geometry lies outside its mathematics, in the fact that it tends to develop a general sense of spatial reality. This can be done more briefly and more effectively, we think, if the effort to develop a systematic structure is abandoned. If generally adopted, this single revision would save nearly half a year in the standard school curriculum.

The example of solid geometry can be repeated, on a smaller scale, in the teaching of algebra, plane geometry, and trigonometry. Among the topics which seem appropriate for condensation or omission are complex numbers,† determinants, logarithmic solutions of triangles, and the geometry of the circle. These are suggestions only; much planning and experience will be needed before teachers of mathematics can determine where to modify and revise their present practice, and it is by no means certain that the problem has a unique solution. All that we assert, and we do so with the authority of our professional advisers, is that much can be squeezed out, to the positive advantage of the basic notions which are now taught.

If we are correct in claiming that there is excess fat on the body mathematical, it is good, in itself, to trim the present curriculum; for it is one of the basic conclusions of our whole study that it is bad to do in two terms what should be

* General Education in a Free Society (Harvard University Press, 1945), p. 163.

† Professor Tucker has suggested to the editor that this reference to complex numbers may have slipped in by accident or in any event was not sufficiently considered by the committee which prepared the report.

done in one. But we are not urging a cut in the traditional mathematics course because we want less time for mathematics. What we are trying to do is at once to bring into visible relief the great ideas of traditional algebra and geometry, and at the same time to make room for two additional sets of notions—those related to the calculus and to statistics. This may be too much to ask, but before we consider the difficulties let us briefly urge the importance of each of these two topics.

The importance of the calculus for all scientists and engineers needs no argument. Its respectability as a discipline in liberal education is equally plain; it is the standard freshman course in the best of our colleges. The only thing new in our position is the suggestion that the elements of this subject can be presented to a large percentage of well-trained schoolboys in the 12th grade. Not very many, perhaps, can meet the standards of a college course, but that is not our first object. What we are eager to do is to get into the minds of the student who is *not* headed for a career in science or mathematics some sense of the power and meaning of the calculus. We are firmly persuaded that this can be done in school, and it seems to us to have a value, in general education, that is greater by several orders of magnitude than the value of drill in the elaboration of solid geometry or determinants. The student who has once grasped the meaning of differentiation and integration sees the world afterward in a larger and more significant way; his exposure to science takes a different shape; his sense of space and motion is enlarged. Obviously not all students take this meaning from a single course in the calculus; not all teachers try to teach it. But for those who can, the opportunity is too great to be sacrificed to anything but necessity, and we are bound to conclude that tradition, not necessity, is what currently limits the teaching of elementary calculus in school.

There is a further and most practical advantage in an expanded teaching of the calculus in school. It is that a student of physics, however good his training in school, finds it extremely difficult to handle advanced physics courses unless he has been introduced to the calculus. The prospective scientist can clear his way in both physics and mathematics if he does solid work in the calculus in school.

In principle, the nonscientist with school training in the calculus might reap a parallel advantage; in his study of science in college he could go much further into the meaning of physics than would otherwise be possible. It is a serious misunderstanding to suppose that general education and mathematical physics are opposites. The best of the college courses for the nonscientist make considerable demands on the mathematical equipment of their students, and their teachers would be happy to build on a knowledge of elementary calculus. But such a development may take time.

To some of our consultants, and in some respects to our Committee as well, the case for statistics is even more powerful than the case for the calculus. The notions of probability, correlation, and sampling are among the fundamentals

of modern social measurement. And since we live in the age of polls, an awareness of the real meaning of these notions is a protection to the consumer as well as a necessity for the producer of information. Moreover, there is in all statistics a salutary concern for the uncertain and the incomplete—for the gray that is real more than for the black and white that is abstraction. It is well for the student to learn both that mathematics has uncertainty and that uncertainty can be mathematically treated. This knowledge is important in many fields; teachers of science and teachers of history alike have their troubles with students who are persuaded that all reasoning is geometrical and all evidence conclusive. All in all, if we had a curriculum to build from the ground up, we cannot suppose that it would omit statistics from a general education.

If we could find the time, we would urge that the standard school curriculum include about one year of the calculus (with a minimal framework of analytic geometry) and half a year of statistics. And if all else were equal, we would urge that the statistics follow the calculus; it has much more meaning and power that way. But time is not easy to find, and all other things are not equal. The teaching of statistics in school can hardly grow very fast until there is a body of teachers with experience in the subject and some core of knowledge on which to build the right kind of course. Moreover, there is a real advantage in leaving the whole of the 12th grade for the calculus—this is the topic that leads directly to advanced work in college, and it is a well-tested and coherent unit which is best taught in a single academic year. Finally, nearly all good school departments of mathematics are well-equipped to teach the calculus; they have been teaching it, on a limited scale, for many years. On balance, therefore, we recommend that the schools should move toward a curriculum in which the basic 12th grade course is the introduction to the calculus. At the same time we hope that there will be intensive experimentation with the teaching of some of the basic concepts of statistics, and we think there is room for this in the second year's study of algebra. Nor do we exclude the possibility that some schools may wish to offer statistics as an alternative to the calculus, or even as an additional elective.

It is our conclusion that the school mathematics curriculum can and should be redesigned to include new areas of instruction; and we think that when this has been done, college mathematics should be a subject for scientists, mathematicians, and really talented amateurs of the topic. For others, there is plenty in the basic course we have outlined, and their college work should be in other fields. And this we feel is as it should be; fundamental mathematics of the sort we have been discussing is taught better—and learned better—in the schools than in the colleges.

SOME REFINEMENTS IN THE THEORY OF SPECIALIZED SPACE CURVES

P. W. GIFFORD, JR., High Mowing School

1. Introduction. The general space curve of this discussion has the parametric vector representation

$$\mathbf{x} = \mathbf{x}(t). \tag{1}$$

Equation (1) consists of the triple of scalar functions

$$x_i = x_i(t); \qquad i = 1, 2, 3, \tag{1'}$$

where the x_i are projections on the axes of a rectangular coordinate system in Cartesian three-space. We require that the domain of equation (1) be a real interval $a \leqq t \leqq b$ and that the functions (1′) be real- and single-valued on this domain. Also, from time to time, we specify that functions (1′) belong to class C^n; $n = 1, 2, 3, \cdots$, that is, to the class of functions with continuous derivatives of order n. Finally, for any two vectors $\mathbf{a}$ and $\mathbf{b}$, the inner or scalar product $(\mathbf{a} \cdot \mathbf{b})$ and the outer or vector product $\mathbf{a} \times \mathbf{b}$ in this discussion have their usual definitions and properties.

Standard texts of differential geometry usually eliminate from the class of space curves all equations of the form (1) which possess singular points. That is, the authors specify that not all the $x_i'(t)$ vanish at any point, when they discuss straight lines and similarly that $\mathbf{x}'(t) \times \mathbf{x}''(t)$ never vanishes when they consider plane curves. Under these restrictions, the standard texts are able to prove that the necessary and sufficient conditions for an equation to represent a straight line are equivalent. The texts give a similar proof for plane curves [**1**, **4**, **7**].

The writer suggests the need for a discussion of rectilinearity and planeness which admits to consideration any equation of the form (1) with the properties ascribed to it in the first paragraph. Let us compare necessary and sufficient conditions for the two geometrical properties within this broad class of equations.

2. Straight lines. Many of the rectifiable curves of differential geometry have a "natural" parametric representation in the sense that the parameter s of this representation expresses the arc length of these curves. Proof of the existence of this representation for a given curve $\mathbf{x}(t)$ requires that the functions (1′) be of class C^1, in which case the curve is rectifiable, and that always at least one of the first derivatives of these functions be different from zero [**7**, pp. 72–73 and 44].

This arc length parametrization $\mathbf{x}(s)$ is mentioned because, within the class of curves which are expressed in this form, the necessary and sufficient conditions for rectilinearity are equivalent. Thus for these special curves only, we follow the pattern of the standard texts in proving

THEOREM I. *The curve* $\mathbf{x}(s)$ *of class* C^2 *and expressed in terms of its arc length*

is a straight line if and only if

$$\mathbf{x}''(s) \equiv 0.$$

If $\mathbf{x}(s)$ is a straight line, it will have the arc length representation of class C^2

$$\mathbf{x} = \mathbf{r} + \mathbf{a}s$$

where $\mathbf{r}$ and $\mathbf{a}$ are constant vectors. Here, arc length s is measured along the line from $\mathbf{r}$ in direction $\mathbf{a}$. We have

$$\mathbf{x}'(s) = \mathbf{a} \neq 0$$
$$\mathbf{x}''(s) \equiv 0.$$

Conversely, if $\mathbf{x}(s)$ is known to be of class C^2 and to satisfy this last relationship, successive component-wise integrations yield

$$\mathbf{x} = \mathbf{r} + \mathbf{a}s.$$

This is interpreted geometrically as a straight line with s as arc length, and the theorem is proved.

Our departure from the standard discussions comes when the curve is expressed as a straight line by using the arbitrary parameter t to write

$$\mathbf{x} = \mathbf{x}(t) = \mathbf{r} + \mathbf{a}\phi(t)$$

in terms of the point $\mathbf{r}$, direction $\mathbf{a}$, and scalar function $\phi(t)$. We assume that $\mathbf{x}(t)$ is of class C^2 so that, differentiating,

$$\mathbf{x}'(t) = \mathbf{a}\phi'(t)$$
$$\mathbf{x}''(t) = \mathbf{a}\phi''(t).$$

Since we exclude points from the class of straight lines,

$$\mathbf{x}'(t) \not\equiv 0.$$

Otherwise $\mathbf{x}(t)$ would reduce to a constant vector, or point. From the expressions for $\mathbf{x}'$ and $\mathbf{x}''$,

$$\mathbf{x}'(t) \times \mathbf{x}''(t) = \mathbf{a} \times \mathbf{a}[\phi'(t)\phi''(t)] \equiv 0,$$

and we state this result in

THEOREM II. *If*:
1. $\mathbf{x}(t)$ is of class C^2,
2. $\mathbf{x}(t)$ *represents a straight line*,

then:
A. $\mathbf{x}'(t) \not\equiv 0$,
B. $\mathbf{x}'(t) \times \mathbf{x}''(t) \equiv 0$.

The example

$$\mathbf{x} = (-t^3, 0, 0), \quad t \leqq 0; \qquad \mathbf{x} = (0, t^3, 0), \quad t > 0$$

demonstrates that the converse of this last theorem is not true. Conclusions A and B above hold for the example which is of class C^2. Yet rectilinearity fails at the origin, where $\mathbf{x}'(t)$ vanishes. Evidently some modification is needed in stating sufficient conditions for rectilinearity for the class of space curves under consideration. As a first approach to the problem, we shall assume differentiability of class C^2, conclusion B of Theorem II, and the additional hypothesis

$$\mathbf{x}'(t) \text{ never vanishes.} \tag{2}$$

Under these conditions, $\mathbf{x}'$ is parallel to $\mathbf{x}''$ or $\mathbf{x}''=0$ so that, for some scalar function $k(t)$,

$$\mathbf{x}''(t) = k(t)\mathbf{x}'(t),$$

which relation may be expressed as

$$\frac{d}{dt} x_i'(t) = k(t) x_i'(t); \qquad i = 1, 2, 3.$$

Condition (2) implies $k(t)$ is continuous so that, from existence and uniqueness theorems of differential equations [5], $x_i'(t)$ is given by

$$x_i'(t) = a_i e^{\int k(t)dt}; \qquad a_i = \frac{x_i'(t_0)}{\left[e^{\int k(t)dt}\right]_{t=t_0}},$$

where t_0 is an arbitrary value of t. Similarly,

$$x_i(t) = a_i f(t) + r_i; \qquad f(t) = \int e^{\int k(t)dt} dt.$$

This has the form of the equation for a straight line, from which we deduced Theorem II. $\mathbf{x}(t)$ cannot reduce to a point by condition (2). We have

THEOREM III. *If*:
1. $\mathbf{x}(t)$ *is of class* C^2,
2. $\mathbf{x}'(t)$ *never vanishes*,
3. $\mathbf{x}'(t) \times \mathbf{x}''(t) \equiv 0$,

then equation (1) *represents a straight line.*

This theorem expresses the fact that sufficient conditions for rectilinearity must be stronger than the necessary conditions of Theorem II. The reader may further satisfy himself on this point by proving that any broken line can be represented by an equation of the form (1) with $\mathbf{x}(t)$ of class C^2 and satisfying conclusions A and B of Theorem II.

3. Plane curves. Necessary and sufficient conditions for planeness follow a

pattern similar to that which we encountered in the case of rectilinearity. The planeness of $\mathbf{x}(t)$ is expressed by writing

$$(\mathbf{a}\cdot\mathbf{x}(t)) = k \tag{3}$$

where $\mathbf{a}$ is the direction of the normal to the plane of the curve, and k is a scalar constant. We assume that $\mathbf{x}(t)$ is of class C^3. From section 2 it is clear that

$$\mathbf{x}'(t) \times \mathbf{x}''(t) \not\equiv 0$$

since the class of plane curves is here restricted to curves which are not (sectionally) straight lines. From (3),

$$(\mathbf{a}\cdot\mathbf{x}') \equiv (\mathbf{a}\cdot\mathbf{x}'') \equiv (\mathbf{a}\cdot\mathbf{x}''') \equiv 0. \tag{4}$$

Now we define

$$(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) = \begin{vmatrix} x_1' & x_1'' & x_1''' \\ x_2' & x_2'' & x_2''' \\ x_3' & x_3'' & x_3''' \end{vmatrix}.$$

Since $\mathbf{a}$ is not the null vector, equations (4) have a non-vanishing solution. It follows that, for all values of t, the determinant of these equations vanishes:

$$(\mathbf{x}'\ \mathbf{x}''\ \mathbf{x}''') \equiv 0.$$

This necessary condition for planeness is expressed in

THEOREM IV. *If:*
1. $\mathbf{x}(t)$ *is of class* C^3,
2. $\mathbf{x}(t)$ *represents a plane curve,*

then:
A. $\mathbf{x}'(t)\times\mathbf{x}''(t)\not\equiv 0$,
B. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t))\equiv 0$.

That the converse of this theorem fails is apparent from the example

$$\mathbf{x} = (t^4, t, 0), \quad t \leqq 0; \qquad \mathbf{x} = (0, t, t^4), \qquad t > 0.$$

Although conclusions A and B above apply, this curve departs from planeness at the origin, where $\mathbf{x}'\times\mathbf{x}''$ vanishes (although $\mathbf{x}'(0)$ is non-zero). To avoid such exceptional behavior, we specify for the present that

$$\mathbf{x}'(t) \times \mathbf{x}''(t) \text{ never vanishes,} \tag{5}$$

as well as a differentiability of C^3 and conclusion B of Theorem IV. Applying the Lagrange identity [7, p. 51] to conclusion B, we have

$$(\mathbf{x}' \times \mathbf{x}''\cdot(\mathbf{x}' \times \mathbf{x}''') \times \mathbf{w}) = (\mathbf{x}'\cdot\mathbf{x}' \times \mathbf{x}''')(\mathbf{x}''\cdot\mathbf{w}) - (\mathbf{x}''\cdot\mathbf{x}' \times \mathbf{x}''')(\mathbf{x}'\cdot\mathbf{w}) \equiv 0$$

where $\mathbf{w}$ is an arbitrary or "dummy" vector. This can be rewritten

$$(\mathbf{x}' \times \mathbf{x}''\ \ \mathbf{x}' \times \mathbf{x}'''\ \ \mathbf{w}) = ((\mathbf{x}' \times \mathbf{x}'') \times (\mathbf{x}' \times \mathbf{x}''')\cdot\mathbf{w}) \equiv 0$$

which implies that

$$(\mathbf{x}' \times \mathbf{x}'') \times (\mathbf{x}' \times \mathbf{x}''') \equiv 0. \tag{6}$$

Next we introduce the transformation

$$\mathbf{y} = \mathbf{x}' \times \mathbf{x}''; \qquad \mathbf{y}' = \mathbf{x}' \times \mathbf{x}''' + \mathbf{x}'' \times \mathbf{x}'' = \mathbf{x}' \times \mathbf{x}'''$$

so that (6) becomes

$$\mathbf{y} \times \mathbf{y}' \equiv 0.$$

Thus, vector $\mathbf{y}$ is parallel to $\mathbf{y}'$, or

$$\mathbf{y}'(t) = k(t)\mathbf{y}(t)$$

in terms of scalar function $k(t)$. We have

$$\frac{d}{dt} y_i(t) = k(t) y_i(t); \qquad i = 1, 2, 3,$$

and, invoking (5) and proceeding as in the proof of Theorem III,

$$y_i(t) = a_i g(t); \qquad g(t) = e^{\int k(t)dt} \not\equiv 0. \tag{7}$$

From the definition of $\mathbf{y}(t)$,

$$(\mathbf{y} \cdot \mathbf{x}') = (\mathbf{x}' \ \mathbf{x}'' \ \mathbf{x}') \equiv 0,$$

and we use relation (7) to obtain

$$(\mathbf{a} \cdot \mathbf{x}'(t)) = \frac{1}{g(t)} (\mathbf{y}(t) \cdot \mathbf{x}'(t)) \equiv 0.$$

Integration gives

$$(\mathbf{a} \cdot \mathbf{x}(t)) = k,$$

so that, recalling (3), $\mathbf{x}(t)$ must be a plane curve, since it cannot be a straight line by Theorem II and condition (5). Sufficient conditions for planeness of space curves are expressed in

THEOREM V. *If*:
1. $\mathbf{x}(t)$ *is of class* C^3,
2. $\mathbf{x}'(t) \times \mathbf{x}''(t)$ *never vanishes*,
3. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) \equiv 0$,

then equation (1) *represents a plane curve.*

4. Refinements. *The analytic case.* We have seen the exceptional behavior exhibited at the origin by the two examples which suggested the proofs of Theorems III and V. These two theorems respectively prohibit the vanishing of $\mathbf{x}'(t)$ and of $\mathbf{x}'(t) \times \mathbf{x}''(t)$ at any point. Yet this complete prohibition is unnecessarily drastic, as a pair of curves will show. The curve

$$\mathbf{x} = (0, t^3, 0)$$

is a straight line throughout, despite the vanishing of $\mathbf{x}'(0)$. Similarly,

$$\mathbf{x} = (0, t, t^4)$$

lies wholly in one plane, even though $\mathbf{x}'(0) \times \mathbf{x}''(0)$ vanishes. Graustein [4] typifies the approach of writers who discuss rectilinearity and planeness with the unnecessary strictness indicated by these two curves.

These remarks suggest that we attempt refinements of the conditions of Theorems III and V in order to admit as representatives of straight lines or plane curves many equations which are at present excluded. The basis for the first refinement will be the requirement that $\mathbf{x}(t)$ (*i.e.*, each of its component functions) be analytic throughout its domain.

We begin by taking the requirement of analyticity together with conclusions A and B of Theorem II as possible sufficient conditions for rectilinearity to replace those of Theorem III. Since $\mathbf{x}(t)$ is analytic, it follows that $\mathbf{x}'(t)$ is also analytic. A standard theorem in analysis states that the zeros of an analytic function are isolated on its domain unless the function vanishes identically [2]. This implies that $\mathbf{x}'(t)$ (*i.e.*, at least one of its components) never vanishes in some sufficiently small neighborhood I of a value t_0 of t which is not a zero of $\mathbf{x}'(t)$. The existence of such a t_0 follows from the assumed conclusion A of Theorem II. For values of t in I, $\mathbf{x}(t)$ is a straight line, since the conditions of Theorem III are fulfilled. Thus in I we can write

$$\mathbf{x}'(t) = \mathbf{a}f(t) \tag{8}$$

where $\mathbf{a}$ is the direction of the straight line represented by $\mathbf{x}(t)$ for t in I, and $f(t)$ is a scalar function.

$$x_i'(t) = a_i f(t); \qquad i = 1, 2, 3 \tag{8'}$$

gives (8) component-wise. If we assume that all the a_i are non-zero, it is clear that the functions

$$\frac{1}{a_i} x_i'(t); \qquad i = 1, 2, 3$$

are analytic in the entire domain of $\mathbf{x}(t)$. Furthermore, by (8′), these functions coincide in the finite interval I. Therefore, they coincide throughout the domain of $\mathbf{x}(t)$, by the identity theorem for analytic functions [6]. We may name this common expression $f(t)$. Thus

$$\frac{1}{a_i} x_i'(t) = f(t); \qquad i = 1, 2, 3$$

from which we obtain an expression of the form (8) for the entire domain of $\mathbf{x}(t)$, implying rectilinearity throughout.

If, in equation (8′), one of the a_i vanishes, say a_j, then still for the two remaining components, (8′) extends to all values of t as before. Since $x_j'(t)$ must vanish in I in this case, the identity theorem assures that, for all t,

$$x_j'(t) \equiv 0 = a_j f(t),$$

so that again equation (8) holds on the entire domain for all components. Further, if two of the a_i vanish in (8′), it is apparent from the identity theorem that the corresponding $x_i'(t)$ vanish for all t so that, interpreting geometrically, $\mathbf{x}'(t)$ has everywhere constant direction. Finally, not all three a_i can vanish since, as we have already observed, at least one component of $\mathbf{x}'(t)$ never vanishes in I. In every case $\mathbf{x}(t)$ is a straight line throughout, and we have established a refined expression of sufficient conditions for rectilinearity:

THEOREM IIIa. *If:*
1. $\mathbf{x}(t)$ *is analytic throughout its domain,*
2. $\mathbf{x}'(t) \not\equiv 0$,
3. $\mathbf{x}'(t) \times \mathbf{x}''(t) \equiv 0$,

then equation (1) *represents a straight line.*

The development of a first revised theorem concerning planeness of $\mathbf{x}(t)$ follows a similar plan. In addition to the analyticity of $\mathbf{x}(t)$, we adopt as hypotheses conclusions A and B of Theorem IV. The vector $\mathbf{x}'(t) \times \mathbf{x}''(t)$ plays a role similar to that of $\mathbf{x}'(t)$ in the foregoing proof. By an argument like that which led to equation (8), it appears that the conditions of Theorem V apply in some interval I of values of t, so that $\mathbf{x}(t)$ is a plane curve in this interval. The vector $\mathbf{x}'(t) \times \mathbf{x}''(t)$ becomes the normal to the plane of $\mathbf{x}(t)$ for t in I. For these values of t,

$$\mathbf{x}'(t) \times \mathbf{x}''(t) = \mathbf{b}g(t)$$

in terms of vector $\mathbf{b}$ and scalar function $g(t)$. This equation is extended over the entire domain of $\mathbf{x}(t)$ by the method applied to equation (8). Thus, $\mathbf{x}'(t) \times \mathbf{x}''(t)$ has everywhere constant direction. This implies that $\mathbf{x}(t)$ is restricted to a single plane, and our revision of sufficient conditions for planeness becomes

THEOREM Va. *If:*
1. $\mathbf{x}(t)$ *is analytic throughout its domain,*
2. $\mathbf{x}'(t) \times \mathbf{x}''(t) \not\equiv 0$,
3. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) \equiv 0$,

then equation (1) *represents a plane curve.*

5. The non-analytic case. A final refinement of the conditions of Theorems III and V arises when we investigate wider classes of non-analytic expressions $\mathbf{x}(t)$ for which rectilinearity or planeness hold. For rectilinearity, we specify that $\mathbf{x}(t)$ is of class C^n, where $n > 1$, and that, at every point, at least one vector derivative of order less than or equal to n is different from zero. Conclusion **B of** Theorem II is also needed.

Under these hypotheses, if t_0 is an arbitrary value of t, there is a value of k for which

$$x_j^{(k)}(t_0) \neq 0; \qquad k \leqq n$$

where j indicates some particular component of $\mathbf{x}^{(k)}(t)$. We may suppose that $k>1$ and is the least value for which such a non-zero relation holds at t_0 in any component. The continuity of $\mathbf{x}^{(k)}(t)$ implies that, within some finite neighborhood I of t_0,

$$x_j^{(k)}(t) = f(t) \neq 0,$$

so that either $f(t)>0$ or $f(t)<0$ in I. The existence and uniqueness theorems [**5**] assure that this last equation has the solution for $x_j'(t)$ of the form [**3**]

$$x_j'(t) = \int_{t_0}^{t} dt \int_{t_0}^{t} dt \cdots \int_{t_0}^{t} f(t)dt, \tag{9}$$

involving $k-1$ integrations. All constants of integration vanish, owing to the relations

$$x_j'(t_0) = x_j''(t_0) = \cdots = x_j^{(k-1)}(t_0) = 0.$$

From the non-zero property of $f(t)$ we see that, at each step of integration, the integral in (9) cannot vanish in I except at t_0. This shows that the zeros of $\mathbf{x}'(t)$ and of its first $k-2$ derivatives are isolated points. Thus $\mathbf{x}(t)$ is at least sectionally straight. Specifically, in I, there is at worst a kink at t_0, since this is the only point where Theorem III breaks down.

It follows from the foregoing that, on the left of t_0 in I, $\mathbf{x}'(t)$ has the form $g(t)(a_1, a_2, a_3)$, where the a_i are the constant components of a unit vector. $g(t)$ is a scalar function. A similar expression involving a unit vector $\mathbf{b}$ holds on the right of t_0 in I. Successive differentiations of these expressions show that the first k derivatives of $\mathbf{x}(t)$ have the direction vectors $\mathbf{a}$ and $\mathbf{b}$ to the left and right of t_0, respectively. But any non-null vector $\mathbf{u}$ has its direction given by components $u_i/\sqrt{\mathbf{u}\cdot\mathbf{u}}$ so that

$$\frac{x_i^{(k)}(t)}{\sqrt{\mathbf{x}^{(k)}(t)\cdot\mathbf{x}^{(k)}(t)}} = a_i,\ t < t_0 \quad \text{and} \quad \frac{x_i^{(k)}(t)}{\sqrt{\mathbf{x}^{(k)}(t)\cdot\mathbf{x}^{(k)}(t)}} = b_i,\ t > t_0; \qquad i = 1, 2, 3.$$

By continuity and the non-zero condition on $\mathbf{x}^{(k)}(t)$ in I, the limits on the left and right at t_0 of the fraction appearing above are equal:

$$a_i = b_i; \qquad i = 1, 2, 3.$$

By this reasoning, we can extend the domain of direction vector $\mathbf{a}$ through all zeros of $\mathbf{x}'(t)$, so that $\mathbf{x}'(t)$ has constant direction throughout. We may state

THEOREM IIIb. *If*:

1. $\mathbf{x}(t)$ *is of class* C^n; $n>1$,
2. *at every point at least one vector derivative* $\mathbf{x}^{(k)}(t)$ *is different from zero*; $k\leqq n$,
3. $\mathbf{x}(t)\times\mathbf{x}''(t)\equiv 0$,

then equation (1) *represents a straight line.*

For planeness of $\mathbf{x}(t)$, the conditions are a differentiability of class C^n, where $n>2$, the existence at every point of some pair of vector derivatives $\mathbf{x}^{(k)}(t)$ and $\mathbf{x}^{(m)}(t)$ (where $k+m-1\leqq n$) whose outer product is different from zero, and conclusion B of Theorem IV. Thus for arbitrary t_0, if the conditions of Theorem V do not hold,

$$\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(m)}(t_0)\neq 0;\qquad k+m>3.$$

It is no restriction to specify that k is some least integer and m a next least integer for which this is true.

From the formula

$$(\mathbf{y}\times\mathbf{z})'=\mathbf{y}\times\mathbf{z}'+\mathbf{y}'\times\mathbf{z}$$

for the derivative of an outer product, the reader will see that the pth derivative of $\mathbf{x}'\times\mathbf{x}''$ is a sum of linear terms in $\mathbf{x}^{(u)}(t)\times\mathbf{x}^{(v)}(t)$ with positive integral coefficients and with

$$u+v=p+3;\qquad 1\leqq u<v.$$

Furthermore, all possible combinations of u and v satisfying this relation will appear in the derivative. The $(k+m-3)$rd derivative of $\mathbf{x}'\times\mathbf{x}''$ therefore has the form

$$K\mathbf{x}^{(k)}(t)\times\mathbf{x}^{(m)}(t)+\text{linear terms in }\mathbf{x}^{(r)}(t)\times\mathbf{x}^{(s)}(t);\qquad r+s=k+m,\tag{10}$$

where K is a positive integer. Now all the expressions $\mathbf{x}^{(r)}\times\mathbf{x}^{(s)}$ fall into one of two categories, as follows.

First, we may have $r<k$. Yet since we specified above that k is the least integer which appears in a non-zero outer product at t_0, it follows that all $\mathbf{x}^{(r)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ in this category vanish.

Second, all remaining terms must have r and s between k and m. Speaking geometrically, if $\mathbf{x}^{(r)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ is non-zero, then $\mathbf{x}^{(r)}(t_0)$ and $\mathbf{x}^{(s)}(t_0)$ are non-null vectors which are not parallel. In the case under consideration, this would imply $\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(r)}(t_0)$ or $\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ or both are different from zero. But any of these alternatives would contradict the hypothesis that m is the least value after k for which $\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(m)}(t_0)$ does not vanish.

We must admit that all the $\mathbf{x}^{(r)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ vanish so that, at t_0, the derivative (10) reduces to the non-null vector $K\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(m)}(t_0)$. By this fact and continuity, we have in a neighborhood I of t_0 for at least one jth component,

$$\frac{d^M(\mathbf{x}'(t) \times \mathbf{x}''(t)_j)}{dt^M} = F(t) \neq 0; \qquad M = k + m - 3.$$

We may employ the same argument used on $x_j^{(k)}(t)$ in establishing equation (9) before. Again, all constants of integration vanish, since it follows from reasoning like that applied above to the Mth derivative of $\mathbf{x}' \times \mathbf{x}''$ that all derivatives of order less than M vanish at t_0. We obtain, after M integrations,

$$\mathbf{x}'(t) \times \mathbf{x}''(t)_j = \int_{t_0}^{t} dt \int_{t_0}^{t} dt \cdots \int_{t_0}^{t} F(t)dt.$$

Reasoning as before, the zeros of $\mathbf{x}' \times \mathbf{x}''$ and of its derivatives are isolated points, and Theorem V holds at least sectionally.

The rest is clear. We treat $\mathbf{x}' \times \mathbf{x}''$ as we did $\mathbf{x}'$ in the straight line case, establishing constant direction vectors on the left and right of t_0. Using continuity and the non-zero property of the Mth derivative in I, we prove equality of these vectors. $\mathbf{x}' \times \mathbf{x}''$ has everywhere constant direction and is normal to the plane of $\mathbf{x}(t)$. This final result is expressed in

THEOREM Vb. *If:*

1. $\mathbf{x}(t)$ *is of class* C^n; $n > 2$,

2. *at every point at least one pair of vector derivatives* $\mathbf{x}^{(k)}(t)$ *and* $\mathbf{x}^{(m)}(t)$ *has an outer product different from zero*; $k+m-1 \leqq n$,

3. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) \equiv 0$,

then equation (1) *represents a plane curve.*

The two pairs of refined theorems admit to the classes of straight lines and plane curves many expressions which the original Theorems III and V exclude. The two curves discussed at the beginning of section 4 are examples. Furthermore, by admitting to consideration curves which possess singular points, our new hypotheses eliminate the unnecessary strictness in the traditional discussion of rectilinearity and planeness.

The writer acknowledges the suggestions of Professor Garrett Birkhoff of Harvard University.

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THE EXPANSION OF A FUNCTION IN TERMS OF ITS VALUES AND DERIVATIVES AT SEVERAL POINTS

W. R. ANDRESS, Canterbury University College, New Zealand

In 1949 under the title *A Generalization of Taylor's Expansion*, Hummel and Seebeck* gave an interesting and useful expansion for $f(x)$ in terms of the derivatives at x, $f(a)$, and the derivatives at $x=a$. Their general theorem is

$$(1)\quad f(x) = f(a) + \sum_{k=1}^{\infty} \frac{(m+n-k)!}{(m+n)!}\left[C_{m,k}f^k(a) - (-1)^k C_{n,k}f^k(x)\right](x-a)^k + R.$$

where

$$R = \frac{(-1)^n m!n!(x-a)^{m+n+1}}{(m+n)!(m+n+1)!} f^{m+n+1}(\theta),$$

and θ lies between x and a. In this expansion the convention that a binomial coefficient $C_{m,k}$ vanishes for all values of k greater than m has been used, so that in the summation, the first term in the square bracket produces m terms whilst the second term in this bracket produces n terms.

In view of the particularly good approximations given by this theorem, it seemed to be of interest to express it, if possible, as a contour integral in the complex plane, and the result obtained is most interesting. The form of the appropriate contour integral is suggested by writing

$$f^k(a) = \frac{k!}{2\pi i}\int_C (t-a)^{-k-1} f(t)dt \text{ in the sum} \sum_{k=0}^{m} \frac{(m+n-k)!}{(m+n)!} C_{m,k} f^k(a)$$

whence, making use of the binomial expansion,

$$(1-u)^{-(n+1)} = \sum_{s=0}^{\infty} C_{n+s,s} u^s,$$

we obtain an integral of the form

$$(2)\qquad I(m, n) = \frac{1}{2\pi i}\int_C \frac{(x-a)^{m+n+1}}{(t-a)^{m+1}(t-x)^{n+1}} f(t)dt.$$

The process now becomes clear; The contour C must include the points x, a, and $f(t)$ must be regular within and on the contour. It then follows by Cauchy's theorem that the integral has a value equal to the sum of the residues of the integrand at the points $t=x$ and $t=a$, and clearly the residue at $t=x$ is expressible in terms of $f(x)$ and its first n derivatives, whilst the residue at $t=a$ is expressible in terms of $f(a)$ and its first m derivatives.

We now calculate the residue for $I(m, n)$ at the point $t=a$ in the usual way by putting $t=a+h$, the residue then being the coefficient of h^m in the expansion

* P. M. Hummel and C. L. Seebeck, Jr., this MONTHLY, vol. 56, 1949, pp. 243–247.

of

$$\frac{(x-a)^{m+n+1}}{(a-x+h)^{n+1}} f(a+h); \quad \text{hence}$$

the residue is equal to the coefficient of h^m in

$$(-1)^{m+1}\left(1+\frac{h}{a-x}\right)^{-(n+1)}(x-a)^m f(a+h)$$

which equals the coefficient of h^m in

$$(-1)^{m+1}(x-a)^m \sum_{s=0}^{\infty} C_{n+s,s}\left(\frac{h}{x-a}\right)^s f(a+h)$$

$$= \sum_{k=0}^{m} (-1)^{m+1}(x-a)^k C_{m+n-k,m-k} \frac{1}{k!} f^k(a)$$

$$= (-1)^{m+1} \frac{(m+n)!}{m!n!} \sum_{k=0}^{m} \frac{(m+n-k)!}{(m+n)!} C_{m,k} f^k(a)(x-a)^k.$$

Similarly, or by interchanging x, a and also m, n, the residue at $t=x$ becomes

$$(-1)^m \frac{(m+n)!}{m!n!} \sum_{k=0}^{n} \frac{(m+n-k)!}{(m+n)!} C_{n,k} f^k(a)(a-x)^k,$$

whence

$$\sum_{k=0}^{\infty} \frac{(m+n-k)!}{(m+n)!} [C_{m,k} f^k(a) - (-1)^k C_{n,k} f^k(x)](x-a)^k$$

$$= (-1)^{m+1} \frac{m!n!}{(m+n)!} I(m, n),$$

which is equivalent to the original theorem (1) with the remainder expressed as a contour integral.

Extensive generalization is now easy. If, for example, we wish to expand $f(x)$ in terms of $f(a)$ and its first $n-1$ derivatives and $f(b)$ and its first $m-1$ derivatives, we take as the appropriate contour integral

$$(3) \qquad I(m, n) = \frac{1}{2\pi i} \int_C \frac{(x-a)^m (x-b)^n}{(t-a)^m (t-b)^n} \frac{f(t)}{(t-x)} dt.$$

The residue at $t=x$ is clearly $f(x)$, and the residues at $t=a$ and $t=b$ follow easily by the usual process. If we introduce a function

$$\overset{n}{g}_s(x) \equiv 1 + nx + \frac{n(n+1)}{1.2} x^2 + \cdots + \frac{n(n+1)\cdots(n+s-1)}{s!} x^s,$$

so that $g_s^n(x)$ is the sum of the first $s+1$ terms in the expansion of $(1-x)^{-n}$, and, for convenience also, write

$$\theta = \frac{x-a}{b-a}, \qquad \phi = \frac{x-b}{a-b},$$

then the corresponding expansion becomes,

$$(4) \qquad \begin{aligned} f(x) &= \phi^n \sum_{k=0}^{m-1} \frac{1}{k!} g_{m-k-1}^n(\theta) f^k(a)(x-a)^k \\ &\quad + \theta^m \sum_{k=0}^{n-1} \frac{1}{k!} g_{n-k-1}^m(\phi) f^k(b)(x-b)^k + I(m, n). \end{aligned}$$

In particular, setting $m=n$, we get the expansion,

$$(5) \qquad \begin{aligned} f(x) = \sum_{k=0}^{n-1} \frac{1}{k!} \{ &\phi^n g_{n-k-1}^n(\theta) f^k(a)(x-a)^k \\ &+ \theta^n g_{n-k-1}^n(\phi) f^k(b)(x-b)^k \} + I(n, n), \end{aligned}$$

which as $n \to \infty$ gives an infinite series whose region of convergence is the interior of a Cassini oval.

As a further particular case we may extend the usual interpolation formula to give an approximation for $f(x)$ in terms of its values and the values of its first derivative at the points $x=a$, $x=b$, $x=c$. The appropriate contour integral will clearly be

$$(6) \qquad I = \frac{1}{2\pi i} \int_C \frac{(x-a)^2(x-b)^2(x-c)^2}{(t-a)^2(t-b)^2(t-c)^2} \frac{f(t)}{(t-x)} dt.$$

The residues at the points $t=x, a, b, c$, are readily calculated, and quickly yield the formula,

$$(7) \qquad \begin{aligned} f(x) = \sum \frac{(x-b)^2(x-c)^2}{(a-b)^2(a-c)^2} \Big[&f(a) + (x-a)f'(a) \\ &+ 2\left\{\frac{a-x}{a-b} + \frac{a-x}{a-c}\right\} f(a) \Big] + I, \end{aligned}$$

in which the summation is taken over cyclic permutations of the letters a, b, c.

The complete generalization is now clear; for each point $x=a$, at which the expansion is to involve $f(a)$ and its first $m-1$ derivatives, we must place a factor $(t-a)^m$ in the denominator, and a corresponding factor $(x-a)^m$ in the numerator, of the integrand of the appropriate contour integral.

SYSTEMS OF DISTINCT REPRESENTATIVES

H. B. MANN and H. J. RYSER, Ohio State University*

1. Introduction. A certain dance is attended by n boys and n girls. Each boy has been previously introduced to exactly j girls, and each girl has been previously introduced to exactly k boys. Here k is necessarily an integer such that $1 \leqq k \leqq n$. No one desires to make any further introductions. Under these assumptions, will it always be possible for the boys and the girls to be paired with each other in such a way that no further introductions are necessary? The dance problem proposed here is actually one of considerable combinatorial significance. The purpose of our present paper is to analyze carefully a very general theorem of Philip Hall, which solves not only the dance problem, but a wide variety of related problems as well. The paper is in part expository, with certain simplifications in proof. However, Theorems 2.4 and 2.5 are new and give further insight into this type of combinatorial problem.

2. Systems of distinct representatives. Let $S_1, S_2, \cdots, S_n$ denote n subsets of a given set M. Let D be a set of n elements of M,

$$D = \{a_1, a_2, \cdots, a_n\},$$

such that the elements a_i are all distinct and such that a_i belongs to S_i for each $i=1, \cdots, n$. Then D is said to be a system of distinct representatives for the subsets $S_1, S_2, \cdots, S_n$, (abbreviated by S.D.R.). Thus, for example, if our elements are integers and if $S_1=\{1, 2, 3\}$, $S_2=\{1, 2\}$, $S_3=\{2, 3\}$, then a suitable D for S_1, S_2, S_3 is $D=\{1, 2, 3\}$. If the sets $S_1, S_2, \cdots, S_n$ have a S.D.R., then it is clear that any k of the sets must contain between them at least k elements. The converse proposition is the combinatorial theorem of Philip Hall [4]. Various elementary proofs of the P. Hall theorem are available (see for example, [1], [4], [5]). Theorem 2.1 which follows is actually a refinement of the P. Hall theorem and gives a lower bound for the number of S.D.R.'s. This bound was first obtained by M. Hall [3]. The proof of Theorem 2.1 given here is essentially the same as the Halmos and Vaughan proof of the P. Hall theorem [5]. Finally, it should be remarked that important extensions of P. Hall's theorem to the case in which the number n of the sets S_i is transfinite have been recently obtained by Everett and Whaples [1] and M .Hall [3]. Halmos and Vaughan [5] have also given an alternative proof of the transfinite form of the P. Hall theorem. We shall not be concerned with these generalizations here, but confine ourselves to the case in which n is finite. No restrictions, however, are imposed on the number of elements in any S_i, nor are any two of the S_i's necessarily even assumed to be distinct.

THEOREM 2.1. *Let $S_1, S_2, \cdots, S_n$ denote n subsets of a set M. For each $k=1, 2, \cdots, n$, suppose that every k of these sets contain between them at least k distinct elements of M. Then there exists a* S.D.R. *for these subsets. Moreover, let r*

* The authors are indebted to the referee for his helpful suggestions.

be a fixed integer less than or equal to the minimal number of elements in each S_i. *Then if* $n \geq r$, *there are at least* $r!$ S.D.R.'s. *If* $n < r$, *then there are at least* $r!/(r-n)!$ S.D.R.'s.

The proof is by induction on n. For $n=1$, the result is trivial. Let $n>1$, and suppose that for each k, $1 \leq k < n$, every set of k S's contains at least $k+1$ distinct elements. Take the set S_1 and select from it any representative a_1. There are at least r choices for a_1. Now form the sets

$$\bar{S}_2 = S_2 - \{a_1\}, \cdots, \bar{S}_n = S_n - \{a_1\}.$$

Any k of the sets $\bar{S}_i$ contain between them at least k elements. By the induction hypothesis, if $r \leq n$, then $r-1 \leq n-1$, and there are at least $(r-1)!$ S.D.R.'s for the $\bar{S}_i$. If $n < r$, then $n-1 < r-1$, and then there are at least $(r-1)!/(r-n)!$ S.D.R.'s. Hence there are at least $r!$ or $r!/(r-n)!$ S.D.R.'s.

Suppose then that for some k, $1 \leq k < n$, there are k S's, say $S_1, \cdots, S_k$, which contain exactly k distinct elements. Here we must have $k \geq r$, and by the induction hypothesis, these sets have at least $r!$ S.D.R.'s. Let $T = \{a_1, \cdots, a_k\}$ be any such S.D.R. Now form the sets

$$\bar{S}_{k+1} = S_{k+1} - T, \cdots, \bar{S}_n = S_n - T.$$

Any h of the $\bar{S}$ sets, $1 \leq h \leq n-k$, cannot contain fewer than h elements. For otherwise the S sets corresponding to these $\bar{S}$ sets along with $S_1, \cdots, S_k$ would contain fewer than $h+k$ elements. An application of the induction hypothesis to the $n-k$ sets $\bar{S}$ completes the proof.

P. Hall has also established the following important consequence of Theorem 2.1.

THEOREM 2.2. *Let* $M = A_1 + \cdots + A_n$ *be a partition of the set* M *into subsets* $A_1, \cdots, A_n$ *where* $A_i \neq 0$ *and** $A_i \cap A_j = 0$ *for all* $i \neq j$. *Let* $B_1, \cdots, B_m$ *be* m *nonempty subsets of* M. *Suppose that for each* k, $1 \leq k \leq m$, *any* k *of the* A*'s contain at most* k *of the* B*'s. Then upon suitably renumbering* m *of the* A*'s,* $A_i \cap B_i \neq 0$ *for* $i = 1, \cdots, m$.

Let S_i be the set of all A_j for which $A_j \cap B_i$ is not null. We assert that any k S's contain between them at least k A's. For if not, then k S's, say $S_1, \cdots, S_k$, contain at most $k-1$ A's. Thus the number of A's each of which intersects at least one of the sets $B_1, \cdots, B_k$ is at most $k-1$. But then the sets $B_1, \cdots, B_k$ are contained in the union of these A's, contrary to hypothesis. Hence by the preceding theorem, there exists a S.D.R. for $S_1, \cdots, S_m$ taken notationally as $A_1, \cdots, A_m$ such that $A_i \cap B_i \neq 0$ for $i = 1, \cdots, m$.

The theorem which follows is a special case of Theorem 2.2, and was first established by D. König [6].

---

* The notation $A_i \cap A_j$ denotes the set of all elements in both A_i and A_j.

THEOREM 2.3. *Let M be a nonempty set with $M = A_1 + \cdots + A_n = B_1 + \cdots + B_n$, where $A_i \cap A_j = B_i \cap B_j = 0$ for $i \neq j$ and where each A_i and B_j contains exactly m elements. Then the A's may be renumbered so that $A_i \cap B_i \neq 0$.*

The following two theorems describe circumstances under which certain specified elements may be selected as part of a S.D.R.

THEOREM 2.4. *Suppose that the sets $S_1, \cdots, S_n$ satisfy the conditions of Theorem 2.1 and suppose further that the elements $e_1, \cdots, e_k$ each occur among the sets $S_1, \cdots, S_n$ at least t times while each of the S_i contains at most t of the elements $e_1, \cdots, e_k$. Then there exists a S.D.R. containing $e_1, \cdots, e_k$.*

We shall call $e_1, \cdots, e_k$ *marginal elements.*

Let Σ be a S.D.R. which contains the highest possible number of marginal elements. We shall show that Σ contains all marginal elements. Suppose to the contrary that Σ contains exactly $e < k$ marginal elements.

The ordered set of sets $S_{j_1}, S_{j_2}, \cdots, S_{j_\alpha}$ will be called a *chain* if for $1 \leqq \beta < \alpha$ the representative in Σ of S_{j_β} occurs in $S_{j_{\beta+1}}$. Note that by this definition one set S_j is a chain.

LEMMA. *Let i be a marginal element which does not occur in Σ. If i is in S_j then there is a chain connecting S_j with a set with a non-marginal representative.*

Deny this and consider all sets $S_{i_1}, \cdots, S_{i_\alpha}$ connected with S_j by a chain. Let $j_1, \cdots, j_\alpha$ be their representatives in Σ. Then $j_1, \cdots, j_\alpha$ are marginal elements and each occurs in the sets $S_{i_1}, \cdots, S_{i_\alpha}$ at least t times, for each set among $S_1, \cdots, S_n$ containing j_β is connected with S_{i_β}. Thus the sets $S_{i_1}, \cdots, S_{i_\alpha}$ contain together at least $\alpha t + 1$ marginal elements whereas by assumption they cannot contain more than αt marginal elements. This proves the lemma.

Now let $S_j = S_{i_1}, S_{i_2}, \cdots, S_{i_\alpha}$ be the shortest chain such that S_{i_α} has a non-marginal representative β. Then $S_j, \cdots, S_{i_{\alpha-1}}$ have marginal representatives in Σ. We construct a system Σ' of representatives of $S_1, \cdots, S_n$ as follows. Let S_j be represented by i. Let S_{i_k} for $1 < k \leqq \alpha$ be represented by the representative of $S_{i_{k-1}}$ in Σ. Let all other sets be represented by their representatives in Σ. Clearly Σ' is a S.D.R. and contains $e+1$ marginal elements contradicting the significance of e and Σ.

THEOREM 2.5. *Let $S_1, \cdots, S_n$ be sets satisfying the conditions of Theorem 2.1 and assume that $e_1, \cdots, e_k$ each occur among the sets $S_1, \cdots, S_n$ at least t times, while each of the sets S_i contains at most t elements. Let α be any element. Then there exists a S.D.R. containing $\alpha, e_1, \cdots, e_k$.*

The proof of Theorem 2.5 is analogous to that of Theorem 2.4. We start with a S.D.R. containing $e_1, \cdots, e_k$ and let α play the role of i in the preceding proof.

3. Applications. We will now consider the applications of the preceding combinatorial theorems to various mathematical investigations. The first application is due to König, and concerns matrices whose elements are composed

only of the integers 0 and 1 [6]. A matrix with a single entry of 1 in each row and in each column and with all other entries 0 is called a permutation matrix.

THEOREM 3.1. *Let A be an n by n matrix of zeros and ones, with exactly k ones in each row and in each column. Then $A = L_1 + \cdots + L_k$, where the L_i are permutation matrices.*

Let the ith row of A have ones in columns $i_1, i_2, \cdots, i_k$, and zeros elsewhere. In this way, we may form n sets $S_i = \{i_1, i_2, \cdots, i_k\}$, each containing k integers from among the integers $1, 2, \cdots, n$. Any r of these sets contain between them at least r distinct elements. For if r sets contained at most $r-1$ elements, then in the corresponding rows of the matrix A there would be at most $k(r-1)$ ones, whereas there are kr ones. Thus the sets S_i fulfill the hypothesis of Theorem 2.1, and a S.D.R. gives rise to a permutation matrix L_1. We may proceed similarly with the matrix $A - L_1$, and Theorem 3.1 then follows by induction. Incidentally, Theorem 3.1 gives a solution to the dance problem described in the introduction. The permutation matrix L_1 gives a desired pairing of previously introduced boys and girls.

Another application of interest arises naturally in the study of symmetrical balanced incomplete block designs. A complete description of such designs may be found in [7]. These designs may be converted into Youden squares [10], and this fact is also a direct consequence of Theorem 2.1.

The next application which we will consider concerns the cosets of a finite group.

THEOREM 3.2. *Let G be a finite group and let H and K be two subgroups of order m. Let $G = Hx_1 + \cdots + Hx_r$ be a right coset decomposition for H and let $G = y_1K + \cdots + y_rK$ be a left coset decomposition for K. Then it is possible to choose $x_1, \cdots, x_r$ so that $G = Hx_1 + \cdots + Hx_r = x_1K + \cdots + x_rK$.*

From the elementary properties of cosets, it is clear that Theorem 3.2 is a direct consequence of Theorem 2.3. The theorem is frequently stated for the case in which $H = K$. Thus the left and right cosets of a subgroup H have a common system of representatives.

Another group-theoretic application of interest was developed by R. Rado, and is specifically concerned with the symmetric group of order $n!$ [8].

The final application which we will consider concerns the subject of latin rectangles. A latin rectangle of order r by s based upon the integers $1, 2, \cdots, n$ is defined as an array of r rows and s columns formed from the integers $1, 2, \cdots, n$ in such a way that the integers in each row and in each column are distinct. The latin rectangle is said to be extendible to an n by n latin square provided that it is possible to adjoin $n-s$ columns and $n-r$ rows in such a way that the resulting array is an n by n latin square. The following theorem concerning the extendibility of latin rectangles to n by n latin squares was first derived for the case of r by n latin rectangles [2] and then extended to the case of r by s latin rectangles [9].

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

ON AN INEQUALITY DUE TO WEINBERGER

RICHARD BELLMAN, The Rand Corporation

In a recent issue of the Proceedings of the National Academy of Sciences, July, 1942, H. F. Weinberger proved the following interesting inequality

$$\sum_{n=1}^{k} (-1)^{n+1} a_n^r \geqq \left(\sum_{n=1}^{k} (-1)^{n+1} a_n \right)^r \tag{1}$$

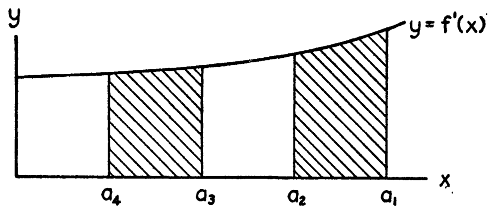
provided that $r \geqq 1$ and

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0. \tag{2}$$

The purpose of this note is to present a generalization of (1), namely

$$\sum_{n=1}^{k} (-1)^{n+1} f(a_n) \geqq f\left(\sum_{n=1}^{k} (-1)^{n+1} a_n \right) \tag{3}$$

which holds provided that (2) is valid and that $f(0), f'(0) \geqq 0$, and $f'(x)$ is monotone increasing. The proof is immediate upon referring to the graph



and comparing areas. It is clear that equality results only if $a_2 = a_3$, $a_4 = a_5$, *etc.*

ON THE CONVERGENCE OF ALTERNATING DOUBLE SERIES

BURNETT MEYER, University of Arizona

1. Introduction. Several years ago a new definition of convergence of double series, σ-convergence, was proposed independently by Sheffer [4] and Amerio [1]. This new method provides a theory more nearly analogous to that of simple series than is obtained by Pringsheim convergence. Somewhat later Wilansky [5] compared σ-convergence with regular convergence and showed that the latter has most of the desirable properties of the former but that the two definitions are not equivalent, σ-convergence being the more stringent requirement. It is our purpose to give a necessary and sufficient condition for the σ-convergence of a monotonic alternating double series and to compare the two definitions

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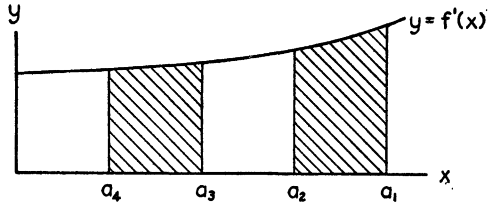

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Now let σ' and σ'' be any two σ-sums (p, p). Then

$$|\sigma' - \sigma''| \leqq |\sigma' - T_p| + |\sigma'' - T_p| < \epsilon/2 + \epsilon/2 = \epsilon,$$

and the series is σ-convergent.

4. An example. It is well-known that for simple series $\sum_{i=1}^{\infty}(-1)^{i+1}i^{-n}$ is convergent for $n>0$ and is absolutely convergent for $n>1$. The analogues of these results for regular and σ-convergence provide an interesting comparison of the two definitions.

We consider the series $\sum_{i,j=1}^{\infty}(-1)^{i+j}(i+j-1)^{-n}$. Since $\int_1^\infty\int_1^\infty (x+y-1)^{-n}\,dxdy$ converges for $n>2$, the above series is absolutely Pringsheim convergent for $n>2$. (See [2], p. 86.) It is shown in [4] that absolute Pringsheim convergence and absolute σ-convergence are equivalent.

Pringsheim has shown [3] that a monotonic alternating double series is Pringsheim convergent if

$$|a_{ij} + a_{i+1,j}| \geqq |a_{i,j+1} + a_{i+1,j+1}|$$

for all i and j. The series $\sum(-1)^{i+j}(i+j-1)^{-n}$ satisfies this condition for $n>0$, since $x^{-n}-(x+1)^{-n}$ is a decreasing function of x for $x\geqq 1$, $n>0$. Hence, the above double series is regularly convergent if $n>0$, and it is easily seen that the series diverges for $n\leqq 0$.

However, if we set $n=1$ and attempt to sum the series by diagonals, we obtain $1-1+1-1+\cdots$; so this regularly convergent series is not σ-convergent. The theorem of the last section shows that the series is σ-convergent if and only if $n>1$.

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A NOTE ON A THEOREM OF PARKER

G. W. MEDLIN, University of North Carolina*

Recently W. V. Parker [2] proved the following theorem:

Let A be an $n\times m$ matrix of rank $r<n$ and let C be an $m\times n$ matrix such that $ACA=kA$ (k a scalar). If B is an $m\times n$ matrix, the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and the characteristic equation of $A(B+C)$ is $x^{n-r}\phi(x-k)=0$.

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Now let σ' and σ'' be any two σ-sums (p, p). Then

$$\left|\sigma' - \sigma''\right| \leqq \left|\sigma' - T_p\right| + \left|\sigma'' - T_p\right| < \epsilon/2 + \epsilon/2 = \epsilon,$$

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It may be remarked that this result can be generalized somewhat in the same manner that W. T. Reid [3] generalized an earlier theorem of Parker [1].

THEOREM 1. *Let A be an $n\times m$ matrix of rank r and D an $n\times n$ matrix such that $DA=kA$. If B is an $m\times n$ matrix, then the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and the characteristic equation of $AB+D$ is $g(x)\phi(x-k)=0$, where $g(x)$ is a polynomial of degree $n-r$.*

Proof. There exist nonsingular matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

If we set

$$PDP^{-1} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

we get

$$PDAQ = PDP^{-1}PAQ = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ D_3 & 0 \end{pmatrix}. \tag{1}$$

But $DA=kA$, hence

$$PDAQ = kPAQ = \begin{pmatrix} kI_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

From (1) and (2) we get

$$D_1 = kI_r \quad \text{and} \quad D_3 = 0;$$

hence

$$PDP^{-1} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} = \begin{pmatrix} kI_r & D_2 \\ 0 & D_4 \end{pmatrix}. \tag{3}$$

Now we apply the similarity transformation P to AB and $AB+D$ and set

$$Q^{-1}BP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

Now

$$PABP^{-1} = PAQQ^{-1}BP^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} \tag{4}$$

and by (3) and (4)

$$(5)\qquad \begin{aligned} P(AB + D)P^{-1} = PABP^{-1} &= \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} kI_r & D_2 \\ 0 & D_4 \end{pmatrix} \\ &= \begin{pmatrix} B_1 + kI_r & B_2 + D_2 \\ 0 & D_4 \end{pmatrix}. \end{aligned}$$

From (4) and (5) we conclude that the characteristic equation of AB is $x^{n-r}\phi(x) = 0$ and that of $AB+D$ is $g(x)\phi(x-k) = 0$ where $\phi(x) = 0$ is the characteristic equation of B_1 and $g(x) = 0$ the characteristic equation of D_4.

COROLLARY 1. *If D is of rank r, and $k \neq 0$, then $g(x) = x^{n-r}$.*

Proof. From (3) we see that $D_4 = 0$ since the rank of D is equal to the sum of the ranks of kI_r and D_4 and the rank of kI_r is r. From (5) we see that

$$g(x) = x^{n-r}.$$

COROLLARY 2. *If A is nonsingular, then $D = kI$.*

In the original form Reid's theorem cannot be generalized. His result shows up in

COROLLARY 3. *If D is nilpotent and $DA = kA$, then $k = 0$.*

Proof. D is nilpotent if and only if PDP^{-1} is. But from (3) we see that PDP^{-1} is reduced; hence each main diagonal factor must be nilpotent. This implies

$$(kI_r)^t = 0,$$

which is possible only if $k = 0$.

Reid's result may be obtained by observing that D_4 is nilpotent; hence it has as characteristic equation $x^{n-r} = 0$.

By exactly the same method Parker [2] used we may prove:

THEOREM 2: *If $D^2 = D$ and $k \neq 0$ (we may assume it to be 1), then the minimum functions of AB, say $g(x)$, and $AB+D$, say $h(x)$, must satisfy one of the four relations*

$$h(x) = g(x-1) \qquad\qquad h(x) = xg(x-1)$$
$$(x-1)h(x) = g(x-1) \qquad\qquad (x-1)h(x) = xg(x-1).$$

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THE NUMBER OF CYCLES ASSOCIATED WITH THE ELEMENTS OF A PERMUTATION GROUP

R. E. GREENWOOD, The University of Texas

Gontcharoff [1] has given several moment generating functions for the number of cycles associated with the elements of the permutation group on N symbols. Recently the author rediscovered the factorial moment generating function for the above problem previously given by Gontcharoff. Since the method of derivation is quite different from that given by Gontcharoff, it may be worth reproducing here.

For each positive integer N, we let G_N denote the group of $N!$ permutations on N symbols (which we may refer to as being cards numbered from 1 to N for convenience). As is well known, each of these permutations may be written as a product of disjoint cycles of lengths $t_1, t_2, \cdots, t_m$ say, and these lengths will satisfy the relation $\sum t_i = N$. We call the permutation $(1, 2, \cdots, N)$ the standard permutation.

The factorial moment generating function found by the author is based on the form

$$M_N = (x + N - 1)^{(N)} \tag{1}$$

where

$$y^{(n)} = y(y-1)\cdots(y-n+1). \tag{2}$$

The form (1) was found by empirical methods; the coefficient of x^p in the expansion of M_N is the number of permutations on N symbols (or elements of G_N) which have p cycles associated with them. This may be trivially verified for $N=1$, for then $M_1 = x$, and there is one cycle with unit length.

The correctness of (1) for the general case may be proved by induction. Assume that $(K+1)$ is the smallest positive integer N for which form (1) fails to provide a correct description of the number of cycles according to the above conventions. Then, by this hypothesis, M_K does provide a correct description for the elements of G_K.

Consider the groups G_{K+1} and G_K. For each of the fixed positions $1, 2, 3, \cdots, K+1$, there are $K!$ elements of G_{K+1} with card $(K+1)$ in that fixed position. We consider the effect of eliminating the card numbered $(K+1)$.

If the card numbered $(K+1)$ is in its standard position, then it of itself constitutes a cycle of length one. The deletion of this card reduces the number of cycles by unity. The remaining K cards are so distributed as to form an element of G_K. Since the form M_K describes the group G_K, the form xM_K (where the factor x adds one to the number of cycles associated with each element of the group G_K) describes the subset of G_{K+1} which has the card $(K+1)$ in its standard position.

If the card numbered $(K+1)$ is not in its standard position, we delete this card and think of the card in position $(K+1)$ as being moved to the position of

$$\text{Variance} = \frac{1}{N!}\left[\frac{d^2M_N}{dx^2}\right]_{x=1} + \mu_1 - (\mu_1)^2$$
$$= 1 + 1/2 + 1/3 + \cdots + 1/N - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}\right).$$

For large N,

$$\text{Variance} \simeq \log N + C - \frac{\pi^2}{6}\cdot$$

Statistics involving higher order moments are obtainable from general formulas associated with the factorial moment generating function [2].

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CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All materials for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ON BLISS' SUBSTITUTE FOR DUHAMEL'S PRINCIPLE

ARTHUR ROSENTHAL, Purdue University

The so-called principle of Duhamel has been criticized and revised by W. F. Osgood [1] and R. L. Moore [2]. Then G. A. Bliss [3] replaced it by a much simpler theorem:

Let F be a continuous function in the closed, bounded, Jordan measurable region V of a Euclidean m-dimensional space. If V is divided into finitely many disjoint, Jordan measurable subregions with maximum diameter less than δ and with volumes denoted by ΔV_k $(k=1, 2, \cdots, n)$ and if in each of these subregions three points p_k, p_k', p_k'' are chosen, then

$$\lim_{\delta\to 0}\sum_{k=1}^{n} F(p_k, p_k', p_k'')\cdot\Delta V_k = \lim_{\delta\to 0}\sum_{k=1}^{n} F(p_k, p_k, p_k)\cdot\Delta V_k = \int_V F dV. \tag{1}$$

The analogue holds if the number of points selected in each subregion were two or more than three.

By using the property of uniform continuity of F in V, the proof of Bliss'

$$\text{Variance} = \frac{1}{N!}\left[\frac{d^2M_N}{dx^2}\right]_{x=1} + \mu_1 - (\mu_1)^2$$
$$= 1 + 1/2 + 1/3 + \cdots + 1/N - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}\right).$$

For large N,

$$\text{Variance} \simeq \log N + C - \frac{\pi^2}{6}\cdot$$

Statistics involving higher order moments are obtainable from general formulas associated with the factorial moment generating function [2].

References

1. V. Gontcharoff, On the field of combinatory analysis, Bulletin de l'Academie des Sciences de l'U.R.S.S., Série Mathématique, vol. 8, no. 1, 1944, p. 1–43.

2. M. G. Kendall, The Advanced Theory of Statistics, vol. i, 4th edition, London, 1948, pp. 56–58.

CLASSROOM NOTES

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hence the last sum on the right hand side of (3) does not occur.

The generalization to Riemann-Stieltjes integrals is immediate.

References

1. W. F. Osgood, Annals of Math., 2, vol. 4, 1903, pp. 161–178.

2. R. L. Moore, Annals of Math., 2, vol. 13, 1912, pp. 161–166. Another formulation of a similar type is stated, *e.g.*, in G. E. F. Sherwood and A. E. Taylor, Calculus (revised edition), New York, 1946, pp. 317–318.

3. G. A. Bliss, A substitute for Duhamel's theorem, Annals of Math., 2, vol. 16, 1914–15, pp. 45–49.

4. See H. B. Fine, Calculus, New York, 1927, pp. 163–164; G. E. F. Sherwood and A. E. Taylor, *loc. cit.* [2], Corollary, pp. 318 and 532; L. L. Smail, Calculus, New York, 1949, pp. 129–130; and other textbooks on Calculus. L. L. Smail, *loc. cit.*, called this case the "theorem of Bliss."

ON A NOWHERE-DENSE SET

ALBERT WILANSKY, Lehigh University

Early in the introduction of Lebesgue measure the natural question arises whether a nowhere-dense set must have measure 0, even whether it must be countable. Cantor's ternary set answers the second question in the negative, and may be modified to answer the first.

If the lecturer does not wish to interrupt the continuity of his presentation he may present the following brief example:

Let I be the closed interval $[0, 1]$. Fix a number t, $0<t<1$. Let $G(t)$ be an open set of measure less than or equal to t, including the set of rational numbers in I. (One has proved, earlier, that a countable set, and, in particular, the set of rational numbers, has measure 0.) Let F be the complement of $G(t)$ with respect to I.

THEOREM. *F is a closed, nowhere-dense (non-countable) set of measure greater than or equal to $1-t$.*

Thus F has some of the properties of the Cantor set. (If one stops to prove that every set of positive measure has a non-measurable subset, this yields the existence of a nowhere-dense non-measurable set.) Notice that the measure of F may be arbitrarily near 1. Notice also that the complement of $\bigcap_{n=1}^{\infty} G(1/n)$ is the well-known example of a set of first category which has measure 1.

AN ELEMENTARY DERIVATION OF THE FORMULAE FOR P_m AND $P_{[m]}$

J. E. FREUND, Alfred University

1. Introduction. The formulae for the probabilities of the realization of m (or at least m) among N events are usually presented by assuming the results and demonstrating that the proper points of the sample space contribute to the expressions while all others do not.* Since this method of proof is a verification

* W. Feller, An Introduction to Probability Theory and Its Applications, New York, 1950, p. 61.

hence the last sum on the right hand side of (3) does not occur.

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* W. Feller, An Introduction to Probability Theory and Its Applications, New York, 1950, p. 61.

Using the well-known identity

$$\binom{m-1+i}{i}+\binom{m-1+i}{i-1}=\binom{m+i}{i},$$

this can also be written as

$$P_{[m]} = S_m + \sum_{i=1}^{N-m} (-1)^i \binom{m+i}{i} S_{m+i}$$

and the formula for $P_{[m]}$, the probability of the realization of exactly m among N events, can, finally, be written as

$$P_{[m]} = \sum_{i=0}^{N-m} (-1)^i \binom{m+i}{i} S_{m+i}. \tag{8}$$

A PROOF OF TAYLOR'S FORMULA

JAMES WOLFE, University of Utah

The following proof of Taylor's formula with remainder may seem more natural than the proofs ordinarily offered in calculus texts.

Suppose $f(x)$ has a continuous $(n-1)$st derivative in the closed interval between a and b and an nth derivative in the open interval between a and b. Then $f(x)$ can be approximated by a polynomial $p(x)$ of degree n which agrees in value with $f(x)$ at a and b and such that the first $n-1$ derivatives of p and f agree at a:

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + k(x-a)^n,$$

where k is determined so that $p(b) = f(b)$. Let $g(x) = f(x) - p(x)$. Then $g(x)$ and its first $n-1$ derivatives vanish at $x=a$ and also $g(b)=0$. Using successive applications of Rolle's theorem, $g'(x_1)=0$ for some x_1 between a and b, $g''(x_2)=0$ for some x_2 between a and x_1, *etc.* and finally $g^{(n)}(x_n)=0$ for some x_n between a and x_{n-1}. But $g^{(n)}(x) = f^{(n)}(x) - n!k$, consequently $k = f^{(n)}(x_n)/n!$ and

$$f(b) = p(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(x_n)}{n!}(b-a)^n.$$

For $n=1$, this proof reduces to the usual proof of the mean value theorem.

1) If x_1 and x_2 are different integers and $[y_1]=[y_2]$, then
$d(P_1, P_2)=|x_1-x_2|+\min(y_1'+y_2', 2-y_1'-y_2')$;
2) if y_1 and y_2 are different integers and $[x_1]=[x_2]$, then
$d(P_1, P_2)=|y_1-y_2|+\min(x_1'+x_2', 2-x_1'-x_2')$.

We wish to describe $C(P, r)$ in case P is not a corner. Let P_1 and P_2 be the corners adjacent to P. Then, if $d(P, P_1)=k$, the set of points $C(P, r)$ is contained in the sum of the sets $C(P_1, r-k)$ and $C(P_2, r+k-1)$. For any corner Q, we denote the interior of the square $C(Q, s)$ as $D(Q, s)$. The circle of radius r about a point P, not a corner, is the intersection of the grid of streets with the octagon $D(P_1, r-k)+D(P_2, r+k-1)$. We see that $C(P, r)$ is not convex.

The accompanying figure illustrates the case $P=(0, 0.6)$, $P_1=(0, 1)$, $P_2=(0, 0)$ and $r=1.8$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 1071. *Proposed by Albert Wilansky, Lehigh University*

If $f(x)=\int_0^x \cos(1/t)dt$, show that $f'(0)=0$.

E 1072. *Proposed by S. J. Jasper, East Tennessee State College*

Find a four digit number in base ten which, when the order of the digits is reversed, becomes an equivalent number in base seven.

E 1073. *Proposed by G. W. Walker, Buffalo, N. Y.*

A polygonal spiral $A_1A_2A_3 \cdots$ of unit segments winds counterclockwise and is constructed in the following manner. Point A_1 is at the origin, point A_2 is at $(1, 0)$, $\measuredangle A_{n-1}A_nA_{n+1}=2\pi/n$ for all $n \geqq 2$. Is there a point lying within the interior of each $\measuredangle A_{n-1}A_nA_{n+1}$? If, so, what are its coordinates?

E 1074. *Proposed by Vern Hoggatt, Oregon State College*

A regular n-gon with center O has a particle of mass m at each vertex. Let PO be a segment perpendicular to the plane of the n-gon and let l be a line

through P parallel to the plane of the n-gon. Show that the sum of the moments of inertia of the n masses about the line l is independent of the (restricted) orientation of l.

E 1075. *Proposed by H. S. Shapiro, Chatham, N. J.*

A sequence of N positive integers contains precisely n distinct numbers. If If $N \geqq 2^n$, show that it is possible to find a block of consecutive terms whose product is a square, and that this result need not hold if $N < 2^n$.

SOLUTIONS

Anagrams

E 1041 [1952, 696]. *Proposed by SHE SLID and O PARISH, Massachusetts Institute of Technology*

Surely the days of anagrams are not dead. Here are four well known mathematicians: (1) A DONUT SHIP, (2) SHE IS A NUT, (3) SEWER STAIRS, (4) FIRE ON SUB.

I. *U TOIL SON by GREW LIGHT SCAR, SOGGY OCEAN EELS LIT CELL.*[1] (1) THUDS PIANO was the father of the RAG BALE.[2] He reveled in indeterminate equations, so appropriately enough THUDS PIANO = A TOP HID SUN = I PUT SHAD ON = DOAN PUSH IT = PAN THIS DUO = A DONUT SHIP = DIOPHANTUS.

(2) SHE IS A NUT who produced THE LAST HOT MAMA'S PANICS.[3] Furthermore, USES A HINT = TIN HAS USE = SHUNS A TIE = THE USA SIN = SHE IS A NUT = STEINHAUS.

(3) WASTERS RISE and SLAY A SIN.[4] Hence we have WASTERS RISE = SERIES WARTS = SWEARS RITES = SEWER STAIRS = WEIERSTRASS.

(4) I BURN FOES = FIE ON BURS = I SOBER FUN = O FINE RUBS = FIRE ON SUB = FROBENIUS made an extensive study of CRUMBS NEAR A BILGE[5] where LAME-BRAIN CURS BEG,[5] BLUE MAN GRABS RICE,[5] and BARE GAL BURNS MICE,[5] A SLUMBERING BRACE.[5]

We also breathe new life into: (5) WENT ON, (6) LARCH STY, (7) ROTTED HUN, (8) REC'D ROE, (9) IV TENS, (10) KIND DEED, (11) THOU DREG, (12) CANE DOG, (13) SEE GUARDS, (14) PELT ONCE, (15) LIE CRUEL APE, (16) BAN ON RICH, (17) A FREE BUCH, (18) RICH CASE, (19) RAM NINE, (20) SING A TUNE, MOOR, (21) A POO IS NULL, (22) I REMOVED, (23) MAR CONE, (24) TY'S REVELS, (25) DEER GLEN, (26) CALL APE, (27) I RIVAL ACE, (28) HIM NOT AL, (29) SET CEDARS, (30) TIRE CHILD, (31) HARM A DAD, (32) I CHARM ALEC, (33) A CURT NO, (34) LEAN MUSE, (35) O NULL BIER, (36) REACH DIMES, (37) HI, SAY AH, (38) DRAB ROC, (39) ONE LIME, (40) CABIN COIF. Finally, we end AVEC CEVA, a pretty palindrome.

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through P parallel to the plane of the n-gon. Show that the sum of the moments of inertia of the n masses about the line l is independent of the (restricted) orientation of l.

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[3] MATHEMATICAL SNAPSHOTS, [4] ANALYSIS, [5] ALGEBRAIC NUMBERS. (5) NEWTON, (6) CHRYSTAL, (7) TODHUNTER, (8) RECORDE, (9) STEVIN, (10) DEDEKIND, (11) OUGHTRED, (12) D'OCAGNE, (13) DESARGUES, (14) PONCELET, (15) PEAUCELLIER, (16) BRIANCHON, (17) FEUERBACH, (18) SACCHERI, (19) RIEMANN, (20) REGIOMONTANUS, (21) APOLLONIUS, (22) DE MOIVRE, (23) CREMONA, (24) SYLVESTER, (25) LEGENDRE, (26) LAPLACE, (27) CAVALIERI, (28) HAMILTON, (29) DESCARTES, (30) DIRICHLET, (31) HADAMARD, (32) CARMICHAEL, (33) COURANT, (34) MENELAUS, (35) BERNOULLI, (36) ARCHIMEDES, (37) HAYASHI, (38) BROCARD, (39) LEMOINE, (40) FIBONACCI

II. *Remarks by CHAS. PULL SINE,*[1] *Cooper Union.* As for the days of anagrams being dead, locate the names of 21 mathematicians of note in the following:

No gem in this or an epic either, yet as she said, "Here Tim, lend me my T-pole," Ess traced a regal nag. "Nice corpus," said lamed Bert, a critic, "you enthuze me—I am on ether, a piner for your recipe." "To be able to do a kind deed is bred in us," Ess argued, "but I lag so." "Ach, you tire, child. Put it down."

Dead indeed!

[1] PAUL L. CHESSIN, no gem = MONGE, or an epic = POINCARE, as she = HASSE, here Tim = HERMITE, lend me = MENDEL, my T-pole = PTOLEMY, Ess traced = DESCARTES, regal nag = LAGRANGE, nice corpus = COPERNICUS, lamed Bert = DALEMBERT, a critic = RICCATI, enthuze = ZEUTHEN, on ether = NOETHER, a piner = NAPIER, recipe = PEIRCE, able = ABEL, kind deed = DEDEKIND, bred in us = BURNSIDE, Ess argued = DESARGUES, I lag so = GALOIS, tire child = DIRICHLET.

III. *Addenda by H. W. Gould, Portsmouth, Va.* How about (1) YOUR SON IS OLD, (2) READ CHIMES, (3) AM SURE NOT GO IN, (4) HE TENSOR EAST, (5) CABIN FOCI, (6) ARYAN CASH A HARK, (7) RUN A CLAIM, (8) WHO IZ KALAMIR and (9) GO PAST HYAR?

(1) DIONYSODORUS, (2) ARCHIMEDES, (3) REGIOMONTANUS, (4) ERATOSTHENES, (5) FIBONACCI, (6) SHANKARACHARYA, (7) MACLAURIN, (8) AL-KHOWARIZMI, (9) PYTHAGORAS.

Also solved by Raphaël Aronson, G. A. Baker, Jr., Leon Bankoff (BONE OF FLANK), T. A. Bickerstaff (STIFFER BACK), Julian Braun (A BURN), W. E. Briggs, J. E. Darraugh, William Douglas, D. M. Dribin, R. P. Eisinger (I IS GREEN), L. R. Ford (DR. OF), J. W. Forman, Louise Gartner, Bennington Gill (ENNOBLING GLINT), J. D. Haggard (HARD GAG), R. Huck, Meyer Jerison, J. H. Means, R. A. Miller (MLLE RI), George Millman, B. E. Mitchell (LITHE CLEM), Leo Moser, Hyman Orlin, A. F. Payton, L. L. Scott (ST. COT), William Small (MALLS), H. S. Wilf, R. H. Wilson, Jr., and the proposers (W. H. Shields and H. S. Shapiro). Late solutions by A. R. Hyde and J. V. Whittaker.

by D. F. Lawden, "On the solution of linear difference equations," *Mathematical Gazette* [1952, 193–196].

Property of Three Concurrent Cevians

E 1043 [1952, 697]. *Proposed by O. J. Ramler, Catholic University, Washington, D. C.*

Prove that the sum of the ratios in which a point within a triangle divides the Cevians of this point is never less than 6 and that the product of the ratios is never less than 8.

Solution by C. W. Trigg, Los Angeles City College. The identity $x+1/x\equiv 2+(x-1)^2/x$ shows that the sum of a positive number and its reciprocal is always $\geqq 2$. Now if the Cevians divide the sides of the triangle in the ratios k, m, n then $kmn=1$ by Ceva's Theorem. Furthermore, the corresponding Cevians are divided by their common point in the ratios $n+1/m$, $k+1/n$, $m+1/k$ (see, *e.g.*, Altshiller-Court, *College Geometry*, 2nd ed. (1952), p. 163, Th. 342). It follows that the sum of these latter ratios,

$$(k + 1/k) + (m + 1/m) + (n + 1/n) \geqq 6.$$

Also, the product of these ratios,

$$(kmn + 1/kmn) + (k + 1/k) + (m + 1/m) + (n + 1/n) \geqq 8.$$

Since $kmn=1$ we observe that the product of the three ratios is always equal to the sum of the three ratios increased by 2. We also note that the minimum values of the sum and product of the three ratios will be attained if and only if the three Cevians are the three medians of the triangle.

Also solved by Leon Bankoff, Fred Discepoli, Harry Furstenberg, Louisa Grinstein, J. D. Haggard, M. S. Klamkin, B. Martin, George Millman, Martin Moliver, T. F. Mulcrone, E. J. Musch, Chih-yi Wang, and the proposer.

A Trigonometrical Product

E 1044 [1952, 697]. *Proposed by J. E. Wilkins, Jr., Nuclear Development Associates, Inc.*

Find

$$\prod_{r=1}^{m-1} \left(2 \sin \frac{\pi r}{m}\right)^r.$$

Solution by J. A. Tierney, U. S. Naval Academy, Annapolis. Set

$$P = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi r}{m}\right)^r.$$

Then we also have

$$P = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi(m-r)}{m}\right)^{m-r} = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi r}{m}\right)^{m-r},$$

whence

$$P^2 = \left(\prod_{r=1}^{m-1} 2 \sin \frac{\pi r}{m}\right)^m.$$

But it is known that

$$\prod_{r=1}^{m-1} 2 \sin \frac{\pi r}{m} = m. \tag{1}$$

Therefore $P = m^{m/2}$.

Also solved by M. J. Antchogno and Milton Handel (jointly), G. A. Baker, Jr., Julian Braun, C. N. Campopiano, L. Carlitz, Harry Furstenberg, H. W. Gould, M. H. Hoehn, Vern Hoggatt, P. G. Kirmser, M. S. Klamkin, A. E. Livingston, William Small, M. R. Spiegel, O. E. Stanaitis, Chih-yi Wang, R. E. Wild, and the proposer. Late solutions by N. J. Fine, S. Parameswaren, and J. V. Whittaker.

The following references were cited where formula (1) may be found: Bromwich, *Theory of Infinite Series*, p. 211; Durell and Robson, *Advanced Trigonometry*, p. 223; Franklin, *Treatise on Advanced Calculus*, prob. 42, p. 582; Nagell, *Introduction to Number Theory*, p. 173; Todhunter, *Plane Trigonometry*, p. 259; Whittaker and Watson, *Modern Analysis*, p. 240.

Gauss's Generalization of Wilson's Theorem

E 1045 [1952, 697]. *Proposed by S. W. Golomb, Harvard University*

Let $n^{\phi}_{!}$ (pronounced "n-phi-torial") denote the product of all integers up to n which are prime to n. Then $n^{\phi}_{!} \equiv \pm 1 \pmod{n}$. Prove this, and determine when the value is $+1$ and when it is -1.

Solution by William Small, University of Rochester. The result, known as Gauss's generalization of Wilson's theorem, is given, with proof, in Oystein Ore, *Number Theory and Its History*, theorem 11-6, p. 266. In the congruence one has the negative sign when $n=4$, p^α, $2p^\alpha$, where p is an odd prime. In all other cases, one has the positive sign, except that for the case $n=2$ it is immaterial which sign is used.

Also solved by J. W. Baldwin, P. T. Bateman, W. E. Briggs, L. Carlitz, L. C. Labowitz, J. Lehner, A. E. Livingston, Leo Moser, J. V. Whittaker, and the proposer.

Other references cited are Hardy and Wright, *Theory of Numbers*, theorem 129, p. 102, and Nagell, *Introduction to Number Theory*, theorem 79, p. 100.

Then we also have

$$P = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi(m-r)}{m}\right)^{m-r} = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi r}{m}\right)^{m-r},$$

whence

$$P^2 = \left(\prod_{r=1}^{m-1} 2 \sin \frac{\pi r}{m}\right)^m.$$

But it is known that

$$(1) \qquad \prod_{r=1}^{m-1} 2 \sin \frac{\pi r}{m} = m.$$

Therefore $P = m^{m/2}$.

Also solved by M. J. Antchogno and Milton Handel (jointly), G. A. Baker, Jr., Julian Braun, C. N. Campopiano, L. Carlitz, Harry Furstenberg, H. W. Gould, M. H. Hoehn, Vern Hoggatt, P. G. Kirmser, M. S. Klamkin, A. E. Livingston, William Small, M. R. Spiegel, O. E. Stanaitis, Chih-yi Wang, R. E. Wild, and the proposer. Late solutions by N. J. Fine, S. Parameswaren, and J. V. Whittaker.

The following references were cited where formula (1) may be found: Bromwich, *Theory of Infinite Series*, p. 211; Durell and Robson, *Advanced Trigonometry*, p. 223; Franklin, *Treatise on Advanced Calculus*, prob. 42, p. 582; Nagell, *Introduction to Number Theory*, p. 173; Todhunter, *Plane Trigonometry*, p. 259; Whittaker and Watson, *Modern Analysis*, p. 240.

Gauss's Generalization of Wilson's Theorem

E 1045 [1952, 697]. *Proposed by S. W. Golomb, Harvard University*

Let n^{ϕ} (pronounced "n-phi-torial") denote the product of all integers up to n which are prime to n. Then $n^{\phi} \equiv \pm 1 \pmod{n}$. Prove this, and determine when the value is $+1$ and when it is -1.

Solution by William Small, University of Rochester. The result, known as Gauss's generalization of Wilson's theorem, is given, with proof, in Oystein Ore, *Number Theory and Its History*, theorem 11-6, p. 266. In the congruence one has the negative sign when $n=4$, p^α, $2p^\alpha$, where p is an odd prime. In all other cases, one has the positive sign, except that for the case $n=2$ it is immaterial which sign is used.

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Other references cited are Hardy and Wright, *Theory of Numbers*, theorem 129, p. 102, and Nagell, *Introduction to Number Theory*, theorem 79, p. 100.

4547. *Proposed by H. S. Shapiro, Chatham, New Jersey*

Let $0 \leqq a_i \leqq 1$, $i=1, \cdots, n$ and put $A = \sum a_i$. Prove

$$\sum_{i=1}^{n} \frac{a_i}{1-a_i} \geqq \frac{nA}{n-A},$$

equality occurring only if all the a_i are equal.

SOLUTIONS

A Summation

4483 [1952, 254]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Find the sum of

$$1 + \frac{1}{2}\,\frac{1}{3^2} + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{5^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{7^2} + \cdots.$$

Solution by Samuel Goldberg, Lehigh University. Consider the integral

$$\int_0^x \frac{\text{arc sin } z}{z}\, dz$$

and expand the integrand in a power series in z. Then

$$\int_0^x \frac{\text{arc sin } z}{z}\, dz = \int_0^x \left(1 + \frac{1}{2}\,\frac{z^2}{3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{z^4}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{z^6}{7} + \cdots\right) dz$$

$$= x + \frac{1}{2}\,\frac{x^3}{3^2} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^5}{5^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^7}{7^2} + \cdots.$$

Inasmuch as the series in the integrand is convergent for $|z| < 1$, the resulting power series in x is convergent for $|x| < 1$. Moreover this series converges for $x=1$ and is then just the series whose sum, S, is to be found. By Abel's theorem,

$$S = \int_0^1 \frac{\text{arc sin } z}{z}\, dz.$$

The substitution $z = \sin t$ yields the result in terms of a standard improper integral

$$S = -\int_0^{\pi/2} \log \sin t\,dt$$

which is easily evaluated by replacing $\sin t$ by $2 \sin \frac{1}{2}t \cos \frac{1}{2}t$. One obtains $S = \frac{1}{2}\pi \log 2$.

Also solved by T. M. Apostol, Ranko Bojanić, Leonard Carlitz, James Clunie, R. V. Esperti, Robert Frucht, Fritz Herzog, A. R. Hyde, H. Kaufman

and J. S. Shipman, J. B. Kelly, R. Kissling, M. S. Klamkin, A. M. Peiser and Sidney Katz, O. J. Ramler, Edward Saibel, O. E. Stanaitis, K. Subbarao and M. Perisastri, C. A. Swanson, F. Underwood, Eugene Usdin, Chih-yi Wang, J. E. Wilkins, Jr., and the Proposer.

Editorial Note. Bojanić and Carlitz point out that the present problem is included in question 330, *Journal of the Indian Mathematical Society*, 10, 1912, p. 32. On pp. 59–61 of the same issue, Ramanujan gives a general method for summing the series

$$1 + \frac{1}{2}\frac{1}{3^n} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{5^n} + \cdots.$$

Rational Approximations

4484 [1952, 254]. *Proposed by Murray Gerstenhaber and R. S. Varga, Harvard University*

Given a real number $y>0$. Let $g_n = h_n/n$ be the rational number which best approximates y, where h_n is a positive integer and $n=1, 2, \cdots$. Let

$$\lambda_n = \begin{cases} + & \text{if} \quad h_n/n - y \geqq 0, \\ - & \text{if} \quad h_n/n - y < 0. \end{cases}$$

Show that λ_n is periodic, *i.e.* there exists a positive integer k such that λ_{mk} is of one sign for all positive integers m, if and only if y is a positive rational number.

Solution by James Clunie, King's College, Aberdeen, Scotland. When y is rational there are positive integers p, q such that $y=p/q$, and therefore $h_n = np/q$ so that λ_n is periodic with period q. When y is irrational then for some arbitrary positive integer k suppose $\lambda_k = -$ ($\lambda_k = +$ is treated in an analogous manner). Hence $y = g_k + \alpha_k/k$ where $0 < \alpha_k < \frac{1}{2}$. There exists an integer m such that $1 > m\alpha_k > \frac{1}{2}$; ($m\alpha_k < \frac{1}{2}$, $(m+1)\alpha_k > 1$ implies $\alpha_k > \frac{1}{2}$.) Therefore

$$y = g_k + \frac{m\alpha_k}{mk} = \frac{mkg_k + 1}{mk} - \frac{(1 - m\alpha_k)}{mk} = g_{mk} - \frac{(1 - m\alpha_k)}{mk},$$

which shows that $\lambda_{mk} = +$. Thus no irrational number can give rise to a periodic λ_n.

Also solved by Henry Furstenberg, Joseph Lehner, Hugh Noland, Judy Richmond, L. M. Weiner, and the Proposers.

Generalization of the Droz-Farney Theorem

4485 [1952, 254]. *Proposed by Simon T. Kao, Catholic University of America, Washington, D. C.*

Through the orthocenter H of a triangle ABC draw any pair of perpendicular lines l_1 and l_2 and let A_1, A_2; B_1, B_2; C_1, C_2 be the respective points of intersection with the three sides BC, CA, AB. Show that the three points P, Q, R which

and J. S. Shipman, J. B. Kelly, R. Kissling, M. S. Klamkin, A. M. Peiser and Sidney Katz, O. J. Ramler, Edward Saibel, O. E. Stanaitis, K. Subbarao and M. Perisastri, C. A. Swanson, F. Underwood, Eugene Usdin, Chih-yi Wang, J. E. Wilkins, Jr., and the Proposer.

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$$\lambda_n = \begin{cases} + & \text{if} \quad h_n/n - y \geqq 0, \\ - & \text{if} \quad h_n/n - y < 0. \end{cases}$$

Show that λ_n is periodic, *i.e.* there exists a positive integer k such that λ_{mk} is of one sign for all positive integers m, if and only if y is a positive rational number.

Solution by James Clunie, King's College, Aberdeen, Scotland. When y is rational there are positive integers p, q such that $y=p/q$, and therefore $h_n = np/q$ so that λ_n is periodic with period q. When y is irrational then for some arbitrary positive integer k suppose $\lambda_k = -$ ($\lambda_k = +$ is treated in an analogous manner). Hence $y = g_k + \alpha_k/k$ where $0<\alpha_k<\frac{1}{2}$. There exists an integer m such that $1>m\alpha_k>\frac{1}{2}$; ($m\alpha_k<\frac{1}{2}$, $(m+1)\alpha_k>1$ implies $\alpha_k>\frac{1}{2}$.) Therefore

$$y = g_k + \frac{m\alpha_k}{mk} = \frac{mkg_k + 1}{mk} - \frac{(1 - m\alpha_k)}{mk} = g_{mk} - \frac{(1 - m\alpha_k)}{mk},$$

which shows that $\lambda_{mk} = +$. Thus no irrational number can give rise to a periodic λ_n.

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4485 [1952, 254]. *Proposed by Simon T. Kao, Catholic University of America, Washington, D. C.*

Through the orthocenter H of a triangle ABC draw any pair of perpendicular lines l_1 and l_2 and let A_1, A_2; B_1, B_2; C_1, C_2 be the respective points of intersection with the three sides BC, CA, AB. Show that the three points P, Q, R which

divide the three segments A_1A_2, B_1B_2, C_1C_2 in the same ratio r lie on a line.

I. *Solution by M. Perisastri, Maharajah's College, Vizianagram, South India.* Let $x/a_i+y/b_i=1$, $i=1, 2, 3$, be the equations of BC, CA, AB. Choose l_1 and l_2 as the x- and y-axes, so that H is $(0, 0)$. Then the coördinates of P, Q, and R are easily found to be

$$\left(\frac{a_i}{r+1}, \frac{rb_i}{r+1}\right).$$

Therefore the familiar determinant

$$|\Delta PQR| = \frac{r}{(1+r)^2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}.$$

Using the condition that H is the orthocenter we get

$$a_1a_2a_3(b_2-b_3)+b_1b_2b_3(a_2-a_3)=0,$$
$$a_1a_2a_3(b_3-b_1)+b_1b_2b_3(a_3-a_1)=0,$$
$$a_1a_2a_3(b_1-b_2)+b_1b_2b_3(a_1-a_2)=0.$$

Dividing these equations respectively by a_2a_3, a_3a_1, a_1a_2 and adding, we have $a_1b_2-a_1b_3+a_2b_3-a_2b_1+a_3b_1-a_3b_2=0$, whence $|\Delta PQR|=0$ and the points P, Q, R are collinear.

II. *Solution by O. J. Ramler, Catholic University of America, Washington, D. C.* Consider the parabola determined by the sides of triangle ABC and a line l_1 through its orthocenter H. Then by an elementary property of the parabola the line l_2 perpendicular to l_1 at H will also be tangent to the parabola. Tangents to the parabola intersect the sides of the triangle in projectively related ranges of points. These projectivities may be considered to be determined by $(A_1A_2I_a)$, $(B_1B_2I_b)$ and $(C_1C_2I_c)$ on sides BC, CA, AB respectively, where I_a, I_b, I_c are the ideal points on those sides. If P_a, P_b, P_c divide segments A_1A_2, B_1B_2, C_1C_2 in the same ratio, then the cross ratios $(A_1A_2P_aI_a)$, $(B_1B_2P_bI_b)$, and $(C_1C_2P_cI_c)$ are equal, thus identifying P_a, P_b, P_c as corresponding points in the three projectively related ranges on the sides of the triangle. Hence P_a, P_b, P_c are collinear, and the envelope of this line is the parabola for varying ratios. Its focus is the Miquel point of the quadrilateral consisting of the sides of the triangle and line l_1.

Also solved by Joseph Langr, Sedalia M. Sims, Roscoe Woods, and the Proposer.

Editorial Note. Woods demonstrates that the orthocenter is the only point which possesses the cited property.

Arbitrarily Large Entire Functions

4486 [1952, 254]. *Proposed by D. J. Newman, Harvard University*

Let $f(x)$ be any function continuous on the positive axis. Prove that there exists an entire function which is always larger than this $f(x)$ along the positive axis. (That is, prove that there are arbitrarily large entire functions.)

Solution by Joshua Barlaz, Rutgers University. The result is classic, occurring as a lemma in a paper by Poincaré, Sur les fonctions à espaces lacunaires, *American Journal of Mathematics*, xiv, 1892, pp. 213–215. A proof by Borel (*Leçons sur les séries à termes positifs*, Paris, 1902, p. 27) is quoted by Hardy, *Orders of Infinity*, Cambridge, 1924, pp. 10–11. Implicit in Poincaré's work is the following generalization: *Let $F(x)$ be continuous for all real x, then there exists an integral function $f(z)$ such that $F(x) \geqq f(x)$ for all real x.* Borel's proof is easily adapted to establish this result.

Define $g(x) = \max\{f(x), f(-x)\}$; $h(x) = \max g(t)$ for $0 \leqq t \leqq x$. $g(x)$ is an even function, $g(x)$ and $h(x)$ are continuous for $x \geqq 0$, $h(x) \geqq g(x)$, $h(x)$ is monotone, and $h(x) \uparrow \infty$ as $x \to \infty$. Let $\phi(x) = h^2(x)$. Then $\phi(x) \geqq h(x)$ for $x \geqq x_0$. Choose two sequences of positive numbers $\{a_n\}$, $\{b_n\}$, such that

$$b_1 < a_1 < b_2 < a_2 < b_3 < a_3 < \cdots$$

and such that $a_n, b_n \to \infty$. We can now choose a sequence of positive even integers ν_n so that both

$$\nu_{n+1} > \nu_n \quad \text{and} \quad \left(\frac{a_n}{b_n}\right)^{\nu_n} > \phi(a_{n+1}).$$

Next, let $F^*(x) = \sum_{n=1}^{\infty} (x/b_n)^{\nu_n}$. The series in convergent for all x. Moreover, if $a_n \leqq x < a_{n+1}$,

$$F^*(x) > \left(\frac{a_n}{b_n}\right)^{\nu_n} > \phi(a_{n+1}) \geqq \phi(x) \geqq h(x)$$

for $x \geqq \max(x_0, 1)$. Let

$$M = \max_{0 \leqq x \leqq \max(x_0, 1)} h(x), \qquad F(x) = F^*(x) + M.$$

Then $F(x) \geqq h(x) \geqq g(x)$ for all $x \geqq 0$. From the fact that $F(x)$ and $g(x)$ are even functions it now follows that $F(x) \geqq f(x)$ for all real x.

Also solved by Ranko Bojanić, Melvin Henriksen, Fritz Herzog, Jacob Korevaar, Jack Kotik and the proposer, George Piranian, W. R. Scott, and E. M. Wright.

Editorial Note. Piranian points out that the original hypothesis should read: let $f(x)$ be any function continuous for $x \geqq 0$.

Divergent Series, Partial Sums

4487 [1952, 254]. *Proposed by Paul Erdös, American University, Washington, D. C.*

Let

$$\sum_{k=1}^{\infty} d_k = \infty, \qquad d_k > 0, \qquad D_n = \sum_{k=1}^{n} d_k;$$

$$\sum_{k=1}^{\infty} a_k < \infty, \qquad a_k > 0.$$

Further assume that $a_k/d_k \geqq a_{k+1}/d_{k+1}$. Prove

$$\lim_{k\to\infty} \frac{a_k D_k}{d_k} = 0.$$

Solution by Ranko Bojanić, Mathematical Institute, Belgrade, Yugoslavia. We prove a somewhat more general result obtained by replacing the hypothesis on $a_k/d_k - a_{k+1}/d_{k+1}$ by the following:

$$\frac{a_k}{d_k} - \frac{a_{k+1}}{d_{k+1}} \geqq -M\frac{d_k}{D_k^2}, \tag{1}$$

where M is a constant $\geqq 0$. Take $\lambda > 1$. Take m such that

$$D_{m-1} \leqq \lambda D_{n-1} < D_m. \tag{2}$$

From the identity

$$\frac{a_m D_m}{d_m}\sum_{k=n}^{m} \frac{d_k}{D_k} = \sum_{\nu=n}^{m}\left(\frac{a_m D_m}{d_m} - \frac{a_\nu D_\nu}{d_\nu}\right)\frac{d_\nu}{D_\nu} + \sum_{k=n}^{m} a_k,$$

it follows that

$$\frac{a_m D_m}{d_m}\sum_{k=n}^{m} \frac{d_k}{D_k} \leqq \max_{n\leqq\nu\leqq m}\left(\frac{a_m D_m}{d_m} - \frac{a_\nu D_\nu}{d_\nu}\right)\sum_{k=n}^{m} \frac{d_k}{D_k} + \sum_{k=n}^{m} a_k.$$

Now the inequality (1) may be written as follows:

$$\frac{a_{k+1}D_{k+1}}{d_{k+1}} - \frac{a_k D_k}{d_k} \leqq M\frac{d_k}{D_k} + a_{k+1}.$$

Since

$$\frac{d_k}{D_k} = 1 - \frac{D_{k-1}}{D_k} \leqq \log\frac{D_k}{D_{k-1}},$$

which follows obviously from $1-x \leqq \log(1/x)$ for $x \leqq 1$, we obtain

$$\frac{a_{k+1}D_{k+1}}{d_{k+1}} - \frac{a_k D_k}{d_k} \leqq M \log \frac{D_k}{D_{k-1}} + a_{k+1}.$$

Summing this inequality from ν to $m-1$ we obtain

$$\frac{a_m D_m}{d_m} - \frac{a_\nu D_\nu}{d_\nu} \leqq M \log \frac{D_{m-1}}{D_{\nu-1}} + \sum_{k=\nu}^{m} a_k.$$

Thus

$$\begin{aligned}\frac{a_m D_m}{d_m} \sum_{k=n}^{m} \frac{d_k}{D_k} &\leqq \max_{n \leqq \nu \leqq m} \left(M \log \frac{D_{m-1}}{D_{\nu-1}} + \sum_{k=\nu}^{m} a_k \right) \sum_{k=n}^{m} \frac{d_k}{D_k} + \sum_{k=n}^{m} a_k \\ &\leqq \left(M \log \frac{D_{m-1}}{D_{n-1}} + \sum_{k=n}^{m} a_k \right) \sum_{k=n}^{m} \frac{d_k}{D_k} + \sum_{k=n}^{m} a_k,\end{aligned}$$

or

$$\frac{a_m D_m}{d_m} \leqq M \log \frac{D_{m-1}}{D_{n-1}} + \sum_{k=n}^{m} a_k + \left(\sum_{k=n}^{m} \frac{d_k}{D_k} \right)^{-1} \sum_{k=n}^{m} a_k.$$

Further, by (2) we have

$$\sum_{k=n}^{m} \frac{d_k}{D_k} \geqq \frac{1}{D_m} \sum_{k=n}^{m} d_k = 1 - \frac{D_{n-1}}{D_m} > 1 - \frac{1}{\lambda} > 0,$$

and $D_{m-1}/D_{n-1} \leqq \lambda$. Hence

$$\frac{a_m D_m}{d_m} \leqq M \log \lambda + \sum_{k=n}^{m} a_k + \frac{\lambda}{\lambda - 1} \sum_{k=n}^{m} a_k.$$

Here let $n \to \infty$. Since $m \to \infty$ with n and $\sum_1^\infty a_k$ converges, we obtain finally

$$\limsup_{m \to \infty} \frac{a_m D_m}{d_m} \leqq M \log \lambda.$$

But λ may be chosen arbitrarily near to 1, and the required result follows.

Also solved by James Clunie, Henry Furstenberg, Cecil Hastings, Jr., Stanley Katz and A. M. Peiser, Jacob Korevaar, A. E. Livingston, C. J. Rajagopal, A. J. Rodriguez, P. Somanadham and M. Perisastri, O. E. Stanaitis, Otto Szász, Gabor Szegö, Chih-yi Wang, R. H. Breusch, and E. M. Wright.

Editorial Note. Results which generalize the above proposal in slightly different ways have been given by C. T. Rajagopol, *Mathematics Student*, vol. 8, pp. 118–123, and by S. Minakshisundaram, *Mathematics Student*, vol. 9, pp. 78–81.

generation of such in the important transportation problem.

In brief, the chief deficiency of the book is its easy superficiality which on the one hand narrows to a seriously restrictive misconception of practical applicability of linear programming (*e.g.*, basic assumption 1, p. 18) and on the other bursts all bounds in identification of the imputed valuations of the dual problem with cost accounting quantities.

Despite its limitations the book is a step in the right direction. There is little doubt that linear programming may be used to extend and enrich the economic theory (and practice) of the business firm.

A. CHARNES
Carnegie Institute of Technology

Statistical Theory with Engineering Applications. By A. Hald. John Wiley and Sons, Inc., New York, 1952. xii+783 pages.

The purpose of the book is to provide engineers with the statistical techniques most important in their work and the theory underlying these techniques. A knowledge of calculus is presupposed.

Before introducing applied techniques, the book presents an introduction to mathematical statistics: elementary probability theory, fundamental properties of distributions, special distributions, limit theorems and sampling distributions. Subsequent chapters are devoted to the applications of special sampling distributions such as the binomial, Poisson, normal, Chi-Square, "t," variance ratio and range. Regression and correlation theory are very thoroughly treated and the standard techniques in the analysis of variance are covered. A chapter is also devoted to sampling from finite populations.

The book is large and remarkably comprehensive; it is chock-full of techniques which are useful in the analysis of data. The chapters on regression, for example, discuss regression when the variance is proportional to a function of the independent variate and give a very interesting discussion of choice between different regression curves. The book contains material on growth curves, fitting truncated and mixed normal distributions, on errors of measurement, on skew distributions and normalizing transformations, on tolerance limits and many other topics of importance in engineering research not ordinarily covered.

The book is unique in combining a comprehensive collection of applied techniques and theory. The customary procedure is simply to list the techniques and, at most, justify them intuitively. On the other hand, many people feel that statistics is a mathematical subject and can be properly understood only if at least some of the basic derivations are studied. It is surprising that more of the statistics books for engineers do not draw on the mathematical background engineers are required to take, and Hald's book appears to be a unique and extremely valuable addition to our reservoir of statistics texts.

A. H. BOWKER
Stanford University

Linear algebra and matrix theory. By. R. R. Stoll. McGraw-Hill Book Company, New York, 1952. 16+272 pp., $6.00.

This is an excellent introduction to the concepts and techniques of modern algebra, via the usual material on linear equations and reduction of matrices to canonical form. For example, in the course of the development the author pauses to define and discuss the abstract concepts of field, vector space, equivalence relation, group, and ring when these concepts arise naturally. The book is written with care and rigor, coupled with enough numerical examples, exercises, and passing from the special to the general to prevent the abstractions from floating out of reach. This nice balance is exemplified by the fact that, while matrices and n-tuples occupy most of the space in the book, one is left with the feeling that linear mappings and abstract vector spaces have been the real objects of study.

The topics covered are: Chapter 1: systems of linear equations; Chapter 2: abstract finite dimensional vector spaces, their dimensions, subspaces, direct sums and quotient spaces (undergraduates unskilled in abstract thinking may find this a little difficult); Chapter 3: matrices, rank, row- and column-equivalence, multiplication; Chapter 4: determinants, introduced axiomatically, following Schreier; Chapter 5: bilinear, quadratic, and Hermitian functions and forms; Chapter 6: generalities about groups and rings of linear transformations; Chapter 7: reduction to canonical form under similarity; Chapter 8: vector spaces with inner products, the principal axis theorem with the noteworthy feature that Hermitian and symmetric transformations are treated as special cases of normal transformations.

The approach in general is fresh but not radical, and should be stimulating to most teachers as well as to students who can handle some degree of abstraction. The over-all tone is one of satisfying completeness.

DANIEL ZELINSKY
Northwestern University

Statistical Theory in Research. By R. L. Anderson and T. A. Bancroft. McGraw-Hill Book Company, New York, 1952, xvi+399 pages. $7.00.

This book is intended as a text and reference book in statistical methods for research workers. A knowledge of calculus is assumed. Useful suggestions are given on the selection of material for a one year course. A well chosen set of problems is given at the end of each chapter. The standard methods of estimation and hypothesis testing are presented in such a way as to give the student a good working knowledge of the subject.

Since the book is not intended for students specializing in statistics, the authors have not tried to give a rigorous presentation, nor a critical discussion of the basic concepts involved. By leaving out the technical details, they are able to give a coherent and very readable account.

One point on which questions may be raised is the omission of recent work

in statistics. Concepts introduced by Wald, such as risk function, minimax estimation and sequential analysis, do not appear. It is the reviewer's opinion that these ideas and some of their simplest applications have an important place in even the most elementary statistical methods book. Less important is the absence of any mention of the modern logical emphasis on statistical decision making as opposed to statistical inference. However, on page 6 where the term "inductive inference" is used it might be well to refer the reader to the introduction of Neyman's *First Course in Probability and Statistics*.

Of minor suggestions and minor errors noted, the following may be mentioned: Page 31—The definitions of $E(x)$ and $E[\theta(x)]$ are contradictory and are given for bounded random variables only, while examples such as 4.21 involve the unbounded case. Additional explanation would be helpful. Page 41—Conditions analogous to the univariate case for $F(x, y)$ to be a cumulative distribution do not exist, as Gnedenko's counter-example shows. Page 94—A consistent estimator need not be unbiased in the limit. Page 101—Additional explanation would prepare the reader for cases, such as the rectangular, in which the maximum likelihood estimate is not a solution of $\partial L/\partial\theta = 0$, and for solutions of $\partial L/\partial\theta = 0$ which do not give a maximum.

The authors are to be congratulated on having written an excellent book for the purposes intended. It will be particularly useful as a text for courses in applied statistics.

C. R. BLYTH
University of Illinois

NEW BOOKS RECEIVED

Student's Workbook for Essential Business Mathematics. Second Edition. By Llewllyn R. Snyder. New York, McGraw-Hill Book Company, 1953. vi+153 pages. $2.50.

An Introduction to Mathematical Thought. By E. R. Stabler. Cambridge, Mass., Addison-Wesley Publishing Company, 1953. 20+268 pages. $4.50.

Elements of Mathematics. By Helen M. Roberts and Doris S. Stockton. Cambridge, Mass., Addison-Wesley Publishing Company, 1953. 8+211 pages. $3.00.

Philosophy and Psycho-Analysis. By John Wisdom. New York, Philosophical Library, 1953. vi+282 pages. $5.75.

Construction and Applications of Conformal Maps. Edited by E. F. Beckenbach. (Proceedings of a Symposium held at the Institute for Numerical Analysis of the National Bureau of Standards at the University of California, June 22–25, 1949.)

Logic for Mathematicians. By J. B. Rosser. New York, McGraw-Hill Book Company, 1953. xiv+530 pages. $10.00.

Demand Analysis, A Study in Econometrics. By Herman Wold in association with Lars Jureen. New York, John Wiley and Sons, Inc., 1953. xvi+358 pages, $7.00.

Complex Analysis. By L. V. Ahlfors. New York, McGraw-Hill Book Company, 1953. xi+247 pages. $5.00.

Solid Geometry. By W. G. Shute, W. W. Shirk, and G. F. Porter. New York, American Book Company, 1953. vii+280 pages. $2.48.

An Introduction to Linear Programming. By A. Charnes, W. W. Cooper, and A. Henderson. New York, John Wiley and Sons, Inc. 1953. ix+74 pages. $2.50.

Numerical Solution of Differential Equations. By W. E. Milne. New York, John Wiley and Sons, Inc., 1953. xi+275 pages. $6.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB NEWS

(Clubs are invited to contribute news items)

The Mathematics Society of the Massachusetts Institute of Technology held a competition for Freshmen on December 6, 1952. Approximately 100 students completed a three hour examination which emphasized concepts rather than technique. The three prize winners were: first, Max A. Plager; second, William G. Strang; third, Donald A. Bavly.

The Kansas Gamma chapter of Kappa Mu Epsilon, Mount St. Scholastica College, announces a new type of prize contest which aims to acquaint students with mathematics journals. An award will be given to the student who demonstrates superior achievement in an examination over the material of a nontechnical nature contained in *The Pentagon, Mathematics Magazine, School Science and Mathematics*, and *The Mathematics Teacher*.

TESTING FOR PRIMALITY

LEONARD CANERS, St. Michael's College,

To determine whether a number C is prime or not, we may use a table of factors and primes.* Such a table, however, is limited to values of C which do not exceed the last entry. The following device enables us to double the scope of the table in a relatively simple manner.

The method depends on the rather elementary fact that if $A+B=C$ and $A-B=R$ (all letters representing positive integers), then if R is a divisor of B it is a divisor of A and therefore of C. On the other hand, if R is not a divisor of B it is not a divisor of A and therefore not of C.

* For instance, see Mathematical Tables from Handbook of Chemistry and Physics.

As an example of the procedure, let our table of factors and primes include numbers from 2 to 2009, and let it be required to test $C=2491$ for primality. As usual, we only need test as divisors the prime numbers not greater than $\sqrt{2491}=49+$. Accordingly, we set in a column the odd primes up to 49; these are the values of R to be tested. Now if $A-B=R$ and $A+B=C$, then $B=\frac{1}{2}(C-R)$, from which the initial value of B can be computed: when $R=3$, $B=\frac{1}{2}(2491-3)=1244$. To complete the B-column, we write in succession 1244, 1243, 1242, $\cdots$, skipping where an odd number is missing in the R-column.

R	B	Prime divisors of B	R	B	Prime divisors of B
3	1244	2, 311	.	.	.
5	1243	11, 113	29	1231	prime
7	1242	2, 3, 23	31	1230	2, 3, 5, 41
.	.	.	.	.	.
11	1240	2, 5, 31	.	.	.
13	1239	3, 7, 59	37	1227	3, 409
.	.	.	.	.	.
17	1237	prime	41	1225	5, 7
19	1236	2, 3, 103	43	1224	2, 3, 17
.	.	.	.	.	.
23	1234	2, 617	47	1222	2, 13, 47
.	.	.			

The prime divisors of B are now read from our table of factors and primes. (In practice, these divisors are merely noted in the table. They are written down here to clarify the method.) If B does not have the prime divisor R opposite it, that value of R can be struck out. If any value of R is not struck out, that number is a divisor of C. If all the values of R are struck out, C is prime. In the example we note that 1222 has the divisor 47, whence 2491 has the divisor 47 and is not prime.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

CONFERENCE ON FUNCTIONS OF A COMPLEX VARIABLE

The Mathematics Department of the University of Michigan is sponsoring a conference on functions of a complex variable with emphasis on topological

As an example of the procedure, let our table of factors and primes include numbers from 2 to 2009, and let it be required to test $C=2491$ for primality. As usual, we only need test as divisors the prime numbers not greater than $\sqrt{2491}=49+$. Accordingly, we set in a column the odd primes up to 49; these are the values of R to be tested. Now if $A-B=R$ and $A+B=C$, then $B=\frac{1}{2}(C-R)$, from which the initial value of B can be computed: when $R=3$, $B=\frac{1}{2}(2491-3)=1244$. To complete the B-column, we write in succession 1244, 1243, 1242, $\cdots$, skipping where an odd number is missing in the R-column.

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·	·	·	·	·	·
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13	1239	3, 7, 59	37	1227	3, 409
·	·	·	·	·	·
17	1237	prime	41	1225	5, 7
19	1236	2, 3, 103	43	1224	2, 3, 17
·	·	·	·	·	·
23	1234	2, 617	47	1222	2, 13, 47
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CONFERENCE ON FUNCTIONS OF A COMPLEX VARIABLE

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methods. The Conference will be held June 17–30, 1953 in Ann Arbor, Michigan.

The program will consist of a group of invited addresses and of sessions for twenty-minute contributed papers. Among those who will present hour addresses are the following: L. V. Ahlfors, L. Bers, A. Beurling, S. Bochner, S. Kakutani, C. Loewner, M. Morse, Z. Nehari, R. Nevanlinna, M. Ohtsuka, P. Rosenbloom, H. L. Royden, L. Sario, D. C. Spencer, G. T. Whyburn.

The Summer Session of the University of Michigan (June 22–August 15) will devote special attention to complex variable theory. In particular, Professor Rolf Nevanlinna will conduct a course on advanced theory of functions of a complex variable and two seminars in this field.

For further information write to Professor Wilfred Kaplan, 274 West Engineering Building, University of Michigan, Ann Arbor, Michigan.

CONFERENCE FOR TEACHERS OF MATHEMATICS

The third annual Conference for Teachers of Mathematics will be held on the Los Angeles Campus of the University of California from July 6–17, 1953. The popular mathematics laboratory course given under the leadership of Miss Ida M. Bernhard will be offered again in conjunction with the Conference. Further information and application blanks may be obtained by writing to Professor Clifford Bell, Mathematics Department, University of California, Los Angeles 24, California.

INSTITUTE FOR MATHEMATICS TEACHERS

An Institute for Mathematics Teachers will be held at the University of Michigan, Ann Arbor, Michigan on August 3–14, 1953. This Institute is sponsored by the School of Education and the Department of Mathematics of the University with the cooperation of the Michigan Council of Teachers of Mathematics. The program includes lectures by representatives of various industries, field trips, discussion and laboratory groups. For further information write to Professor P. S. Jones, Angell Hall, University of Michigan, Ann Arbor, Michigan.

SUMMER SESSION AT REED COLLEGE

Under sponsorship of the Fund for the Advancement of Education, Reed College and the Portland (Oregon) Public Schools are cooperating to enrich the school program for gifted children. A special seminar in mathematics and physical science will be offered this summer at Reed College, primarily for teachers in the program, but other interested teachers will be welcome. Directors of the seminar are Professor Emeritus E. T. Bell of California Institute of Technology, Professors K. E. Davis and R. A. Rosenbaum of Reed College. For further information write to the Director of the Summer Session, Reed College, Portland, Oregon.

PERSONAL ITEMS

Associate Professor Arthur Bernhart of the University of Oklahoma has received a five hundred dollar award "for extraordinary excellence in student counseling and in the teaching of freshman and sophomore students." A grant of $25,000 to the University will make it possible to present ten similar awards each year during the next five years.

Professor O. E. Neugebauer of Brown University has been awarded the first Dannie Heineman Prize for his book *The Exact Sciences in Antiquity* by the Heineman Foundation for Research, Educational, Charitable and Scientific Purposes. This Foundation will continue to present a prize of $5,000 every three years to the author of an outstanding book on a high scientific level in the mathematical and physical sciences.

Associate Professor Marguerite Lehr of Bryn Mawr College and Professor I. J. Schoenberg of the University of Pennsylvania represented the Association at the meeting of the American Academy of Political and Social Sciences in Philadelphia, Pennsylvania on April 10–11, 1953.

Professor S. Grace Smyth, Associate Professor of Mathematics and Dean of Women at Knox College, served as the representative of the Association at the inauguration of President R. W. Gibson of Monmouth College on April 17, 1953.

Dr. E. L. Arnoff, formerly Aeronautical Research Scientist with the National Advisory Committee for Aeronautics, Cleveland, Ohio, is now a research associate with the Operations Research Group, Case Institute of Technology.

Mr. R. S. Barton has accepted a position as a senior mathematician with International Business Machines Corporation, Poughkeepsie, New York.

Dr. R. L. Beinert of Hobart and William Smith Colleges has been promoted to an assistant professorship.

Mr. James Bercos, who was formerly with the United States Naval Ordnance Plant, Macon, Georgia, is now a personnel statistician with Lockheed Aircraft Corporation, Marietta, Georgia.

Professor L. M. Blumenthal has returned to his position at the University of Missouri after serving as a consultant at the Institute for Numerical Analysis.

Mr. O. C. Braune, previously a student at St. Mary's University, has accepted a position as an electrical engineer with the Sandia Corporation, Albuquerque, New Mexico.

Mr. J. E. Brown of Florida State University has been appointed to an instructorship at the University of Georgia.

Dr. W. B. Brown, formerly with the National Advisory Committee for Aeronautics, Cleveland, Ohio, is now a design specialist with Northrop Aircraft, Hawthorne, California.

Assistant Professor Emalou Brumfield of Kent State University has been on leave of absence from the University during 1952–53 and has been doing graduate work at Ohio State University.

Mr. G. R. Bushyeager has been appointed Professor and Head of the Department of Mathematics of Morningside College.

Dr. R. E. Carr, who was formerly a research engineer with North American Aviation, Downey, California, has been appointed Head of the Computer Operations Group, Jet Propulsion Laboratory, California Institute of Technology.

Mr. J. L. Connors, previously a graduate student at Rutgers University, has a position as a mathematician with Project Meteor, Massachusetts Institute of Technology.

Mr. G. R. Costello has accepted a position as Aerophysics Project Engineer with Chance-Vought Aircraft, Dallas, Texas.

Mr. J. I. Derr of Southern Methodist University has a position as Aerophysics Engineer with Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. G. M. Dillon is a mathematics specialist in the Treasury Division of E. I. du Pont de Nemours and Company, Wilmington, Delaware.

Mr. Raymond Doby, formerly a development engineer with Sylvania Electric Products, Kew Gardens, New York, has accepted a position as a research engineer with Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Dr. I. A. Dodes, who has been teaching at Stuyvesant High School, New York City, has been appointed Chairman of the Department of Mathematics of Morris High School, Bronx, New York.

Professor R. D. Doner of Alabama Polytechnic Institute is now Chief Training Officer at Redstone Arsenal, Huntsville, Alabama.

Graduate Assistant Edwin Duda of West Virginia University has been promoted to an instructorship.

Mr. R. P. Eddy, previously of the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland, has a position as a mathematician at the David Taylor Model Basin, Washington, D. C.

Mr. Harvey Eisenberg, who was located previously at Fort Monmouth, New Jersey, has accepted a position as a mathematician with Evans Signal Laboratory, Belmar, New Jersey.

Mr. G. V. Emerson, formerly a student at Hobart College, is engaged now as an applied mathematician with Atlantic Research Corporation, Alexandria, Virginia.

Assistant Professor Virginia I. Felder of Mississippi Southern College is teaching in the United States Armed Forces Institute, Japan.

Mr. E. A. Franz, who has been on active military duty in the Signal Corps, United States Army, has returned to Culver-Stockton College as Assistant Professor of Mathematics.

Assistant Professor I. C. Gentry of Wake Forest College has been promoted to an associate professorship.

Mr. R. D. Glauz, previously a research assistant at Brown University, is now a staff member at the Los Alamos Scientific Laboratory.

Mr. W. A. Golomski has been appointed to an instructorship at St. Louis University.

Dr. R. E. Graves, who was formerly at White Sands Proving Ground, Las Cruces, New Mexico, has accepted a position as a mathematician with Goodyear Aircraft Corporation, Litchfield, Arizona.

Mr. E. LeR. Grindall has a position as a project engineer at the Ordnance Research Laboratory, Pennsylvania State College.

Mr. J. P. Gwin, previously a student at the University of Tulsa, is now a computer with the Seismograph Service Corporation, Tulsa, Oklahoma.

Mr. W. C. Hamilton has received an appointment at the Crellin Laboratory of Chemistry, California Institute of Technology.

Mr. J. O. Harrison, Jr., of the Eckert-Mauchly Division of Remington Rand, Incorporated, has accepted a position as Head of the Computation Laboratory, Operations Research Office, Johns Hopkins University, Chevy Chase, Maryland.

Mr. F. C. Hatfield of Blackburn College has been appointed to an instructorship at State Teachers College, Mankato, Minnesota.

Associate Professor I. I. Hirschman of Washington University has been at the Institute for Advanced Study during the academic year 1952–53.

Professor W. G. Hubert of the City College of the City of New York has retired with the title of Professor Emeritus.

Mr. C. A. Jacokes of Knox College has been appointed to an instructorship at North Park College.

Mr. C. E. Kerr, previously a graduate assistant at the University of Delaware, has been appointed to an instructorship at Lafayette College.

Dr. Nathan Keyfitz of the Dominion Bureau of Statistics, Ottawa, Canada, is spending the year 1953 in Djakarta, Indonesia, as a statistical consultant.

Mr. F. A. Kros, who was formerly at White Sands Proving Ground, Las Cruces, New Mexico, is engaged now as a data reduction engineer with Land-Air, Incorporated, Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. M. R. Laudante, previously a student at Polytechnic Institute of Brooklyn, is now a research analyst with North American Aviation, Los Angeles, California.

Mr. E. A. Lew of the Metropolitan Life Insurance Company has been promoted to the position of Associate Actuary and Statistician.

Mr. S. L. Lida, who has been at the Los Alamos Scientific Laboratory, has accepted a position as Applied Science Representative, International Business Machines Corporation, New York City.

Assistant Professor F. W. Light of Johns Hopkins University has a position now as Chief, Wound Assessment Branch, Biophysics Division, Chemical Corps Medical Laboratories, Army Chemical Center, Maryland.

Mr. L. I. Lowell, formerly a graduate student at Michigan State College, is employed as a junior engineer with Boeing Airplane Company, Seattle, Washington.

Mr. J. R. Macy, previously a student at the University of Chicago, has accepted a position as a member of the technical staff of Bell Telephone Laboratories, New York City.

Mr. G. W. Medlin, who was a part-time instructor at the University of North Carolina, has been appointed to an instructorship at Wake Forest College.

Mr. George Millman, who was employed previously as a mathematician with the Army Map Service, has transferred to the position of Analytical Statistician, Office of the Quartermaster General, Department of the Army, Washington, D. C.

Dr. Knox Millsaps has a position as a research physicist with the Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. Dewey Moore, formerly with the National Advisory Committee for Aeronautics, Langley Field, Virginia, has been appointed Physicist at the Naval Ordnance Laboratory, White Oak, Maryland.

Dr. J. T. Moore has been appointed to an assistant professorship at Georgia Institute of Technology.

Mr. B. J. Morse, who has been doing graduate work at Columbia University is now engaged as a mathematician with the Vitro Corporation of America, Verona, New Jersey.

Mr. P. M. Moskowitz, formerly a physicist at the New York Naval Shipyard, is with the California Research and Development Company, Livermore, California.

Dr. W. L. Murdock has accepted a position as Applied Science Representative with International Business Machines Corporation, New York City.

Mr. A. B. Neale, previously at the Naval Aircraft Factory, Philadelphia, Pennsylvania, has a position as a senior engineer with Marquardt Aircraft Company, Van Nuys, California.

Assistant Professor W. V. Neisius of Georgia Institute of Technology has accepted a position with Logistics Research Company, Redondo Beach, California.

Associate Professor Ruth E. O'Donnell of Duquesne University is employed now as Senior Scientist with the Atomic Power Division of Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Mr. R. D. Oeder, formerly a research assistant at Los Alamos Scientific Laboratory, is a mathematician at the Radiation Laboratory, University of California at Livermore.

Mr. S. R. Orr, who has been a research physicist at the Mound Laboratories, Miamisburg, Ohio, has accepted a position as a staff member at the Los Alamos Scientific Laboratory.

Mr. W. D. Paxton, Jr., has a position as Aerodynamicist-Mathematician with North American Aviation, Los Angeles, California.

Mr. S. R. Peterson of the University of New Hampshire has been appointed to an instructorship at Union College.

Mr. R. M. Pinkerton, previously with the National Advisory Committee for Aeronautics, Langley Field, Virginia, has been appointed Professor of Mechanical Engineering at North Carolina State College.

Mr. C. A. Pursel of California State Polytechnic College has a position as a process engineer, General Electric Company, Richland, Washington.

Professor W. T. Reid has returned to Northwestern University after serving as a staff member at Sandia Corporation, Albuquerque, New Mexico.

Professor W. D. Rice of Olivet Nazarene College has been promoted to the position of Chairman of the Department of Physics and Mathematics.

Mr. F. P. J. Sansom, formerly a student at Baker University, has accepted a position at Wright Air Development Center, Dayton, Ohio.

Mr. Arthur Schach of the City College of the City of New York has been appointed Assistant Editor, United States Quarterly Book Review, Library of Congress, Washington, D. C.

Mr. Seymour Schuster has been appointed to an instructorship at Polytechnic Institute of Brooklyn effective September 1, 1953.

Dr. W. R. Seugling, who has been a research assistant at the University of California at Los Angeles, is engaged now as a structural engineer with Yoh Engineering, Incorporated, Los Angeles, California and Wichita, Kansas.

Reverend R. J. Swords, who has been a student at Rathfornham Castle, Dublin, Ireland, has been appointed to an assistant professorship at College of the Holy Cross.

Mr. A. V. Sylwester of California Concordia College is now at the Weather Observer School, Chanute Air Force Base, Illinois.

Mr. R. T. Tear, who has been in military service, has returned to his position as an instructor at Rensselaer Polytechnic Institute.

Assistant Professor B. T. Wade of Clemson Agricultural College has accepted a position as a mathematician at Aberdeen Proving Ground, Maryland.

Miss Frances E. Walsh of Ursuline College has been appointed to an instructorship at the College of Saint Teresa, Winona, Minnesota.

Miss Betty J. Whaley, previously a student at Tennessee Polytechnic Institute, is employed as an engineering aide with the General Electric Company, Schenectady, New York.

Associate Professor R. M. Whitmore of Southwestern University is now a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Mr. R. F. Willis, who has been doing graduate work at Columbia University, has accepted a position as a mathematician with the Operations Research Group of Arthur D. Little, Incorporated, Cambridge, Massachusetts.

Mr. W. W. Youden, formerly a student at Brown University, is engaged as an electronic scientist in the Electronic Computers Laboratory, National Bureau of Standards, Washington, D. C.

Dr. D. F. Campbell of Chicago, Illinois, died on February 5, 1953. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 103 persons have been elected to membership by the Board of Governors on applications duly certified.

E. F. ASSMUS, Student, Oberlin College.

L. F. BABCOCK, Ed.M. (Harvard) Teacher, Woodmere Academy, Woodmere, N. Y.

J. H. BARRETT, Ph.D. (Texas) Asst. Professor, University of Delaware.

JOSEPH BASILE, C.E. (Brussels) General Manager, Ateliers de Constructions Basile, Brussels, Belgium.

M. ISOBEL BLYTH, Ph.D. (Michigan) Asst. Professor, Michigan State College.

R. C. BOLES, Ed.D. (Florida) Asso. Professor, Florence State Teachers College, Ala.

F. E. BRADLEY, Design Engineer, McDonnell Aircraft Corporation, St. Louis, Mo.

D. G. BRENNAN, Student, Massachusetts Institute of Technology.

E. H. BRYANT, M.S. (Mississippi) Head of Department, Northwest Mississippi Junior College, Senatobia, Miss.

JAMES WINFIELD BUTLER, M.A. (Kent State) Junior Development Engineer, Goodyear Aircraft, Akron, Ohio.

L. A. CANERS, M.A. (Manitoba) M.A. (Minnesota) Asst. Professor, St. Michael's College, Winooski, Vt.

MR. DALE CARPENTER, B.S.E.E. (Ohio Northern) Mathematics Supervisor, Board of Education, Los Angeles, Calif.

T. F. CARROLL, B.S. (Tulane) Assistant Engineer, Celotex Corporation, Marrero, La.

E. W. CHENEY, B.S. (Lehigh) Teaching Fellow, University of Kansas.

T. S. CHIHARA, M.S. (Purdue) Grad. Teaching Assistant, Purdue University.

R. A. CLARK, Ph.D. (M.I.T.) Asst. Professor, Case Institute of Technology.

PATRICIA A. CLINES, B.S. (Nazareth C.) Grad. Student, Catholic University.

J. W. COLBY, Student, University of Louisville.

G. R. COSTELLO, Aerophysics Project Engineer, Chance-Vought Aircraft, Dallas, Tex.

G. L. CRUMLEY, B.S. (Kansas S.T.C., Emporia) Grad. Student, Kansas State Teachers College, Emporia, Kan.

H. B. CURTIS, JR., M.A. (Arkansas) Instr., Texas Agricultural and Mechanical College.

BEVERLY J. DALE, B.A. (Buffalo) Engineering Computer, Bell Aircraft Corporation, Niagara Falls, N. Y.

C. H. DALTON, M.A. (Michigan) Asst. Professor, Southeast Missouri State College, Cape Girardeau, Mo.

R. B. DAVIS, Ph.D. (M.I.T.) Asst. Professor, University of New Hampshire.

CAROLINA D. DEL MAR, B.S. in Ed. (Philippines) Grad. Student, St. Louis University.

W. E. DESKINS, Ph.D. (Wisconsin) Instr., University of Wisconsin.

J. P. DIMICELI, Student, St. John's University.

VIDA E. DUNBAR, M.A. (Kansas) Instr., Northwest Missouri State College, Maryville, Mo.

R. P. EDDY, M.S. (Brown) Mathematician, David Taylor Model Basin, Washington, D. C.

JOANNE ELLIOTT, Ph.D. (Cornell) Asst. Professor, Mount Holyoke College.

MRS. KATHRYN P. ELLIS, M.S. (Iowa) Instr., State University of Iowa.

WADE ELLIS, Ph.D. (Michigan) Asso. Professor, Oberlin College.

ROMEO FARGNOLI, Student, University of Rhode Island.

CATHERINE S. FEELY, B.A. (St. Xavier's C.) Grad. Student, University of Notre Dame.

B. A. FLEISHMAN, Ph.D. (N.Y.U.) Senior Mathematician, Applied Physics Laboratory, Johns Hopkins University.

L. A. FULK, B.A. (Illinois) United States Air Force.

I. C. GENTRY, Ph.D. (Duke) Asso. Professor, Wake Forest College.

E. P. Rozycki, B.A. (Buffalo) Grad. Student, University of Buffalo.

W. H. Roudebush, B.A. (Cincinnati) Aeronautical Research Scientist, Lewis Flight Propulsion Laboratory, Cleveland, Ohio.

Mildred G. Schlapkohl, Student, Northwestern University.

J. G. Sharp, Student, University of Detroit.

R. M. Simonson, Student, Gonzaga University.

Sister Florence Marie (Knob), M.A. (Catholic) Instr., D'Youville College.

Sister Marie Loretta (Bentz), M.A. (Catholic) Instr., Barry College, Miami, Fla.

Sister Mary deChantal, B.A. (St. Teresa's C.) Teacher, Central Catholic High School, Toledo, Ohio.

R. P. Smith, B.A. (Drake) Grad. Assistant, Kansas State College.

S. M. Spencer, Jr., Ph.D. (Duke) Asso. Professor, Louisiana College, Pineville, La.

W. L. Strother, Ph.D. (Tulane) Asst. Professor, University of Miami.

M. C. Suman, Student, San Jose State College.

V. E. Thomas, B.A. (American International) Grad. Student, University of Massachusetts.

J. S. Thorpe, B.S. (Pittsburgh) Teacher, Shady Side Academy, Pittsburgh, Pa.

G. R. Trimble, Jr., M.A. (Delaware) Mathematician, International Business Machines Corporation, Endicott, N. Y.

R. E. Wheeler, Ph.D. (Kentucky) Asst. Professor, Florida State University.

N. K. Williamson, B.S. (Citadel) Asst. Professor, Newberry College.

R. A. Willoughby, Ph.D. (California) Asst. Professor, Georgia Institute of Technology.

R. P. Winter, Ed.M. (St. Thomas C.) Instr., College of St. Thomas.

F. J. Witt, M.A. (Duke) Asst. Professor, Tennessee Polytechnic Institute.

E. M. Wright, Ph.D. (Oxford) Professor and Head of Department, University of Aberdeen, Scotland.

Paul Yacynych, Student, University of Pittsburgh.

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, Canada, August 31–September 1, 1953.

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

Allegheny Mountain

Illinois

Indiana

Iowa

Kansas

Kentucky

Louisiana-Mississippi, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

Maryland-District of Columbia-Virginia

Metropolitan New York

Michigan

Minnesota, Bemidji State Teachers College, October 10, 1953.

Missouri

Nebraska

Northern California

Ohio

Oklahoma, Oklahoma City, October, 1953.

Pacific Northwest, Montana State University, Missoula, June 19, 1953.

Philadelphia, Drexel Institute of Technology, Philadelphia, November 28, 1953.

Rocky Mountain

Southeastern, University of South Carolina, Columbia, March 12–13, 1954.

Southern California

Southwestern

Texas

Upper New York State

Wisconsin

E. P. ROZYCKI, B.A. (Buffalo) Grad. Student, University of Buffalo.

W. H. ROUDEBUSH, B.A. (Cincinnati) Aeronautical Research Scientist, Lewis Flight Propulsion Laboratory, Cleveland, Ohio.

MILDRED G. SCHLAPKOHL, Student, Northwestern University.

J. G. SHARP, Student, University of Detroit.

R. M. SIMONSON, Student, Gonzaga University.

SISTER FLORENCE MARIE (Knob), M.A. (Catholic) Instr., D'Youville College.

SISTER MARIE LORETTA (Bentz), M.A. (Catholic) Instr., Barry College, Miami, Fla.

SISTER MARY DECHANTAL, B.A. (St. Teresa's C.) Teacher, Central Catholic High School, Toledo, Ohio.

R. P. SMITH, B.A. (Drake) Grad. Assistant, Kansas State College.

S. M. SPENCER, JR., Ph.D. (Duke) Asso. Professor, Louisiana College, Pineville, La.

W. L. STROTHER, Ph.D. (Tulane) Asst. Professor, University of Miami.

M. C. SUMAN, Student, San Jose State College.

V. E. THOMAS, B.A. (American International) Grad. Student, University of Massachusetts.

J. S. THORPE, B.S. (Pittsburgh) Teacher, Shady Side Academy, Pittsburgh, Pa.

G. R. TRIMBLE, JR., M.A. (Delaware) Mathematician, International Business Machines Corporation, Endicott, N. Y.

R. E. WHEELER, Ph.D. (Kentucky) Asst. Professor, Florida State University.

N. K. WILLIAMSON, B.S. (Citadel) Asst. Professor, Newberry College.

R. A. WILLOUGHBY, Ph.D. (California) Asst. Professor, Georgia Institute of Technology.

R. P. WINTER, Ed.M. (St. Thomas C.) Instr., College of St. Thomas.

F. J. WITT, M.A. (Duke) Asst. Professor, Tennessee Polytechnic Institute.

E. M. WRIGHT, Ph.D. (Oxford) Professor and Head of Department, University of Aberdeen, Scotland.

PAUL YACYNYCH, Student, University of Pittsburgh.

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, Canada, August 31–September 1, 1953.

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS

INDIANA

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, Bemidji State Teachers College, October 10, 1953.

MISSOURI

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City, October, 1953.

PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

ROCKY MOUNTAIN

SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE

WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 8

CONTENTS

OCTOBER 1953

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

ELEMENTARY AND SECONDARY SCHOOL TRAINING IN MATHEMATICS*

S. S. CAIRNS, University of Illinois

1. Introduction. The general problem to which these comments pertain is that of adapting our public schools to the needs of society and the nation. Attention is restricted, however, primarily to the question of mathematical instruction, the field in which the writer is best qualified to offer advice. This is also a phase of the general problem to which a peculiar importance attaches as a result of the national emergency and the associated critical shortage of scientifically trained manpower. The shortage is expected to become worse before it gets better, since industry, the government, and the military services are making increasingly heavy demands with no signs of a corresponding increase in the supply. A substantial improvement in the situation could be effected by remedying some of the serious defects in elementary and secondary school mathematical training. It is the present object to support this assertion by discussing such defects and suggesting remedial measures.

2. The principal questions. Recent discussions of educational problems have raised subsidiary questions which becloud some of the main issues and present a danger to effective progress. These troublesome and almost irrelevant problems include (1) whether certain subjects are better taught now than at some previous time (2) who is to blame for some of the recognized shortcomings of our schools and (3) whether we are preparing most students to meet their expected needs in later life. We should rather concentrate on the magnitude and nature of our national needs, on the obstacles to meeting them and on methods for overcoming these obstacles.

3. Mathematical shortcomings of our schools. To commence with generalizations, our high schools are sadly deficient both in preparing students for college and in offering adequate education to those not bound for college. Our elementary schools, in turn, are deficient in preparing students for high school.

Children of average to superior abilities, in the earliest grades, are frequently (perhaps generally) offered no encouragement to proceed at their natural pace in learning those aspects of arithmetic which appeal to them. At a slightly later stage, they are introduced to the fundamental operations of addition, subtraction, multiplication and division, but they are generally not drilled in such basic necessities as the multiplication tables. As a consequence they enter high school severely handicapped, save for that small proportion who learn so readily that they need no drill. In high school, the mathematics courses hit a slow pace, partly because the students have inadequate backgrounds, partly because there is no genuine incentive for the schools to provide suitable courses

* This is a revision of a statement presented to the School Problems Commission of the State of Illinois, meeting at Carbondale, March 6, 1952.

or for the students to take them. As a consequence, the colleges lose a year or two of mathematical training by having to teach to the freshmen, and frequently to the sophomores, courses which properly belong in the high schools. For students in the humanities, this situation is less serious than for those in the sciences and engineering. The latter are delayed to such an extent in commencing their essential studies that (1) a net loss of at least a year is difficult to avoid before their training is complete and (2) their programs are so crowded as to preclude many of the broadening studies which should form part of their general education. The delay carries over into the graduate schools, where we find ourselves teaching large proportions of our students material which belongs in a good undergraduate program.

Before substantiating the foregoing remarks, let it be noted that exceptional schools exist, generally in certain urban areas, and that in other schools exceptional teachers can be found who somehow partially counteract the general difficulties.

While a wealth of data could be offered in support of the preceding statement of shortcomings, a selection will be made, for convenience and brevity, from experience with two categories of students at the University of Illinois: (1) students who enter the College of Liberal Arts and Sciences with deficiencies in mathematics and (2) students in the elementary schools program of the College of Education. There were 234 students in the first group in the period 1949 to 1951 and 268 in the second between 1947 and 1952. Deficiencies in mathematics imply only failure to have taken one full year each of high school algebra and plane geometry. The lack may be due to lack of opportunity or to a deliberate avoidance of the subject for one reason or another. There is evidence that a good proportion of the students with such deficiencies (constituting about 10% of the L. A. S. freshmen) are suitable, though poorly prepared, college material. Identical standardized arithmetic tests have been administered to both groups. The results reveal a shocking inability to handle elementary arithmetic. To mention a few examples from data supplied by Mr. Clarence Phillips of the Mathematics Department, only 41% of the first group and 59% of the second correctly figured one year's interest at 6% on \$175; the percentages of success were 34 and 55 in computing $7-6+2-4$, and 30% and 53% in arranging the numbers .40, 2.5 and .875 in order of magnitude. The difference between the two groups is due to the fact that the second group (1) had more mathematics in high school, (2) is more selective as to admission and (3) contains 77% seniors and graduate students, while the first group is almost all freshmen.

Of the students entering with mathematical deficiencies 50% fail to attain sophomore standing. This percentage is far out of line with the native abilities within the group and reveals the handicap of a student who is so poorly prepared by his high school.

The College of Education students just mentioned are required to take an arithmetic course in the Mathematics Department, intended to deepen their

understanding of what they will soon be teaching. Many of them are deplorably weak on the fundamentals, will hesitate over such things as eight times seven or seven plus six and will frankly express their easily understood fear of teaching arithmetic. This often takes place in the second semester of the senior year, after they have done practice teaching and a few months before they will be on the job, perhaps unconsciously transmitting their own aversions and lack of confidence to their students.

Turning to the College of Engineering, suffice it to remark that the inadequate mathematical preparation of the entering freshmen recently led to the establishment of a joint committee from that College, the College of Education and the Department of Mathematics. The work of this committee culminated in a pamphlet entitled *Mathematical Needs of Prospective Students at the College of Engineering of the University of Illinois*, which has been widely circulated among Illinois high schools. At present, a similarly composed committee, under the chairmanship of one of our University High School mathematics teachers, is studying means of adjusting the high school program to meet these needs. This cooperative effort is encouraging. It is to be hoped that the work of the committee will lead to widespread improvements in the teaching of mathematics and will serve as a model for cooperation elsewhere.

4. Underlying causes for shortcomings. This partly speculative section could run to great lengths. To avoid that, a few false principles will be listed, with brief comments and with no effort to estimate how widely these principles are accepted by those responsible for administering our schools.

a. *The theory that drill and deliberate memorizing must be avoided, especially in the lower grades.* This clearly works to the detriment of (1) learning the multiplication tables, (2) learning the alphabet at the proper time, (3) learning to spell, (4) learning the essentials of English grammar, (5) at a somewhat later stage developing a vocabulary when studying a foreign language and (6) acquiring the study habits demanded by effective college work. This theory is generally associated with the unwarranted belief that techniques will be incidentally acquired.

b. *The theory that local needs and desires should dominate in determining curricula, to the practical exclusion of needs on a national scale.*

c. *The theory that high schools should limit their programs to those skills and manipulations that some group of individuals finds necessary to the average adult.*

d. *The belief that the less successful students should be kept in the same classes with the more successful.*

e. *The aversion to competition among students.* This, and the previous item have a deadening effect on those who should be stimulated and encouraged.

5. A proposed guiding principle. To quote from a letter by Professor J. W.

specialized needs of parts of the student body when the latter effort is taken at the expense of a good program of general education." (p. 13.)

This is at variance with the guiding principle suggested above. It implies that, although talent is uniformly distributed throughout the population, students in certain communities are to be denied the opportunity for their full development.

ON SEQUENCES OF OPERATIONS IN COMPLETE VECTOR SPACES

I. S. GÁL, Institute for Advanced Study

1. Introduction. The first purpose of this paper is to describe how various results of classical analysis can be summarized in two general theorems concerning linear operations from a Banach space into a normed vector space. The two theorems are known as the *principle of uniform boundedness* and the *principle of condensation of singularities*. Our second object is to show how one can generalize these principles in the case of certain sequences of non-linear operations. In order to formulate our principles we start from simple examples, namely the summation of infinite sequences and the divergence problem of Fourier series of continuous functions.

2. The consistency of summation methods. As is well known, a real matrix-summation correlates with any sequence of real numbers $\{x_n\}$ a new sequence $\{u_m\}$ by means of a fixed infinite square matrix (a_{mn}) $(m, n=1, 2, \cdots)$; more precisely

$$u_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + \cdots \qquad (m = 1, 2, \cdots), \tag{1}$$

where the infinite series on the right hand side must be considered as a formal expression. It may diverge if the sequence $\{x_n\}$ is not sufficiently regular. However, it is natural to require of a summation method at least that it be efficient in the case of any convergent sequence, *i.e.* if $x_n \to \xi$ then all formal series (1) must converge and also $u_m \to \xi$ as $m \to \infty$. If a matrix-summation method (a_{mn}) has this basic property it is called a *consistent* or *regular* summation process. Perhaps the simplest non-trivial example of such consistent methods is the summation by arithmetical means.

There is a simple necessary and sufficient condition in order that a matrix-summation method should be consistent. Namely, according to a theorem of H. Steinhaus and O. Toeplitz [1], a summation process (a_{mn}) is regular if and only if

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(1) *all series* $\sum_n a_{mn}$ *converge and their sequence tends to* 1 *as* $m\to\infty$;
(2) $\sum_n |a_{mn}| \leqq H < \infty$, *where* $H>0$ *is independent of* m;
(3) *for every fixed value of* n, $a_{mn}\to 0$ *as* $m\to\infty$.

In the case of summation by the first or higher arithmetical means one can easily see that the above conditions are all satisfied.

The sufficiency of the conditions (1)–(3) can easily be proved using everyday tricks of analysis. The necessity of conditions (1) and (3) is evident: If (a_{mn}) is a consistent method we must have $u_m\to 1$ for the convergent sequence $(1, 1, 1, \cdots)$, that is to say $u_m=\sum_n a_{mn}\to 1$ as $m\to\infty$. Similarly u_m must tend to zero in the case of $(0, 0, \cdots, 0, 1, 0, 0, \cdots)$, in other words $u_m=a_{mn}\to 0$ as $m\to\infty$. The necessity of condition (2) causes more difficulties in the proof, but, as a matter of fact, this condition is the crux of the Steinhaus-Toeplitz theorem, and it is worthwhile to investigate it more closely.

Before, and not too long after the discovery of this theorem, other results were found in various parts of analysis each of them involving certain conditions which are very similar to (2). The first of these results is the famous theorem of H. Hahn [2] concerning interpolations, then followed Pólya's theorem on the convergence of various quadrature processes [3], H. Hahn's and I. Schur's theorems about the singular integrals and summation processes (respectively [4], [5]) and other results.

It becomes obvious very soon that all these conditions must be special cases of some very general rule, and, indeed, in 1927 St. Banach and H. Steinhaus published a theorem [6] which includes all former results of the above type. This theorem is commonly known today as the principle of uniform boundedness of linear operations or the Banach-Steinhaus theorem.

3. Linear operations in vector spaces. It is evident that the formulation of such a summarizing result requires the introduction of new mathematical concepts. In our case the discovery of the importance of *complete vector spaces* played a major rôle. A *vector space* E is a set of elements (denoted by x, y, z, *etc.*) for which two basic operations are defined: the *addition* of two arbitrary elements and the *multiplication* of an arbitrary element by real numbers. Thus $x+y$ and λx, with λ real, are defined as elements of E. It is natural to require $1\cdot x=x$, moreover it is assumed that these two operations satisfy the usual laws of arithmetic (commutativity and associativity of the two operations and the distributive law). Then it follows immediately that a *zero element* $\theta\in E$ exists, which has the property that $x+\theta=x$ and $0\cdot x=\theta$ for every $x\in E$.

Let us suppose that with every element x of the vector space E we have associated a non-negative number, which we shall call the *norm* of x and denote by $\|x\|$. We require that the norm should always be positive, except in the case of the zero element θ for which $\|\theta\|=0$, and that it should satisfy the triangle inequality, *i.e.* $\|x+y\|\leqq\|x\|+\|y\|$ for every x, $y\in E$, and finally that $\|\lambda x\|=|\lambda|\cdot\|x\|$ for every real λ. If a vector space E has these properties we say

that the space is *normed.* In this way our space E becomes a *metric space* because we can define the *distance* of any two elements x, $y \in E$ as $d(x, y) = \|x-y\|$; obviously this function $d(x, y)$ has the usual properties of a distance function. Now we can introduce a kind of analysis in the space, and we may speak about *convergent sequences, infinite series, limit elements* and so on.

We say that a sequence of elements $x_n \in E$ converges to ξ if $\|x_n - \xi\| \to 0$ as $n \to \infty$. The triangle inequality implies that a convergent sequence always satisfies the *Cauchy convergence criterion*, that is $\|x_m - x_n\| \to 0$ as m, $n \to \infty$ simultaneously. However the converse is not true in general, because there are vector spaces where the Cauchy condition is satisfied for some sequence $\{x_n\}$ but no element $\xi \in E$ exists such that $\|x_n - \xi\| \to 0$ $(n \to \infty)$ would be true. If a normed vector space has the property that the Cauchy condition always implies the existence of a limit element ξ, then it is called a *complete vector space* or a *Banach space.* The importance of these spaces was discovered almost simultaneously by St. Banach [7], H. Hahn [4], T. H. Hildebrandt [8], and N. Wiener [9], but Banach was the one who developed their theory in detail.

Let us consider now two normed vector spaces E and E'. If a mapping of E into E' is given, *i.e.*, if to every element x of E there corresponds an element $u = u(x)$ of the second space E', we speak about an *operation* $u(x)$ from E into E', and we say that $u(x)$ is the image of the element x. The operation $u(x)$ is called *additive* if $u(x+y) = u(x) + u(y)$ for every couple x, y. Another important concept is the boundedness of an operation: $u(x)$ is *bounded* if there exists a positive constant M such that $\|u(x)\| \leqq M \cdot \|x\|$ for every $x \in E$. It is easy to see that an additive and bounded operation satisfies the relation $u(\lambda x) = \lambda \cdot u(x)$ for every real λ and $x \in E$, *i.e.*, one may "multiply out" with a numerical factor λ. This is the reason that additive and bounded operations are called *linear.* It follows from the boundedness condition that $\text{l.u.b.}_{\|x\|=1} \|u(x)\|$ is finite, and one can see immediately that

$$\underset{\|x\|=1}{\text{l.u.b.}} \|u(x)\| = \underset{\|x\|\leqq 1}{\text{l.u.b.}} \|u(x)\| = \underset{x \neq 0}{\text{l.u.b.}} \frac{\|u(x)\|}{\|x\|} \cdot$$

This important number is the *norm of the operation* $u(x)$, usually denoted by $|u|$.

4. The principle of uniform boundedness. Now we are able to state the first of our principles in the generality in which it was proved by Banach and Steinhaus: *Consider a sequence of linear operations* $u_m(x)$ $(m = 1, 2, \cdots)$ *from a fixed complete vector space E into one common or into different normed vector spaces. If the sequence* $\{u_m(x)\}$ *is such that* $\lim\sup_{m\to\infty} \|u_m(x)\| < \infty$ *for every element* $x \in E$, *i.e. if the image points of each fixed point x remain in a finite neighborhood of the origin, then* $|u_m| \leqq H \leqq \infty$ $(m = 1, 2, \cdots)$, *that is to say the sequence of the norms of the operations is bounded.*

We can easily prove this theorem by contradiction: Let us suppose that all

the conditions are satisfied, but the conclusion is not true, *i.e.*,

$$\limsup_{m\to\infty} |u_m| = +\infty. \tag{2}$$

Then we shall construct an element $x \in E$ such that $\limsup_{m\to\infty} \|u_m(x)\| = +\infty$, which would contradict one of our hypotheses. To do so, we first notice that for every operation $u_m(x)$ there exists at least one element $x_m \in E$ for which

$$|u_m| \geqq \|u_m(x_m)\| \geqq \tfrac{1}{2}|u_m| \quad \text{and} \quad \|x_m\| = 1. \tag{3}$$

This follows immediately from the definition of the norm of the operation $u_m(x)$.

To start the construction, we notice that according to (2) and (3) we can choose an index $m_1 \geqq 1$ such that $\|u_{m_1}(x_{m_1})\| \geqq \frac{1}{2}|u_{m_1}| \geqq 1$. Let us suppose that the indices $m_1 < m_2 < \cdots < m_{i-1}$ $(i \geqq 2)$ have already been determined in a convenient way. We shall now define m_i: first let

$$y_i = x_{m_1} + \frac{x_{m_2}}{2|u_{m_1}|} + \cdots + \frac{x_{m_{i-1}}}{2^{i-2}|u_{m_{i-2}}|}.$$

Since the conditions of the principle are supposed to be true we have

$$\|u_m(y_i)\| \leqq H(y_i) < +\infty \qquad (m = 1, 2, \cdots), \tag{4}$$

for a sufficiently large value of $H(y_i) > 0$. Now we choose the index $m_i > m_{i-1}$ so large that

$$\|u_{m_i}(x_{m_i})\| \geqq 2^{i-1} \cdot |u_{m_{i-1}}| \cdot (H(y_i) + i + 1). \tag{5}$$

Since the right hand side is independent of m_i, this choice is possible by (2) and (3). In this way we obtain a subsequence $\{u_{m_i}(x)\}$ $(i = 1, 2, \cdots)$ which satisfies the relations (4) and (5).

Finally we define x in the form of an infinite series

$$x = x_{m_1} + \frac{x_{m_2}}{2|u_{m_1}|} + \cdots + \frac{x_{m_i}}{2^{i-1}|u_{m_{i-1}}|} + \cdots.$$

The partial sums of this series were denoted earlier by y_i. It will be convenient to abbreviate also the remainder terms writing

$$z_i = \frac{x_{m_{i+1}}}{2^i \cdot |u_{m_i}|} + \frac{x_{m_{i+2}}}{2^{i+1} \cdot |u_{m_{i+1}}|} + \cdots, \qquad (i = 1, 2, \cdots).$$

Thus we have for every $i \geqq 2$

$$x = y_i + \frac{x_{m_i}}{2^{i-1} \cdot |u_{m_{i-1}}|} + z_i.$$

The convergence of these series is evident, because the space E is complete and the Cauchy criterion is satisfied. More precisely, using $\|x_i\| = 1$ and $\|u_{m_i}\| > 1$ we can deduce the inequality $|u_{m_i}| \cdot \|z_i\| < 1$ $(i = 2, 3, \cdots)$, and (3) and (5)

show that $|u_{m_i}| \to \infty$ as $i \to \infty$.

Now it is easy to prove that $\lim\sup_{m\to\infty} \|u_m(x)\| = +\infty$ which contradicts the hypotheses of our principle. For, we obtain from (6)

$$\|u_{m_i}(x)\| = \left\| u_{m_i}(y_i) + u_{m_i}\left(\frac{x_{m_i}}{2^{i-1}\cdot |u_{m_{i-1}}|}\right) + u_{m_i}(z_i) \right\|$$
$$\geqq \frac{\|u_{m_i}(x_{m_i})\|}{2^{i-1}\cdot |u_{m_{i-1}}|} - \|u_{m_i}(y_i)\| - \|u_{m_i}(z_i)\|.$$

Hence using (4), (5) and the inequality $\|u_{m_i}(z_i)\| \leqq |u_{m_i}| \cdot \|z_i\| < 1$ we get

$$\|u_{m_i}(x)\| > (H(y_i) + i + 1) - H(y_i) - 1 = i$$

for every $i = 1, 2, \cdots$. Thus, indeed $\lim\sup_{m\to\infty} \|u_m(x)\| = +\infty$, which completes the proof of the principle of uniform boundedness.

5. Examples and applications. Let us now return to the consistency question of matrix-summation methods. As the first application of our principle we shall prove the necessity of condition (2) of the Steinhaus-Toeplitz theorem: Let E be the set of all convergent real sequences $\{x_n\}$ which we shall from now on denote by small Roman letters (x, y, *etc.*) instead of $\{x_n\}$, $\{y_n\}$, *etc.* We define the sum of two arbitrary sequences and the product of a sequence and a real number in the usual way as $x+y=\{x_n\}+\{y_n\}=\{x_n+y_n\}$ and $\lambda x = \lambda\cdot\{x_n\} = \{\lambda x_n\}$, respectively. Obviously $x+y\in E$ and $\lambda x \in E$, and one can also see that these operations satisfy the usual requirements. Thus the set of all convergent real sequences forms a vector space E. The space will be normed if we introduce the norm $\|x\| = \|\{x_n\}\| = \text{l.u.b.}_{(n)} |x_n|$. This norm is finite because the sequence x is supposed to be convergent. It is easy to verify that this normed vector space is complete.

A matrix-summation (a_{mn}) can now be considered as a sequence of linear operations

$$u_m(x) = u_m(\{x_n\}) = a_{m1}x_1 + a_{m2}x_2 + \cdots, \quad (m = 1, 2, \cdots),$$

from the space E into the space of all real numbers R. Let us suppose that $\{u_m(x)\}$ $(m\to\infty)$ converges to the same limit as the arbitrary convergent sequence $\{x_n\}$. Hence the sequence $\{u_m(x)\}$ is *a priori* bounded for every element $x = \{x_n\} \in E$, so that the main condition of the principle of uniform boundedness is satisfied. Since E is complete we can apply the principle and it follows that the sequence of the norms $\{|u_m|\}$ is bounded.

Let us now evaluate the norm of $u_m(x)$, and thus obtain a necessary condition for the regularity of the summation process (a_{mn}). Since the sequence

$$y_n = (\text{sign } a_{m1}, \text{sign } a_{m2}, \cdots, \text{sign } a_{mn}, 0, 0, \cdots) \to 0$$

belongs to E and has norm 1 we see that

$$|u_m| \geqq \|u_m(y_n)\| = |a_{m1}| + |a_{m2}| + \cdots + |a_{mn}|.$$

$n \geqq 1$ being arbitrary it follows that $|u_m| \geqq \sum_{n=1} |a_{mn}|$. On the other hand it is obvious that

$$\|u_m(x)\| = \left| \sum_{n=1}^{\infty} a_{mn} x_n \right| \leqq \sum_{n=1}^{\infty} |a_{mn}|,$$

provided $|x_n| \leqq 1$ for $n=1, 2, \cdots$, *i.e.*, if $\|x\| \leqq 1$. Finally we see that $|u_m| = \sum_{n=1}^{\infty} |a_{mn}|$, and we have the necessity of condition (2).

As a second example we consider the divergence problem of the Fourier series of continuous functions at a given point. It has been known for quite a long time that there exist periodic functions with period 2π which are continuous everywhere, but whose Fourier series diverge in a prescribed point. Of course one can assume that the prescribed point is the origin $t=0$. The first example of such functions was given by P. du Bois Reymond [10], and the simplest known construction is due to L. Fejér [11]. H. Lebesgue [12] gave a proof of the existence of such functions which is very remarkable from our point of view. For what Lebesgue proves is nothing but the converse of the principle of uniform boundedness for the special case of Fourier series. It is unfortunate that Lebesgue did not investigate in what generality his method of proof works, for then the principle of uniform boundedness would have been known much earlier.

Having proved the general principle we can go on and use it to show the existence of periodic continuous functions with period 2π whose Fourier series diverge at the origin. For, the m-th partial sum of the Fourier series of $x(t)$ can be written in the form of a Dirichlet integral, and if $t=0$

$$s_m(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t) D_m(t) dt,$$

where for simplicity $D_m(t) = (\sin (m+\frac{1}{2})t)/\sin \frac{1}{2}t$. (See for example A. Zygmund's textbook on trigonometrical series, pp. 20–21.) Evidently the set of all 2π-periodic and continuous functions $x(t)$ forms a vector space E which can be normed by $\|x\| = \max |x(t)|$. It is easy to see that E is complete. Hence we can consider the value of the m-th partial sum at $t=0$ as a linear operation $s_m(x)$ from the space E into the space of all real numbers R. If $\{s_m(x)\}$ $(m \to \infty)$ were convergent for every element $x = x(t) \in E$, then according to the principle of uniform boundedness the sequence $\{|s_m|\}$ would be bounded. However we shall immediately see that this is not the case, so that there exist continuous functions whose Fourier series diverge at the origin.

It is obvious that

$$|s_m| \leqq L_m = \frac{1}{2\pi} \int_0^{2\pi} |D_m(t)| dt.$$

On the other hand, let $\epsilon<\pi/(2m+1)$ and consider the 2π-periodic and continuous function

$$x_\epsilon(t) = \begin{cases} \operatorname{sign} D_m(t) \quad \text{for} \quad \dfrac{2k}{2m+1} + \epsilon \leqq t \leqq \dfrac{2(k+1)}{2m+1} - \epsilon \\ \text{linear otherwise} \quad (k = 0, \pm 1, \pm 2, \cdots). \end{cases}$$

We have

$$\begin{aligned} \|s_m(x_\epsilon)\| &\geqq \frac{1}{2\pi}\int_0^{2\pi} |D_m(t)|\,dt - \frac{1}{\pi}\sum_{k=0}^{2m}\int_{(2k\pi)/(2m+1)-\epsilon}^{(2k\pi)/(2m+1)+\epsilon} |D_m(t)|\,dt \\ &\geqq L_m - \frac{2\epsilon}{\pi}(2m+1)^2, \end{aligned}$$

because $|D_m(t)| = |1+2\cos t + \cdots + 2\cos mt| \leqq 2m+1$. Since $\epsilon>0$ is arbitrarily small, we obtain $|s_m| \geqq \lim_{\epsilon\to 0} \|s_m(x_\epsilon)\| = L_m$ and finally the norm of the operation $s_m(x)$ turns out to be $|s_m| = L_m$. This quantity is called the m-th *Lebesgue constant* and it plays an important rôle in the theory of Fourier series.

A rough estimate is now sufficient to prove that $|s_m| \to \infty$: for, we have

$$L_m > \frac{1}{2\pi}\sum_{k=0}^{2m}\int_{\alpha_k}^{\beta_k} \frac{|\sin(m+\frac{1}{2})t|}{|\sin\frac{1}{2}t|}\,dt \geqq \frac{\sqrt{2}}{4\pi}\sum_{k=0}^{2m}\int_{\alpha_k}^{\beta_k} \frac{1}{|\sin\frac{1}{2}t|}\,dt,$$

where for the sake of simplicity $\alpha_k = (4k+1)\pi/(4m+2)$ and $\beta_k = (4k+3)\pi/(4m+2)$, because $\sin t \geqq 1/\sqrt{2}$ in the interval $(\pi/4, 3\pi/4)$. Using $|\sin t| \leqq t$ $(t \geqq 0)$ it follows that

$$L_m > \frac{\sqrt{2}}{\pi}\sum_{k=0}^{2m}\frac{1}{4k+3}.$$

This last inequality clearly shows that $L_m \to \infty$ as $m \to \infty$. Thus there exist continuous functions whose Fourier series diverge at any prescribed point. We note briefly that a precise estimate for L_m due to F. Fejér [**13**], states $L_m = 4/\pi^2 \log m + O(1)$.

6. **The principle of condensation of singularities.** Instead of giving further applications of the principle of uniform boundedness (as for example the summability of Fourier series by arithmetic means), let us continue our investigations concerning the divergence problem of Fourier series. After the preceding considerations one may expect that the divergence property is much more general than the result discussed above about divergence in finitely many prescribed points. Indeed, P. du Bois Reymond [**10**] proved the following: *Let S be an arbitrary countable set of real numbers t $(0 \leqq t \leqq 2\pi)$. Then there exists a periodic continuous function with period 2π such that its Fourier series diverges at every $t \in S$.* Thus, for example, we can construct continuous functions whose set of divergence points is everywhere dense.

The theorem can be proved in several ways. However all proofs have one thing in common: they all use the fact that there exist continuous functions with Fourier series diverging at a point (or what is the same, $L_m \to \infty$), but only very few of the other properties of Fourier series are used. These facts suggest that the theorem can be extended to a greater generality than the case of Fourier series. As a matter of fact, the method used in the proofs goes back at least to the time of Riemann (Habilitationsschr., 1854) and was later improved by several authors. H. Hankel [14] was the first one who considered the problem independently of its origin and tried to present the method and the results obtained by it in the greatest possible generality. He also introduced the name: *principle of condensation of singularities* (Condensation der Singularitäten).

The introduction of the concept of complete vector spaces was the step necessary for a better understanding. In fact, St. Banach and H. Steinhaus [6] succeeded in proving a general principle which includes the theorem on Fourier series and many other results and saves the application of the method in each particular case. It is the following: *Consider a double-sequence of linear operations* $u_{mn}(x)$ $(m, n=1, 2, \cdots)$ *from a complete vector space* E *into arbitrary normed vector spaces* E'_{mn}. *Suppose that for every* $m \geqq 1$ *there exists an element* x_m *of the space* E *such that* $\limsup_{n\to\infty} \|u_{mn}(x_m)\| = +\infty$. *Then there exists a common element* $x \in E$ *such that* $\limsup_{n\to\infty} \|u_{mn}(x)\| = +\infty$ *for every* $m \geqq 1$.

The connection of this theorem with the divergence of Fourier series on enumerable sets is evident: Let $\{t_1, t_2, \cdots, t_m, \cdots\}$ be the set given in advance and let $s_{mn}(x)$ denote the n-th partial sum of the Fourier series of $x(t)$ in the point t_m. According to the principle of uniform boundedness there exists a continuous function $x_m(t)$ whose Fourier series diverges to infinity at the given point t_m. Thus the conditions of our second principle are satisfied and it follows that there exists a continuous function $x(t)$ such that all sequences $\{s_{mn}(x)\}$ (m fixed, $n \to \infty$) diverge, in other words the Fourier series of $x(t)$ diverges in all points t_m $(m \geqq 1)$.

This example shows clearly that in the applications one usually applies the two principles simultaneously. First, using the principle of uniform boundedness we show that the conditions of the principle of condensation are satisfied, and then we apply the second principle. This is the situation, for instance, in the case of the *Lagrange interpolation* which we shall investigate more closely. The proof of the principle itself is similar to the proof given in section **4**, and for this reason we shall omit it.

7. The divergence of the Lagrange interpolation. For simplicity let us consider only the case of the equidistant interpolation of continuous functions $x(t)$ $(0 \leqq t \leqq 1)$. As is well known, we define the n-th Lagrange approximation of $x(t)$ as

$$l_n(x; t) = \sum_{k=0}^{n} x\left(\frac{k}{n}\right)\lambda_k^{(n)}(t), \qquad (n = 1, 2, \cdots),$$

where

$$\lambda_k^{(n)}(t) = (-1)^{n-k}\frac{n^n}{k!(n-k!)}\cdot t\left(t-\frac{1}{n}\right)\left(t-\frac{2}{n}\right)\cdots(t-1)\left(t-\frac{k}{n}\right)^{-1}.$$

Let C denote the space of all continuous functions $x(t)$, $0\leqq t\leqq 1$. If we introduce the norm $\|x\|=\|x(t)\|=\max|x(t)|$, then C becomes a complete vector space. Now we can consider $l_n(x;t)$ as a linear operation from C into C itself, or if we wish, we can also say that for every fixed value of t, $(0\leqq t\leqq 1)$, $l_n(x;t)$ is a linear operation from C into the space of real numbers R. Let us make the second choice, so that we obtain a sequence of operations $\{l_n(x;\tau)\}$ $(n\to\infty)$ depending on the parameter τ $(0\leqq\tau\leqq 1)$.

The norm of the operation $l_n(x;\tau)$ can be determined very easily: It is obvious that $|l_n(\tau)|\leqq\sum_0^n|\lambda_k(\tau)|$. On the other hand let $x_n=x_n(t)=\text{sign }\lambda_k(t)$ for $t=k/n$, $(k=0, 1, \cdots, n)$, and linear otherwise. We have $x_n\in C$, $\|x_n\|=1$, and hence $|l_n(\tau)|\geqq\|l_n(x_n;\tau)\|=\sum_0^n|\lambda_k(\tau)|$. Thus it follows that $|l_n(\tau)|=\sum_0^n|\lambda_k(\tau)|$. We next investigate the boundedness of the sequence $\{|l_n(\tau)|\}$. It will turn out that $\limsup_{n\to\infty}|l_n(\tau)|=+\infty$ for every $0<\tau<1$. According to the principle of uniform boundedness therefore there exists a continuous function whose Lagrange interpolation diverges to infinity for any given τ, $0<\tau<1$.

Let us suppose that $\tau\neq\frac{1}{2}$ and $0<\tau<1$. In order to estimate $|l_n(\tau)|$ we choose the integer $a=a_n(\tau)$ so that $a/n<\tau<(a+1)/n$. If τ is a rational number this inequality gives some restriction also on the possible choice of $n\geqq 1$. However we shall get rid of this restriction immediately. According to the definition of $\lambda_k(\tau)$ we have

$$\begin{aligned}|\lambda_k(\tau)| &\geqq \frac{n^n}{k!(n-k)!}\cdot\frac{a}{n}\cdot\frac{a-1}{n}\cdot\;\cdots\;\cdot\frac{1}{n}\left(\tau-\frac{a}{n}\right)\left(\frac{a+1}{n}-\tau\right)\\ &\quad\cdot\frac{1}{n}\cdot\frac{2}{n}\cdot\;\cdots\;\cdot\frac{n-a-1}{n}\Big/\left|\tau-\frac{k}{n}\right|\\ &\geqq\frac{a!(n-a)!}{k!(n-k)!}\left(\tau-\frac{a}{n}\right)\left(\frac{a+1}{n}-\tau\right)\\ &\geqq\frac{1}{2n}\cdot\frac{a!(n-a)!}{k!(n-k)!}\underset{0\leqq m\leqq n}{\text{minimum}}\left|\tau-\frac{m}{n}\right|,\end{aligned}$$

because $|\tau-k/n|\leqq 1$, and $(n-a-1)!n\geqq(n-a)!$. Evidently this inequality holds for every choice of $n\geqq 1$. Using the relation $\sum_0^n\binom{n}{k}=2^n$ we obtain from the above inequality

$$|l_n(\tau)|=\sum_{k=0}^n|\lambda_k(\tau)|\geqq\frac{2^n}{2n}\cdot\frac{a!(n-a)!}{n!}\underset{0\leqq m\leqq n}{\text{minimum}}\left|\tau-\frac{m}{n}\right|.$$

Using the elementary inequalities $(\nu/e)^\nu < \nu! < 4\nu(\nu/e)^\nu$ we see that

$$|l_n(\tau)| \geqq \frac{2^n}{8n^2}\cdot\frac{a^a(n-a)^{n-a}}{n^n} \underset{0\leqq m\leqq n}{\text{minimum}} \left|\tau - \frac{m}{n}\right|$$

$$= \frac{1}{8n^2}\,(2\alpha(1-\alpha)^{1-\alpha})^n \underset{0\leqq m\leqq n}{\text{minimum}} \left|\tau - \frac{m}{n}\right|,$$

where $\alpha=\alpha_n(\tau)=a/n$. Since $\tau\neq\frac{1}{2}$ and $0\leqq \tau - a/n = \tau-\alpha_n < 1/n$, we have $|\alpha_n-\frac{1}{2}| > \frac{1}{2}|\tau-\frac{1}{2}|>0$, provided n is large enough. Moreover, since the convex function $x(t)=2t^t(1-t)^{1-t}$, $0\leqq t\leqq 1$, has its minimum at $t=\frac{1}{2}$ with $x(\frac{1}{2})=1$, we see that there is a constant $\eta=\eta(\tau)>1$ such that $2\alpha_n^{\alpha_n}(1-\alpha_n)^{1-\alpha_n}\geqq\eta>1$ holds for every sufficiently large $n=1, 2, \cdots$. Therefore for every sufficiently large n

$$|l_n(\tau)| > \frac{\eta^n}{8n^2} \underset{0\leqq m\leqq n}{\text{minimum}} \left|\tau - \frac{m}{n}\right|; \qquad \eta = \eta(\tau) > 1.$$

To finish the proof we show that there are infinitely many n such that $\min_{(m)} |\tau - m/n| > 1/4n^2$. For, let us suppose that n is fixed, $n=p\geqq 2$, and $\min |\tau-m/p| = |\tau-m_1/p|$ and $\min|\tau-m/(p+1)| = |\tau-m_2/(p+1)|$. Since $0<\tau<1$ we have $1\leqq m_1\leqq p-1$ and $1\leqq m_2\leqq p$, provided p is large enough; $p\geqq n_0(\tau)\geqq 2$.

Since p and $p+1$ are relatively prime, it follows that $|m_1(p+1)-m_2p|\geqq 1$. Consequently

$$\left|\tau-\frac{m_1}{p}\right| + \left|\tau - \frac{m_2}{p+1}\right| \geqq \left|\frac{m_1}{p}-\frac{m_2}{p+1}\right| \geqq \frac{1}{p(p+1)},$$

and so

$$\min_{(m)}\left|\tau-\frac{m}{p}\right| + \min_{(m)}\left|\tau-\frac{m}{p+1}\right| \geqq \frac{1}{2p^2}\cdot$$

Hence for every $p\geqq 2$, $n=p$ or $n=p+1$ satisfies the required inequality, and so $|l_n(\tau)|\geqq \eta^n/32n^4$, $(\eta=\eta(\tau)>1)$, is valid for infinitely many indices. Consequently $\limsup_{n\to\infty} |l_n(\tau)| = \infty$ $(\tau\neq 0, \frac{1}{2}, 1)$. If $\tau=\frac{1}{2}$ one must use a slightly different estimation and one can show that $|l_{2n+1}(\frac{1}{2})|\sim \log n/\pi$.

Using the principle of uniform boundedness we see that for every $0<\tau<1$ there exists an element $x_\tau\in C$ such that $\limsup_{n\to\infty}|l_n(x_\tau;\tau)| = +\infty$. Therefore we can apply the principle of condensation and obtain: Given an arbitrary enumerable set of abscissae t, $0<t<1$, there exists a continuous function whose equidistant Lagrange interpolation diverges at each point of the prescribed set. It is easy to see that at the points $t=0$ and $t=1$ the interpolation is convergent.

8. Generalizations of the principles. Although our principles are stated

in a general form which includes many theorems as special cases, it is nevertheless desirable to obtain further extensions. Let us consider first the principle of condensation: Using Fejér's example it is a simple matter to construct a continuous function $x(t)$ with Fourier series divergent in a non-denumerable set of points. It is conjectured that there exist continuous functions whose Fourier series diverge everywhere. In the case of the Lagrange interpolation the corresponding statement has already been proved by G. Grünwald [15]. However we have as yet been unable to clear up the situation completely. The most general form of the principle of condensation to date is due to W. Orlicz [16] and states: *Consider a sequence of linear operations $u_n(x;\tau)$ depending on a continuous parameter $\tau(0 \leqq \tau \leqq 1)$. Suppose that the operations are continuous with respect to τ. Suppose moreover that for every $\tau(0 \leqq \tau \leqq 1)$ there exists an x_τ such that* $\limsup_{n\to\infty} \|u_n(x_\tau;\ \tau)\| = +\infty$. *Then there exists also an element x so that* $\limsup_{n\to\infty} \|u_n(x;\tau)\|$ *holds for a non-denumerable (perfect) set of the values of τ.*

It is easier to generalize our principles in another direction, namely by assuming instead of linearity some less restrictive conditions on the operations $u_m(x)$ and $u_{mn}(x)$: If we investigate the original proof of Banach and Steinhaus we see that the principle of uniform boundedness can be extended to the case of bounded, homogeneous and subadditive (convex) operations. By definition these operations satisfy the conditions: $\|u(x)\| \leqq M\|x\|$, $u(\lambda x) = \lambda u(x)$ and $\|u(x+y)\| \leqq \|u(x)\| + \|u(y)\|$ for every choice of x, $y \in E$. It is easy to verify that these conditions are indeed less restrictive than linearity. We can go still somewhat further and replace homogeneity by the weaker condition $\|u(\lambda x)\| = |\lambda| \cdot \|u(x)\|$, but this is about the most general form of the principle one can obtain by the method of Banach and Steinhaus.

There is a recent result concerning almost everywhere approximation of measurable functions by polynomials [17] which is evidently closely related to the principle of uniform boundedness. However this last result does not follow from the form of the principle which is stated above, but suggests that the method of section 4 might be applied successfully in this case. This method can indeed be used to prove a still further extension of the principle [18]: A sequence of bounded operations $u_m(x)$ is called *asymptotically subadditive* if $\|u_m(\lambda x)\| = |\lambda| \cdot \|u_m(x)\|$ for every real λ, and

$$\|u_m(x + y)\| \leqq \|u_m(x)\| + O(|u_m| \cdot \|y\|)$$

uniformly for every x, y with $\|x\|$, $\|y\| \leqq 1$, and if also

$$\operatorname*{g.l.b.}_{\|y\| \leqq 1} [\|u_m(x + y)\| + \|u_m(x)\| - \|u_m(y)\|] \geqq o(|u_m|)$$

as $m \to \infty$, but not necessarily uniformly in $x \in E$. (In other words the left hand side of the last formula can be negative, but must be greater than $-c_m|u_m|$ where $c_m \to 0$ as $m \to \infty$. The O has its usual meaning, *i.e.* it denotes a function of $|u_m|$ and $\|y\|$ which in absolute value is less than $c|u_m| \cdot \|y\|$, where $c > 0$ denotes a constant.)

The result is the following: If the sequence of operations $u_m(x)$ is asymptotically subadditive, then the principle of uniform boundedness is true. The situation is similar in the case of the principle of condensation [18]: If every single sequence $\{u_{mn}(x)\}$ (m fixed, $n \to \infty$) is asymptotically subadditive, then the principle can be applied.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, College of St. Thomas

The following results of the thirteenth William Lowell Putnam Mathematical Competition held March 23, 1953, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Norman Bauman, Marshall L. Freimer and Ralph M. Krause; to each of these a prize of forty dollars is awarded.

There was a tie for second place between The City College, New York, New York, and Cornell University, Ithaca, New York. A prize of two hundred and fifty dollars is awarded the Departments of Mathematics of each of these institutions. The members of the team from The City College, New York, New York, were Arnold M. Benson, Irwin Friedman and Samuel Jacob Klein; to each of these a prize of twenty-five dollars is awarded. The members of the team from Cornell University, Ithaca, New York, were Leonard Evens, David Hertzig and Daniel J. Kleitman; to each of these a prize of twenty-five dollars is awarded.

The fourth prize, one hundred dollars is awarded to the Department of Mathematics of the University of California, Berkeley, California. The members of the team were James Brown Herreshoff 4th, Eva Marianne Kallin and Harold Weitzner; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were Norman Bauman, Harvard University; Marshall L. Freimer, Harvard University; James Brown Herreshoff 4th, University of California (Berkeley); Samuel Jacob Klein, The City College (New York); Tai Tsun Wu, University of Minnesota. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, were George A. Baker, California Institute of Technology; William Morton Kahan, University of Toronto; Henry J. Landau, Harvard University; John S. Lew, Yale University; Richard Gordon Swan, Princeton University. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: Brooklyn College, Brooklyn, New York, the members of the team being Donald Solitar, Aaron Stein, Jack Towber; California Institute of Technology, Pasadena, California, the members of the team being George A. Baker, Robert D. Ryan, David F. Stevens; Columbia College, New York, New York, the members of the team being Lee Abramson, S. David Berkowitz, Elihu Lubkin; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being Louis de Branges, Redmond O'Brien, Monroe Weinstein.

Fifteen individuals were given honorable mention. The names are listed in alphabetical order. Leonard E. Baum, Harvard University; Arnold M. Benson, The City College (N. Y.); David S. Berkowitz, Columbia College (N. Y.); Robert Hamilton Boyer, Carnegie Institute of Technology; Glen Eugene Bredon, Stanford University; Eugene Butkov, University of British Columbia; Louis de Branges, Massachusetts Institute of Technology; Wilfred Keith Hastings, University of Toronto; Daniel J. Kleitman, Cornell University; Ralph M. Krause, Harvard University; John Randolph Manning, Ursinus College (Collegeville, Pa.); Benjamin Muckenhaupt, Harvard University; Gion Carlo Rota, Princeton University; Donald Solitar, Brooklyn College; Harold Weitzner, University of California (Berkeley).

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: Arizona State College, Brooklyn College, Brown University, California Institute of Technology, Carnegie Institute of Technology, Carleton College, Catholic University of America, City College (N. Y.), Coe College, College of St. Thomas, Columbia College (N. Y.), Cooper Union, Cornell University, Harvard University, Haverford College, Holy Cross College, Iowa State College, Kenyon College, Knox College, LaSalle College, Laval University, Lynchburg College, Massachusetts Institute of Technology, McGill University, McMaster University, Memphis State College, New York University, Oregon State College, Polytechnic Institute of Brooklyn, Princeton University, Queen's University, Radcliffe College, Rosemont College, Rutgers University, Saint Joseph College, Stanford University, Syracuse University, U. S. Naval Academy, University of British Columbia, University of California (Berkeley), University of Detroit, University of Illinois, University of Miami, University of Minnesota, University of Rochester, University of Toronto, Ursinus College, Washington University (Saint Louis), Yale University,

The following additional colleges and universities entered individual contestants only: Barat College, Boston University, Clemson College, Hartwick College, Lincoln Memorial University, Loyola College, Ohio State University, Pomona College, Purdue University, University of Arizona, University of Kentucky, University of Omaha, University of Washington (Seattle), University of Wisconsin.

A total of 256 undergraduates representing 63 institutions took part in the Competition.

The departments of mathematics of any of the competing institutions may obtain the rankings of their individual contestants (except that the relative rankings of the first five will not be given) by writing to the Director. These rankings may now be communicated to the individual contestants by their departments. Any other departments of mathematics may obtain individual rankings for the purpose of selecting graduate students.

Participants in the Competition were given the following lists of problems:

Part I

MORNING SESSION: 9:00 A.M. TO 12:00 NOON

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. Prove that, for every positive integer n,

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$$

is more than $\frac{2}{3}n\sqrt{n}$ and less than

$$\frac{4n+3}{6}\sqrt{n}.$$

2. Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color.
3. If x_1, x_2, x_3 are real numbers and the sum of any two is greater than the third, show that

$$\frac{2}{3}\sum_{i=1}^{3} x_i \sum_{i=1}^{3} x_i^2 > \sum_{i=1}^{3} x_i^3 + x_1x_2x_3.$$

4. From the identity

$$\int_0^{\pi/2} \log \sin 2x dx = \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx + \int_0^{\pi/2} \log 2 dx,$$

deduce the value of

$$\int_0^{\pi/2} \log \sin x dx.$$

5. Let P be a point from which three distinct normals can be drawn to a parabola. Show that the sum of the angles which these three normals make with the axis exceeds by a multiple of π the angle which the line joining P to the focus makes with the axis.
6. Show that the sequence

$$\sqrt{7},\ \sqrt{7-\sqrt{7}},\ \sqrt{7-\sqrt{7+\sqrt{7}}},\ \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}},\ \cdots$$

converges, and evaluate the limit.
7. Assuming that the roots of $x^3+px^2+qx+r=0$ are all real and positive, find

the relation between p, q and r which is a necessary and sufficient condition that the roots may be the cosines of the angles of a triangle.

PART II

AFTERNOON SESSION: 2:00 TO 5:00 P.M.

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. Is the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^{(n+1)/n}}$$
convergent? Prove your statement.
2. Let $a_0, a_1, \cdots, a_n$ be real numbers and let $f(x)=a_0+a_1x+\cdots+a_nx^n$. Suppose that, for every integer i, $f(i)$ is an integer. Prove that $n!\cdot a_k$ is an integer for each k.
3. Solve the equations
$$\frac{dy}{dx} = z(y+z)^n \qquad \frac{dz}{dx} = y(y+z)^n,$$
given the initial conditions $y=1$ and $z=0$ when $x=0$.
4. Determine the equation of a surface in three dimensional cartesian space which has the following properties: (a) it passes through the point (1, 1, 1); and (b) if the tangent plane be drawn at any point P, and A, B and C are the intersections of this plane with the x, y and z axes respectively, then P is the orthocenter (intersection of the altitudes) of the triangle ABC.
5. Show that the roots of $x^4+ax^3+bx^2+cx+d=0$, if suitably numbered, satisfy the relation $r_1/r_2=r_3/r_4$, provided $a^2d=c^2\neq 0$.
6. P and Q are any points inside a circle (C) with center C, such that $CP=CQ$. Determine the location of a point Z on (C) such that $PZ=QZ$* shall be a minimum.
7. Let w be an irrational number of $0<w<1$. Prove that w has a unique convergent expansion of the form
$$w = \frac{1}{p_0} - \frac{1}{p_0p_1} + \frac{1}{p_0p_1p_2} - \frac{1}{p_0p_1p_2p_3} + \cdots,$$
where $p_0, p_1, p_2, \cdots$ are integers and $1\leqq p_0<p_1<p_2<\cdots$. If $w=\frac{1}{2}\sqrt{2}$, find p_0, p_1, p_2.

* The examination committee intended that this should read $PZ+QZ$, but the problem was put before the contestants as shown above.

A NOTE ON THE SYLVESTER-FRANKE THEOREM

HARLEY FLANDERS, University of California, Berkeley

L. Tornheim [1] has recently given a proof of the Sylvester-Franke theorem based on elementary transformations. We shall present here another proof which is based on the Grassmann exterior algebra [2].

Let L be a linear space of dimension n over a field of scalars k. For each non-negative integer p, there is associated to L in an intrinsic manner a new vector space, denoted by $\Lambda^p L$, whose elements are called p-vectors. This space may be characterized as follows. There is a function f which carries ordered p-tuples of vectors of L into $\Lambda^p L$, *i.e.*,

$$(x_1, \cdots, x_p) \to f(x_1, \cdots, x_p) \in \Lambda^p L \qquad \text{for } x_i \in L,$$

such that:

(a) f is linear in each variable. In short, f is multilinear.

(b) f is alternating. This means that if π is any permutation of the indices $1, 2, \cdots, p$, then

$$f(x_{\pi(1)}, \cdots, x_{\pi(p)}) = (\operatorname{sgn} \pi) f(x_1, \cdots, x_p)$$

for all elements $x_1, \cdots, x_p$ of L. Here sgn π denotes the sign of the permutation π. It is 1 if π is even and -1 if π is odd.

(c) The collection of all images $f(x_1, \cdots, x_p)$ of p-tuples of L generates the space $\Lambda^p L$.

(d) If $x_1, \cdots, x_n$ is a basis of L, then the $C_{n,p}$ elements $f(x_{i_1}, \cdots, x_{i_p})$, $(1 \leqq i_1 < i_2 < \cdots < i_p \leqq n)$ are linearly independent in $\Lambda^p L$.

The following are consequences of (a)–(d):

(e) The $C_{n,p}$ elements of (d) form a basis of $\Lambda^p L$.

(f) If $x_i = x_j$ for some $i \neq j$, then $f(x_1, \cdots, x_n) = 0$.

We now use the following notation. The *exterior product* of $x_1, \cdots, x_p$ is

$$x_1 \wedge \cdots \wedge x_p = f(x_1, \cdots, x_p).$$

Let M be a second linear space over k and A a linear transformation on L into M. This induces a linear transformation, denoted by $\Lambda^p A$, on $\Lambda^p L$ into $\Lambda^p M$ which is determined as follows: If $x_1, \cdots, x_p$ are in L, then

$$(1) \qquad (\Lambda^p A)(x_1 \wedge \cdots \wedge x_p) = A x_1 \wedge \cdots \wedge A x_p.$$

In terms of matrices, this exterior power $\Lambda^p A$ of A is precisely the pth compound of A.

We can now state and prove the Sylvester-Franke theorem.

THEOREM. *If A is a linear transformation on L into itself, then*

$$|\Lambda^p A| = |A|^{C_{n-1,p-1}}. \tag{5}$$

Case 1. The matrix of A is diagonal. Then we have a basis $x_1, \cdots, x_n$ of L such that $Ax_i = a_i x_i$ for scalars a_i. Thus $(\Lambda^p A)x_H = a_H x_H$ where $a_H = a_{i_1} \cdots a_{i_h}$, and so

$$|\Lambda^p A| = \prod_H a_H = (a_1 \cdots a_n)^{C_{n-1,p-1}} = |A|^{C_{n-1,p-1}}.$$

The exponent represents the number of combinations H out of N which contain a specific a_i.

Case 2. The general case. If our field k were the field of complex numbers, we could finish the proof by a continuity argument, using the fact that the matrices with distinct characteristic roots are dense amongst all matrices. But each of these is similar to a diagonal matrix, so the result follows from Case 1 and the Corollary above. An algebraic proof that always works is analogous to this continuity proof. The desired result (4) is an algebraic identity amongst the coefficients of A. Thus it suffices to prove (4) in case $A = \|x_{ij}\|$ is a matrix with independent variables as coefficients. But such a matrix has distinct characteristic roots in an algebraic extension of the field $k(x_{ij})$, hence $PAP^{-1} = B$, a diagonal matrix, and our conclusion follows.

References

1. L. Tornheim, The Sylvester-Franke theorem, this MONTHLY, vol. 59, 1952, pp. 389–91.
2. N. Bourbaki, Algèbre Multilinéaire, Paris, 1948.

CROSS-ASSOCIATIVITY AND ESSENTIAL SIMILARITY†

DAVID ELLIS, The University of Florida

1. Let G be a group under each of the operations a^*b and $a\#b$. We consider the two conditions:

(CA) $a\#(b^*c) = (a\#b)^*c$; for all a, b, c in G.

(ES) There is an element y in G so that for all a, c in G: $a^*c = a\#y\#c$.

It is to be noted that we might also consider a dual of (CA) with the cross and star interchanged. As we will show, however, that (CA) and (ES) are equivalent, and since (ES) is symmetric in a and c, it will follow that (CA) and its dual are equivalent.

We denote notions relative to the two operations by subscripts. For example, 1_* is the identity element for $G(^*)$ and $(a)_\#^{-1}$ is the inverse of a in $G(\#)$.

† Presented to the Mathematical Association of America; Spring, 1952.

2. THEOREM 1. (CA) *implies* (ES) *where the* y *in* (ES) *is* $(1_*)_{\#}^{-1}$.

Proof. Write $z=1_*$ so that $y\#z=1_{\#}$. Then, by (CA), $a^*c=(a\#1_{\#})^*c=(a\#y\#z)^*c=(a\#y)\#(z^*c)=(a\#y)\#c$.

THEOREM 2. (ES) *implies* (CA) *and if* (ES) *subsists, the* y *in* (ES) *is necessarily* $(1_*)_{\#}^{-1}$.

Proof. By direct computation from (ES):

$$(a\#b)^*c = (a\#b)\#(y\#c). \tag{1}$$

$$a\#(b^*c) = a\#(b\#y\#c). \tag{2}$$

Thus, (CA) subsists when (ES) does. But if (CA) subsists, $y=(1_*)_{\#}^{-1}$ by Theorem 1 and cancellation.

THEOREM 3. (ES) *and* (CA) *are equivalent and each implies isomorphism of* $G(^*)$ *and* $G(\#)$.

Proof. The first statement combines Theorems 1 and 2. For the remainder, note that if (ES) subsists, then $f(a)=a\#y$ is an isomorphism of $G(^*)$ onto $G(\#)$.

3. We have taken $G(^*)$ and $G(\#)$ to be groups merely for purposes of familiarity. To obtain the three results, it actually suffices to require only that $G(^*)$ and $G(\#)$ are semigroups with identity elements and that the identity of $G(^*)$ has an inverse in $G(\#)$. Finally we offer the

CONJECTURE. *If* $G(+,\ ^*)$ *and* $G(+,\ \#)$ *are s-fields, then* (ES) *subsists and, hence,* $G(^*)$ *and* $G(\#)$ *are isomorphic.*

CORRECTION

ALBERT WILANSKY, Lehigh University

The following correction should be made in my note, *Two examples in real variables*, this MONTHLY, vol. 60, 1953, p. 317:

For lines 12–15, read: "latter class follows from example 1, the function there constructed being of Baire class I."

Professor Goffman kindly pointed out the inaccuracy of the given text.

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Consider the triangles BZX and EDC. Corresponding sides are seen to meet on l. Hence by the converse of Desargues' Theorem, BE, ZD, and $CX=l_0$ meet in the point F.

Right distributivity may be proved in a similar manner.

FLOW OF FLUID THROUGH A SYSTEM OF VESSELS IN SERIES

S. Kulik, Claremont Men's College

Introduction. The problem arose in connection with some chemical process.

A fluid is treated by filling up a vessel and emptying it at regular intervals. The time it is in the vessel is the time of treatment. In order to accelerate the process it was suggested that the fluid flow continuously through one or several vessels in series, being thoroughly mixed in each vessel.

The distribution function of the time of treatment of the fluid is given, assuming that each element of the fluid has the same chance of leaving the vessel.

Case of one vessel. A fluid flows through a vessel of a given volume v_0 at a constant rate a in unit time. The empty vessel would be filled up in t_0 units of time, where

$$v_0 = at_0 \tag{1}$$

Considering the process from the initial time $t=0$ there will be in the vessel at a time t, (1) a volume v of the fluid which has flowed in, (2) a volume v_0-v of the fluid which was in the vessel at time $t=0$.

The amount of the fluid flowing out of the vessel during the increment of time dt is composed of two parts, each of which is proportional to the corresponding component of the fluid in the vessel at time t and also to dt.

Thus, it can be written

$$d(v_0 - v) = - k(v_0 - v)dt \tag{2}$$

where, as it can be shown, the coefficient $k=a/v_0=1/t_0$; the initial condition is $v=0$ at $t=0$. Therefore, the solution of the equation is

$$v = v_0(1 - e^{-kt}). \tag{3}$$

The original fluid will theoretically be expelled completely from the vessel only in an infinite period of time, because v reaches the value v_0 only when $t=\infty$. Practically, however, the difference v_0-v may be considered negligible at some finite value of t.

When the initial fluid is expelled the process is considered as steady, and the proportion of each unit of the volume which had been treated for not less than time t is, according to equation (3),

$$F(t) = 1 - e^{-kt} \tag{4}$$

it follows that $ZD=ZF$, proving left distributivity.

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the variance

$$V_n = \int_0^\infty (t - nt_0)f_n(t)dt = nt_0^2 = n/k^2 \tag{12}$$

and the standard deviation

$$\sigma_n = t_0\sqrt{n}. \tag{13}$$

Thus when n vessels in series are used the average time of treatment equals the time required to fill up with fluid all the empty vessels at a given rate of flow, and the standard deviation is $\sqrt{n}$ of this time.

Hence, it can be deduced that if n vessels have a total constant volume equal to that in the case of one vessel, the formulas (10)–(13) must be rewritten as follows:

$$f_n(t) = \frac{(nk)^n}{(n-1)!} t^{n-1}e^{-nkt} \tag{14}$$

$$\mu_{1,n} = 1/k = t_0 \tag{15}$$

$$V_n = 1/nk^2 = t_0^2 n \tag{16}$$

$$\sigma_n = 1/k\sqrt{n} = t_0/\sqrt{n}. \tag{17}$$

Thus the average remains constant while the variance is one nth of that for one vessel.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1081. *Proposed by A. E. Livingston, Institute for Advanced Study, Princeton, N. J.*

Let $f(x_1, \cdots, x_n)$ be the nth order determinant $|a_{ij}|$ with $a_{ii}=x_i$ and $a_{ij}=1$ for $i\neq j$. Clearly, $f(x_1, \cdots, x_n)$ is symmetric in $x_1, \cdots, x_n$ Find its representation in terms of the elementary symmetric functions in $x_1, \cdots, x_n$.

E 1082. *Proposed by L. L. Pennisi, University of Illinois*

Show that $\lim_{n\to\infty} n(\sqrt{n}/n!)^{1/n} = e$.

E 1083. *Proposed by F. J. Duarte, Caracas, Venezuela*

Suppose that $x^n + y^n = z^n$, where n is a positive integer and x, y, z are positive real numbers. Show that $(xy/z^2)^n < 2/5$.

E 1084. *Proposed by R. E. Shafer, University of California*

Trygve Nagell in his book *Elementary Number Theory* shows that

$$\sum_{p\leqq x} (1/p) > \log\log x - 1,$$

where the summation is over all primes p not exceeding x. Establish the sharper inequality

$$\sum_{p\leqq x} (1/p) > \log\log x + 1 - \pi^2/6.$$

E 1085. *Proposed by Josef Langr, Prague, Czechoslovakia*

The perpendicular bisectors of the sides of a quadrailateral Q form a quadrilateral Q_1, and the perpendicular bisectors of the sides of Q_1 form a quadrilateral Q_2. Show that Q_2 is similar to Q and find the ratio of similitude.

SOLUTIONS

A Transcendental Number

E 1050 [1953, 40]. *Proposed by S. H. Gould, Purdue University*

If $c_0 = 1$, $c_{i+1} = 2^{c_i}$, prove that $\sum_{i=1}^{\infty} c_i^{-1}$ is transcendental.

Solution by J. V. Whittaker, U. C. L. A. If $k \geqq 1$, then $k \leqq c_{k-1}$ and $kc_{k-1} \leqq c_{k-1}^2 \leqq 2^{c_{k-1}} = c_k$. Hence, $2^{kc_{k-1}} \leqq 2^{c_k}$, or $c_k^k \leqq c_{k+1}$. Suppose $\xi = \sum_{i=1}^{\infty} 1/c_i$ were algebraic of degree $n > 1$, and let $\xi_k = \sum_{i=1}^{k} 1/c_i = p_k/q_k$, where p_k and $q_k = c_k$ are integers. Then $\xi - p_k/q_k = \sum_{i=k+1}^{\infty} 1/c_i < 2/c_{k+1} \leqq 2/c_k^k$, and $\lim_{k\to\infty} q_k^n(\xi - p_k/q_k) = \lim_{k\to\infty} 2q_k^n/q_k^k = 0$. But this contradicts the hypothesis that ξ was algebraic of degree $n > 1$, by Liouville's theorem. Now suppose $\xi = p/q$, p and q integers, and let $c_k > q$. Then the integer $\xi qc_k - \xi_k qc_k = qc_k \sum_{i=k+1}^{\infty} 1/c_i < 2qc_k/c_{k+1} \leqq 2q/c_k^{k-1} < 2/c_k^{k-2} < 1$ if $k \geqq 3$, which is a contradiction. Therefore, ξ is transcendental.

Also solved by W. E. Briggs, Vern Hoggatt, John Jones, Jr., M. S. Klamkin, A. E. Livingston, Albert Wilansky, and the proposer.

N Objects in B Boxes

E 1051 [1953, 114]. *Proposed by S. W. Golomb, Harvard University*

Given N objects and B boxes, what is a necessary and sufficient condition for at least two boxes to contain the same number of objects?

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If no two of the boxes contain the same number of objects, then the total number of objects is

$$0 + 1 + 2 + \cdots + (B - 1) = B(B - 1)/2$$

or any greater integer. Therefore a necessary and sufficient condition for at least two boxes to contain the same number of objects is that $N < B(B-1)/2$.

Also solved by N. J. Fine, L. A. Fulk, H. M. Gehman, Leo Moser, Azriel Rosenfeld, I. R. Savage, and the proposer.

Three Related Loci

E 1052 [1953, 114]. *Proposed by H. H. Berry, U. S. Army Engineering Corps*

Let AOB be a fixed diameter of a given circle (O), and let P be any point on the circle. Denote by Q the foot of the perpendicular from P on AB and by R the foot of the perpendicular from O on PA. Let PQ and RO intersect in N, and let QR and PO intersect NA in L and M, respectively. Find the loci of points L, M, and N as P moves along the given circle.

Solution by D. C. B. Marsh, University of Colorado. Take O as origin and OB as positive x-axis. If r is the radius of the given circle and (u, v) the coordinates of point P, we readily find the coordinates of M, N, L to be

$$M:\quad (-u^2/r,\ -uv/r),$$
$$N:\quad (u,\ -uv/(r - u)),$$
$$L:\quad (u(2r - u)/(r - 2u),\ -uv/(r - 2u)).$$

Eliminating u and v from each of these we get:

$$\text{locus of } M:\quad x^2 + y^2 + rx = 0,$$
$$\text{locus of } N:\quad y^2(r - x) = x^2(r + x),$$
$$\text{locus of } L:\quad [(x + r)^2 + y^2](x^2 - 3y^2) = r^2y^2.$$

Also solved by W. B. Carver, Charlotte Froese, J. D. Haggard, J. R. Hatcher, Douglas Holdridge, A. E. Livingston, B. Martin, L. V. Mead, C. S. Ogilvy, M. Perisastri, A. Sisk, O. E. Stanaitis, and the proposer.

Editorial Note. The locus of M is a circle on AO as diameter. The locus of N is a right strophoid with the end of its loop at A, its node at O, and its asymptote along the tangent at B to the given circle. The locus of L is a unicursal quartic symmetrical in the x-axis and having the lines $x^2 - 3y^2 = 0$ as asymptotes; it has a crunode at O and a tacnode at A.

Evaluation of a Certain Symmetrical Determinant

E 1053 [1953, 115]. *Proposed by H. S. Shapiro, Chatham, N.J.*

Evaluate the determinant

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If no two of the boxes contain the same number of objects, then the total number of objects is

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Evaluation of a Certain Symmetrical Determinant

E 1053 [1953, 115]. *Proposed by H. S. Shapiro, Chatham, N.J.*

Evaluate the determinant

Quadrangles with Incircles

E 1055 [1953, 115]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The center of the four circles passing through triples of vertices of a quadrangle $ABCD$ inscribed in (circumscribed about) a circle (O) are the vertices of a quadrangle $A'B'C'D'$ inscribed in (circumscribed about) a circle (O').

Editorial Note. The part of the theorem not in parentheses is trivially true, since in this case the quadrangle $A'B'C'D'$ reduces to the point O.

To prove the part of the theorem in parentheses let AB and CD intersect in E and let (O'') be the circle inscribed in triangle BCE. Let CB'', parallel to DB, intersect BE in B'', and let BC'', parallel to AC, intersect CE in C''. Then $B''C''$ and DA are corresponding lines under the negative homothety set up by circles (O'') and (O). It follows that $B''C''$ is tangent to (O''), or that $CC''B''B$ possesses an incircle. Now one can easily show that $A'B'C'D'$ is similar to $CC''B''B$, since corresponding sides and diagonals are perpendicular so as to make corresponding angles equal. Therefore $A'B'C'D'$ possesses an incircle.

Also solved, trigonometrically, by Josef Langr.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4553. *Proposed by D. J. Newman, Harvard University*

Consider any sequence $\{a_n\}$ of real numbers. Prove

$$\sum_{n=1}^{\infty} a_n \leqq \sqrt{2} \sum_{n=1}^{\infty} \sqrt{\frac{a_n^2 + a_{n+1}^2 + \cdots}{n}}\,.$$

4554. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Find the general solution of the differential equation

$$\left(\frac{dy}{dx} + Py\right)^P = R\left(\frac{dy}{dx} + Qy\right)^Q,$$

where P, Q and R are constants.

4555. *Proposed by R. M. Redheffer, University of California at Los Angeles*

If $\tanh u = \tanh^2 (\pi\sqrt{z} \sin \theta)$ and $\tanh v = \tan^2 (\pi\sqrt{z} \cos \theta)$, then the function

$$F(z) = \int_{-\pi}^{\pi} \coth (u + v) d\theta - 2 \cot \pi z$$

has simple poles at the (positive and negative) composite integers and is regular everywhere else.

4556. *Proposed by H. D. Grossman, New York City*

Suppose an election results in km votes for A and kn votes for B. In how many orders may votes be cast so that A's vote is always at least m/n times B's? Prove the following formula (cited in *Scripta Math.*, v. 16, pp. 207–212, and in *Math. Reviews*, October 1951, p. 665)

$$p_k = \sum \frac{F_1^{k_1} F_2^{k_2} \cdots}{k_1! k_2!},$$

in which $k_1 + 2k_2 + \cdots = k$ and $F_j = \binom{jm+jn}{jm} / j(m+n)$, and the summation is taken over all partitions of k.

4557. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $f(n)$ be the number of ways the integer n can be decomposed into dissimilar factors [e.g. $f(1) = 1, f(45) = 3$ because $1 \cdot 45 = 3 \cdot 15 = 5 \cdot 9$ but not $3 \cdot 3 \cdot 5$]. Evaluate

$$\sum_{n=0}^{\infty} \frac{f(2n + 1)}{(2n + 1)^2}.$$

SOLUTIONS

Basis in a Banach Space

4478 [1952, 186]. *Proposed by Albert Wilansky, Lehigh University*

A basis in a Banach space B is a set of elements $\{x_n\}$ such that every element in the space is a unique, finite or infinite, linear combination of elements of the set.

(1) Prove that the x_n are isolated.

(2) If $\sum x_n$ is an element of B, show that there is a divergent sequence $\{s_n\}$ of numbers such that $\sum s_n x_n$ is an element of B.

Correction by G. G. Lorentz. The solution of (1) as given in the May issue [1953, 338] is incomplete and should be replaced by the following: Let

$$x = \sum_{i=1}^{\infty} a_i x_i$$

be the representation of an element x of the space by means of the basis. Then $a_i = a_i(x)$ is a bounded linear functional (Banach, *Théorie des operations linéaires*, Warsaw 1932, p. 111), which vanishes for $x = x_j$, $j \neq i$, and is equal to 1 for $x = x_i$. Hence (*loc. cit.*, p. 58) x_i is at a positive distance from the linear set spanned by the x_j, $j \neq i$, and is therefore isolated from these x_j.

Linear Combination of Vectors with Non-negative Coefficients

4490 [1952, 332]. *Proposed by L. M. Blumenthal, University of Missouri*

Let S be a set of more than $2n$ pairwise distinct (non-null) vectors of Euclidean n-space E_n. If each vector of E_n is expressible as a linear combination of vectors of S with non-negative coefficients, then S contains a subset R of at most $2n-1$ vectors such that R has this same property. (This is an extension of no. 4395 [1952, 46].)

Solution by V. L. Klee, Jr., The Institute for Advanced Study. For a subset X of E_n, the following assertions are equivalent: (a) each vector of E_n is expressible as a linear combination of vectors of X with non-negative coefficients: (b) the origin is interior to the convex hull of X. Now with S as given, it is clear that some finite subset X of S has property (a). But then, since (a) is equivalent to (b), the desired conclusion is implied by the following result of C. V. Robinson (Spherical theorems of Helly type and congruence indices of spherical caps, *Amer. Jour. Math.*, vol. 64 (1942), p. 263): *If q is interior to a convex polyhedron P of E_n with vertices $p_1, p_2, \cdots, p_m (m \geqq 2n-1)$ then either* (1) *q is interior to a sub-polyhedron of P with $2n-1$ or fewer vertices, or* (2) *P has exactly $2n$ vertices collinear in pairs with q.*

Also solved by J. W. Gaddum and the Proposer.

The Brocard Angle

4491 [1952, 333]. *Proposed by C. S. Venkataraman, Tricur, South India*

If ω denotes the Brocard angle of a triangle ABC, prove that

(i) the sides are equal when $\cot \omega = \sqrt{3}$.

(ii) the squares of the lengths of the sides, a^2, b^2, c^2, are in arithmetic progression when $\cot \omega = 3 \cot B$.

Solution by W. B. Carver, Cornell University. It is well known (see Johnson, *Modern Geometry*, p. 266) that

$$\cot \omega = \cot A + \cot B + \cot C,$$
$$\cot A = (b^2 + c^2 - a^2)/4\Delta,$$

where Δ is the area of the triangle ABC, with analogous formulas for $\cot B$ and $\cot C$. Therefore $\cot \omega = (a^2+b^2+c^2)/4\Delta$.

(i) When cot $\omega=\sqrt{3}$, we have, after expressing Δ in terms of the sides and simplifying,

$$(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2=0.$$

Since a, b, c are real and positive, this implies $a=b=c$.

(ii) From cot $\omega=3$ cot B, we obtain

$$2b^2=a^2+c^2,$$

which makes b^2 the arithmetic mean between a^2 and c^2.

Both conditions are necessary and sufficient.

Also solved by F. A. Alfieri, Louisa S. Grinstein, L. M. Kelly, M. S. Klamkin, J. D. E. Konhauser, Josef Langr, Margaret Olmsted, M. Perisastri, L. A. Ringenberg, F. Underwood, Chih-yi Wang, Alan Wayne, and the Proposer.

A Locally Increasing Function

4492 [1952, 333]. *Proposed by G. Lumer, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

A real function $f(x)$ is said to be locally increasing at the point x_0 if there exists a neighborhood U of x_0 such that for $x\in U$,

$$\frac{f(x)-f(x_0)}{x-x_0}\geqq 0.$$

Show that there exist functions $f(x)$, defined and continuous on a closed interval $[a, b]$, locally increasing on a set D, everywhere dense on $[a, b]$, but such that $f(b)<f(a)$.

Solution by E. M. Beesley, University of Nevada. For $0\leqq x\leqq 1$, let $F(x)=-C(x)$ where C is the well known Cantor function. (See Titchmarsh, *Theory of Functions*, p. 366; or Hobson, *the Theory of Functions of a Real Variable*, v. I, p. 368.) This function is continuous and has a derivative almost everywhere. Then $F(1)=-1<0=F(0)$. If x is not in Cantor's set, $F'(x)=-C'(x)=0$. Since Cantor's set is nowhere dense, F is a solution to the proposed problem.

If we let $F(x)=-C(x)+ax$, where $0<a<1$, we have a better result in the sense that $F'(x)>0$ on the same everywhere dense set.

Also solved by Henry Furstenberg, L. M. Kelly, M. S. Klamkin, J. H. Michael, C. L. Perry and W. S. Snyder, George Piranian, Edgar Reich, L. A. Ringenberg, Arthur Rosenthal, W. R. Scott, and the Proposer.

An Irrational Number

4493 [1952, 412]. *Proposed by Paul Erdös, American University, Washington, D. C.*

For each integer n let $\sigma(n)$ be the sum of its divisors. Prove that

(i) When cot $\omega=\sqrt{3}$, we have, after expressing Δ in terms of the sides and simplifying,

$$(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2 = 0.$$

Since a, b, c are real and positive, this implies $a=b=c$.

(ii) From cot $\omega=3$ cot B, we obtain

$$2b^2 = a^2 + c^2,$$

which makes b^2 the arithmetic mean between a^2 and c^2.

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If we let $F(x) = -C(x) + ax$, where $0 < a < 1$, we have a better result in the sense that $F'(x) > 0$ on the same everywhere dense set.

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An Irrational Number

4493 [1952, 412]. *Proposed by Paul Erdös, American University, Washington, D. C.*

For each integer n let $\sigma(n)$ be the sum of its divisors. Prove that

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!}$$

is irrational.

Solution by J. B. Kelly, Michigan State College. Suppose that $h = \sum_{n=1}^{\infty}\sigma(n)/n! = r/s$, where r and s are positive integers and $(r, s) = 1$. Let p be any prime larger than s and larger than 6. Then

$$h = \sum_{n=1}^{p-1} \frac{\sigma(n)}{n!} + \sum_{n=p}^{\infty} \frac{\sigma(n)}{n!},$$

whence

$$(p-1)!h = (p-1)!\sum_{n=1}^{p-1} \frac{\sigma(n)}{n!} + \sum_{c=0}^{\infty} \frac{\sigma(p+c)}{p(p+1)\cdots(p+c)}.$$

Now $(p-1)!h$ and $(p-1)!\sum_{n=1}^{p-1}\sigma(n)/n!$ are integers. We shall obtain a contradiction by showing that

$$k = \sum_{c=0}^{\infty} \frac{\sigma(p+c)}{p(p+1)\cdots(p+c)} \tag{1}$$

cannot be an integer.

We have

$$\sigma(p)/p = 1 + 1/p, \tag{2}$$

while $\sigma(p+c) < 1+2+3+\cdots+(p+c) = \frac{1}{2}(p+c)(p+c+1)$, so that

$$\begin{aligned}\sum_{c=1}^{\infty} \frac{\sigma(p+c)}{p(p+1)\cdots(p+c)} &< \sum_{c=1}^{\infty} \frac{p+c+1}{2p(p+1)\cdots(p+c-1)} \\ &< \sum_{c=1}^{\infty} \frac{p+2}{2p^c} < \frac{p+2}{2(p-1)}.\end{aligned} \tag{3}$$

Also for $p>6$, $(p+2)/2(p-1) < (p-1)/p$. Finally we have from (1), (2) and (3),

$$1 < k < 1 + \frac{1}{p} + \frac{p+2}{2(p-1)} < 2.$$

Also solved by Robert Breusch, Leonard Carlitz, N. J. Fine, Fritz Herzog, Leo Moser, D. J. Newman, A. Oppenheim, and L. L. Pennisi.

Editorial Note. Several contributors address themselves to other problems of the form $h = \sum a_n/n!$. Moser proves that h is irrational for $a_n = \phi(n)$. Oppenheim proves h irrational for $a_n = \phi(n)\sigma(n)$. See also the Proposer's problem 4518, [1953, 47]. The Proposer is also interested in determining whether or not $\sum\sigma(n)/2^n$ is rational.

sion of consistency, independence, completeness, and categoricalness of postulates, with illustrations, including various field postulates. 9. Groups, rings, and Boolean algebras. 10. General and equivalence relations, linearly and partially ordered sets, and lattices. 11. A survey of the foundations of mathematics, with brief sketches of the various schools. The exercises are interesting and well chosen, and the text is alive.

There are very few misprints, and only the one on the bottom of page 96, where "$x \notin A$" should be replaced by "$x \in A$," might cause the student a little trouble. On the bottom of page 143, it is stated that "the lines of projective geometry are thought of as made up continuously of points"; the existence of finite projective geometries shows, however, that this is false. On page 183, the author should have been more specific as to the applications of the theory of finite groups in the theory of algebraic equations. Bolyai's contribution to non-Euclidean geometry is somewhat slighted, the absence of biographical notice in connection with the names Abel and De Morgan is conspicuous, and bouquets are thrown to mathematicians prominent in America. These, however, are minor imperfections in the book of many merits.

F. BAGEMIHL
Rochester, N. Y.

Advanced Mathematics in Physics and Engineering. By Arthur Bronwell. McGraw-Hill Book Company, 1953. xvi+475 pages. $6.00.

This text by an electrical engineer is intended for senior or graduate use in the fields mentioned. Chapter headings are as follows: 1. Infinite Series. 2. Complex Numbers and Hyperbolic Functions. 3. Fourier Series and Fourier Integral. 4. Ordinary Differential Equations. 5. Series Solution of Differential Equations—Bessel and Legendre Equations. 6. Partial Differentiation. 7. Elastic Vibrations and Electric Oscillations—Systems with Lumped Elements. 8. Vibrations in Systems with Distributed Elements. 9. Lagrange's Equations. 10. Vector Analysis. 11. Solutions of the Wave Equation. 12. Heat Flow. 13. Dynamics of Fluids. 14. Electromagnetic Theory. 15. Functions of a Complex Variable. 16. Complex Roots of Polynomials and Dynamic Stability. 17. Laplace Transformations. Chapters 2 and 6 are short, chapters 5, 15 and 17 are long. Each chapter ends with a bibliography, and, except 16, with an adequate set of problems, some with answers. Illustrative examples however are few and untypically easy.

The major comment called for by this text concerns the generally unsatisfactory quality of the exposition. There is much carelessness, linguistic and mathematical, mostly in the first half of the book. Many statements are wrong, still more are meaningless from lack of stated assumptions, and quite often a sentence means just the opposite of what the author intended. Chapter 11, signalled out in the preface as an innovation, attempts too many things simultaneously and is merely confusing. Errors range all the way from confusion of necessary with sufficient conditions to the implication that drums have rec-

tangular membranes.

The last chapters are less careless than the rest, and the reviewer profited from the discussions of wave guides and of dynamic stability. There is about the whole a commendable balance as between fields treated; and the attempt to do justice and bring unity to such diverse problems in less than 500 uncrowded pages is laudable. However this reviewer would not recommend this book as a whole to any beginner in this area; some of the later sections should have value for the experienced and careful reader.

E. H. CUTLER
Lehigh University

NEW BOOKS RECEIVED

Logic and Language. By A. G. N. Flew. New York, Philosophical Library, 1953. 8+242 pages. $4.75.

Elements of Algebra. By Vaughn B. Caris. Boston, Ginn and Company, 1953. 5+307 pages. $3.30.

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CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

THE ELIMINATION OF THE HIGH COST OF COMPTOMETRY

E. I. GALE, Norland, Ontario

"I took a few pieces of string and an old tin can and made myself a Ford car and the darn thing ran."—The late Dr. Jas. S. Gale of Korea.

His nephew has just assembled the same primitive materials to evolve a remarkably simple and efficient method of carrying out multiplications of astronomical magnitudes. The scheme in a nutshell is to combine the principle of Genaille's rods (see this MONTHLY, February, 1953) and the suanpan (see your Chinese laundry man). The former invention yields the partial products, while the abacus is well adapted for assembling them into the final product.

An example in addition may be helpful to those readers unfamiliar with the oriental method of using the abacus for addition. The bar running athwart the rods carrying the beads separates "heaven" from "hell." On the "heaven" side sliding on each rod are a pair of "quints" while on each rod in "hell" are five units. The rods in order from the right follow the Arabic system of counting units, tens, hundreds, *etc.* To "clear" the board, push quints and units as far from the central bar as possible. The board now reads zero. Let us add 269. Bring a quint down and four units up on the units rod for the 9. On the tens rod bring down a quint and raise a unit for the 6. Finally on the hundreds rod raise two units for the 2. The suanpan reads 269. We now will add 584. Obviously we can't raise four units on the units rod so we depress the other quint and lower a

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unit. Now a universal abacus rule forbids two quints to rest at the central bar so we raise these two quints and raise a compensating unit on the tens rod. The board now reads 273. To add 9 tens the same method is followed, *viz.*, lower a quint on the tens rod and raise three units. However by the rule mentioned above the two tens quints are pushed to the top and a unit raised on the hundreds rod. The board now reads 353. The reader will observe here that it would be more economical with respect to beads to replace the five units on the tens rod by pulling down a quint on the same rod. This we do. This is a second rule and is invariably followed. The state of the board will now be 3 units on the hundreds rod, 1 quint on the tens rod, and 3 units on the units rod. Finally depress a quint on the hundreds rod yielding the desired 853. In practice this operation is performed much more quickly than the description would lead one to suppose.

The abacus is kept in the horizontal plane while being manipulated as is also the author's modification of Genaille's rods, which will be explained here. To follow the description use the diagram on page 140 of the February number of this MONTHLY. Genaille's arrows indicate the significant figures which are used in the computation (note an error in the "one" rod, row 8: 8 and 9 should point to 6, while 0 to 5 should point to 7).

In the author's instrument, shields cover all figures which are not involved in the operation of partial multiplication. These shields are simply a left-hand projection on each rod of width equal to the width of the columns which support the small figures. Single width and double width rectangular notches are cut in the projecting flaps to expose the single or double numerals as indicated by the single or double arrows respectively. Column (Rod) 1 however carries no projecting shield, but a special rod is brought into juxtaposition with the other rods at the extreme right. This rod carries a left-hand shield having eight single notches. These notches are placed so they would expose 6, 9, 2, 5, 8, 1, 4 and 7 if applied to the diagram depicted in the MONTHLY. The notches are now converted into rectangular holes by fastening a thin wire along the lip of each flap. Each wire is previously threaded through small glass beads, so that two beads are left rotatable in the single holes and three in the larger holes.

Provision is now made for shielding all the remaining unshielded small numerals. This is effected in a very simple manner. A rectangular piece of thin metal (tin is ideal) is bent about a wire slightly larger in diameter than the wires forming the left-hand boundaries of the rectangular holes. These pieces are pinched by a suitable tool around the wire to form a cylindrical hole. A cotter pin opened to 90 degrees gives the idea of the "edge-on" view of the L-shaped tin shutters. These little shutters are snapped over the supporting wires with a bead at each end of the hinge to reduce friction. An even better assembly technique would be to position them before attaching the long hinge wire. The "rods" themselves are kept in exact relationship to each other by a pair of holes part way through in the back of each which fit over studs projecting from a backing board. Now if the left-hand end of the supporting board is raised all the

biguity the successive figures in the partial product and moreover the shutters operated upon fall into a position of stable equilibrium in their respective laterally inverted L-position as viewed from the bottom of the board.

The resulting partial product is immediately set up on the abacus and the Genaille board tilted to blank out all the numerals. The stylus is again drawn across the board along the row corresponding to the tens figure of the multiplier. The result is again registered on the abacus only starting this time on the tens rod. It is quite evident that the recording of successive partial products must progressively be recorded rod by rod and more and more to the left on the abacus.

OBITUARY

DAVID RAYMOND CURTISS

In Memoriam

David Raymond Curtiss passed away at his home in Redlands, California on April 29 at the age of seventy-five. He had retired from his professorship of mathematics at Northwestern University in 1943 and moved to Redlands where he had spent his boyhood days. Here he built a delightful hillside home overlooking nearby orange groves and having a magnificent view of the not distant mountains. His charming flower garden attested his love of beauty and his ability to coax nature to yield her utmost for him. Unfortunately the final months of his life were marred by the serious illness of his wife who passed away with him from asphyxiation in their garage.

He was born in Derby, Connecticut, on January 12, 1878. Shortly thereafter the Curtiss family moved to California. After graduation from the Redlands high school, Raymond entered the University of California, from which he received his bachelor's degree in 1899. He spent a year teaching in a secondary school before returning to the university for a masters degree, which he received in 1901. By this time his enthusiasm for mathematics had become a controlling passion, and he went to Harvard for more advanced work. There, under the primary influence of Bocher and Osgood, he proceeded to the doctorate (1903), one of the first of a distinguished group to receive the degree under their guidance in the early years of the century. His success was rewarded by the granting of a travelling fellowship, and he went to France to study at the École Normale Supérieure in 1903–04.

After returning to this country he was instructor in mathematics at Yale for the year 1904–05. From there he went to Northwestern University where he served as head of the mathematics department for over twenty of the thirty-eight years of his active service. On leave of absence he returned to Harvard

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as lecturer in 1920–21, and he taught for one summer session at California.

In 1907 he married Sigrid Eckman who died in 1941, and in 1943 he married Ruth Kneen. From the first union he has three children: John Hamilton, Margaret Eckman, and Alice Judson (now Mrs. A. W. Tucker). Raymond's scientific talents were shared by his brother, Ralph, who was a distinguished astronomer at Michigan. His son has won high recognition for his mathematical work, and his son-in-law is now chairman of the mathematics department at Princeton. A grandson is reported to have unusual mathematical ability. This is reminiscent of the famous Struve family of astronomers.

At Northwestern Curtiss was a leading member of the faculty, always highly respected by his colleagues and by the administration. As a result, he found it necessary to devote endless hours to committee work, at which he was eminently successful. Likewise the American Mathematical Society called upon him extensively for help, as member of the Council, as vice-president, as editor of the Transactions (1914–19), as editor of the Bulletin (1928–38), and as a member of many special committees. He was a charter member of the Mathematical Association of America, to which he devoted much energy as member of the finance committee and of the Board of Governors, serving as its president for two years (1935–36). For about twenty years he was on the Editorial Committee on Carus Monographs sponsored by the Association. He was a fellow of the American Association for the Advancement of Science, was vice-president (Section A) in 1921, and served on the Executive Committee for four years. He was also an editor of the Transactions of the International Mathematical Congress at Toronto (1924).

He was always interested in the teaching of mathematics, and in addition to being an exceptionally conscientious and successful teacher he wrote articles on the subject. He prepared a report for an International Commission of the Teaching of Mathematics, portions of which were published in the Bulletin of the American Mathematical Society (vol. 1911, pp. 122–137). In this MONTHLY (vol. 48, pp. 224–228) we find one of his papers, entitled *The professional interests of mathematical instructors in Junior Colleges*. His interest lay in the training of teachers, not in the methods of teaching. He always took great interest in those students who showed especial ability in mathematics. Among the many whom he encouraged to go on in advanced study, there were several whom he helped financially until they attained doctorates.

Curtiss's editorships of important mathematical publications, spanning about thirty years, clearly indicate his breadth of knowledge and his soundness of judgment. His devotion to the general interests of the mathematical world doubtless militated against his own productivity, but he did find time to write about twenty research papers and a half dozen books. His first paper, *Note on the sufficient conditions for an analytic function* (Bull. Amer. Math. Soc., vol. 8, 1902, pp. 329–331), and his second, *On the invariants of a homogeneous quadratic differential equation of the second order* (Amer. J. Math., vol. 15, 1903, pp. 355–362), were written while he was a graduate student at Harvard. His third,

Binary families in a triply connected region, with special reference to hypergeometric families (Memoirs of the American Academy of Arts and Sciences, vol. 13, 1904, pp. 2–59), was his doctoral dissertation. Somewhat closely related to these papers were three others: *Theorems converse to Riemann's on linear differential equations* (Trans. Amer. Math. Soc., vol. 7, 1906, pp. 99–106); *Sur la théorie des fonctions hypergéométriques* (Ann. École Norm. (3), vol. 22, 1906, pp. 121–143); *Relations between kindred Riemannian P and Q functions* (Bull. Amer. Math. Soc., vol. 29, 1923, pp. 154–160).

In the period from 1906 to 1911 he published four short papers in function theory: *A proof of the theorem concerning artificial singularities* (Ann. of Math. (2), vol. 7, 1906, pp. 161–164); *On certain theorems of mean value for analytic functions of a complex variable* (Ann. of Math. (2), vol. 8, 1907, pp. 118–126); *The vanishing of the Wronskian and the problem of linear dependence* (Math. Ann., vol. 65, 1908, pp. 282–298); and *Relations between the Gramian, the Wronskian, and a third determinant connected with the problem of linear dependence* (Bull. Amer. Math. Soc., vol. 17, 1911, pp. 462–467).

In 1911 he turned his attention to certain problems in the theory of equations, which resulted in a series of six papers: *An extension of Descartes' rule of signs* (Math. Ann., vol. 73, 1912, pp. 424–435); *The degree of a cartesian multiplier* (Bull. Amer. Math. Soc., vol. 20, 1913, pp. 19–26); *Extensions of Descartes' rule of signs connected with a problem suggested by Laguerre* (Trans. Amer. Math. Soc., vol. 16, 1915, pp. 350–360); *Recent extensions of Descartes' rule of signs* (Ann. of Math., vol. 19, 1918, pp. 251–278); *A mechanical analogy in the theory of equations* (Science, vol. 55, 1922, pp. 189–194); and *A note on the preceding paper* (Trans. Amer. Math. Soc., vol. 24, 1922, pp. 181–194). We do not have space to discuss any of these papers, but will remark that in Curtiss's terminology a given polynomial with real coefficients, $f(x)$, has the cartesian multiplier $f_1(x)$, a polynomial, if the Descartes rule of signs applied to the product $f(x)f_1(x)$ gives exactly the number of positive real roots of $f(x)$. He gave proofs of the existence of cartesian multipliers, and established bounds for the degree of a multiplier for a given $f(x)$; he remarked, "It is not likely that the method here given, even if improved, will have much value for numerical calculation."

In 1922, Curtiss solved a problem given in this MONTHLY by Kellogg (vol. 29, pp. 300–303) which led him to write a paper in diophantine analysis, *Classes of diophantine equations whose positive integral solutions are bounded* (Bull. Amer. Math. Soc., vol. 35, 1929, pp. 859–865).

In 1941 he published a valuable paper on *Maxima and minima of functions of two or more variables* (Mathematical Monographs, Northwestern University, vol. 1, pp. 3–43). An unpublished manuscript on *Geographical maps and their scales* was written for class-room purposes in 1943. We find that he presented a paper to the Society in 1922 entitled *On the zeros of successive polars of a binary form*, but do not find a published paper under that title.

Two non-technical papers should also be mentioned. In 1917 he was chairman of a committee which drafted a chapter on *Mathematics* for a book entitled

Science and Learning in France. And when he retired as president of the Mathematical Association he gave an instructive, and at times amusing, address on *Fashions in mathematics*, published in this MONTHLY (vol. 44, 1937, pp. 559–566).

With regard to his many papers we shall merely remark that he demonstrated a clear insight into interesting problems and presented solutions in a finished form. He did not hurry into print with embryonic ideas and subsequently add greatly to his list of titles by a succession of corrections, amendments, and extensions. He preferred to digest his subject before publication rather than to chew his cud in public.

We now turn to the books of which Curtiss was author or co-author. His first was *Analytic Functions of a Complex Variable*, which appeared in 1926 as the second of the Carus Mathematical Monographs. In this little book of 173 pages he gives a clear and concise development of the fundamental theorems of his subject, with a brief discussion of a few of its applications. In the preface he states: "Numerous citations of authorities are favorite means for an author to show his erudition—or someone else's. Perhaps this book goes to the opposite extreme and might well give a more complete statement of sources. However, the plan used in the following pages has been adopted because the author believes that the greatest service he can do for those for whom the book was intended is to induce them to supplement the reading of the present monograph by a study of accessible standard treatises on the subject."

In collaboration with a colleague, E. J. Moulton, Curtiss wrote a number of textbooks in trigonometry and analytic geometry:

Trigonometry, Plane and Spherical. 1927. 276 pp.
High School Trigonometry. 1928. 216 pp.
Analytic Geometry. 1930. 338 pp.
Exercises in Trigonometry. 1931. 223 pp.
A Brief Course in Trigonometry. 1940. 118 pp.
Essentials of Trigonometry with Applications. 1942. 174 pp.
Plane Trigonometry (for the United States Armed Forces Institute). 1943. 206 pp.
Essentials of Analytic Geometry. 1947. 259 pp.

In addition to these books of which he was co-author, Curtiss contributed a chapter on applied mathematics for a book entitled *Basic Mathematics* which appeared in 1944.

Curtiss was noted among his colleagues for his wit, often barbed, always pointed. His speech was inclined to be slow, but his mind was most active. His interests were wide and his memory superlative. Like many mathematicians, he was a lover of music. The esthetic aspects of mathematics and of the world about him appealed to him strongly; perhaps one could well say that they were dominant features of his life.

E. J. MOULTON

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MEETING OF MATHEMATICS DIVISION OF A.S.E.E.

The Mathematics Division of the American Society for Engineering Education met on June 24–26, 1953 at the University of Florida, Gainesville, Florida. Four sessions of the Division were held. The following new officers for the Mathematics Division were elected: Chairman of the Division, Dr. R. S. Burington, Chief Mathematician, Bureau of Ordnance, Navy Department, Washington, D. C.; Director of the Division, Dr. C. V. Newsom, Associate Commissioner for Higher Education, State University of New York; Member of Council of A.S.E.E., Professor H. K. Justice, University of Cincinnati.

The next meeting of the Mathematics Division of A.S.E.E. will be held June 14–18, 1954 at the University of Illinois, Urbana, Illinois. For further information write to Dr. R. S. Burington, 1834 North Hartford Street, Arlington 1, Virginia.

TRAVEL GRANTS OF THE NATIONAL SCIENCE FOUNDATION

The National Science Foundation will receive applications for a limited number of grants to assist in payment of travel expenses to the International Mathematical Congress and International Mathematical Union, Amsterdam and The Hague, Netherlands, August 30–September 9, 1954. Applications should be received by January 1, 1954. Application forms may be obtained upon request from the National Science Foundation, Washington 25, D. C.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1953 Preliminary Actuarial Examinations are as follows:

First Prize of $200

Broadwin, Emile Bernard........................Harvard University

Additional Prizes of $100

Fredkin, Donald R................................New York University
Gassner, Betty J.................................New York University
Hessenthaler, Ruth...............................Pembroke College
Speake, Neal M...................................University of Michigan
Steiner, Lisa....................................Swarthmore College
Traynor, Edwin A.................................Holy Cross College
Watson, Charles B. H.............................University of Toronto

Zinger, Alexis. . . . . University of Montreal

The Society of Actuaries has authorized a similar set of nine prizes for the 1954 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. *Language Aptitude Examination.*
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. *General Mathematics Examination.*
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. *Special Mathematics Examination.*
(Finite differences, probability and statistics.)

The 1954 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 12, 1954. The closing date for applications is March 15, 1954.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

The eighth annual Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. 53, no. 7, pp. 406–409, 1946) was held April 11, 1953 in the principal high schools of the states of California, Oregon, Washington and Arizona. This was the first year when the examination was given outside of California. Stanford put aside two scholarships of $660 each for the best papers of this competition. The following problems were proposed:

1. Bob has 10 pockets and 44 silver dollars. He wants to put his dollars into his pockets so distributed that each pocket contains a different number of dollars.
   (*a*) Can he do so?
   (*b*) Generalize the problem, considering p pockets and n dollars. The problem is the most interesting when

$$n = \frac{(p+1)(p-2)}{2}.$$

   Why?

2. Observe that the value of

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$$

is $\frac{1}{2}$, $\frac{5}{6}$, $\frac{23}{24}$ for $n=1, 2, 3$, respectively, guess the general law (by observing more values if necessary) and prove your guess.

3. Find x, y, u, and v satisfying the system of four equations

$$\begin{aligned} x + 7y + 3v + 5u &= 16 \\ 8x + 4y + 6v + 2u &= -16 \\ 2x + 6y + 4v + 8u &= 16 \\ 5x + 3y + 7v + u &= -16 \end{aligned}$$

(This may look long and boring: look for a shortcut.)

4. The four points G, H, V, and U are (in this order) the four corners of a quadrilateral. A surveyor wants to find the length $UV=x$. He knows the length $GH=l$ and measures the four angles

$$\angle GUH = \alpha, \qquad \angle HUV = \beta, \qquad \angle UVG = \gamma, \qquad \angle GVH = \delta.$$

(a) Express x in terms of α, β, γ, δ, and l.
(b) Find some way to test the correctness of the result.
(c) If you had a clear plan to do (a) characterize it in one short sentence.

The first prize went to Mr. Peter Stauffer from the Palo Alto High School, Palo Alto, California. No paper was found worthy for the second prize. Twelve prizes of "Honorable Mention" were awarded; these persons received a book as a gift from the Department of Mathematics.

PERSONAL ITEMS

The honorary degree of Doctor of Science has been conferred upon Professor R. B. McClenon by Grinnell College; Professor McClenon has retired with the title Professor Emeritus of Mathematics.

The Thirteenth William Lowell Putnam Mathematical Competition scholarship at Harvard University has been awarded to Mr. Tai Tsun Wu of the University of Minnesota.

Alabama Polytechnic Institute announces the following: Professor W. V. Parker has been appointed Dean of the Graduate School; Professor Parker will continue to serve as the Head of the Department of Mathematics; Mr. C. W. McArthur of the University of Maryland and Dr. H. D. Sprinkle of the University of Florida have been appointed to assistant professorships.

Brown University reports the following: Professor Herbert Federer has been working under a grant from the National Science Foundation since October 1, 1952; Professor David Gale has been working under a similar grant since January 1, 1953 and Professor Bjarni Jonsson under a similar grant since February 1, 1953; Professor Gale has received a Fulbright award for research in mathematical economics to be carried on in the University of Copenhagen and the Applied Mathematics Institute of the Technical University of Denmark

during the year 1953–54.

DePaul University announces: Dr. Pasquale Porcelli of the University of Texas has been appointed to an assistant professorship; Dr. W. F. Darsow has been promoted from Lecturer to Instructor.

At Illinois Institute of Technology: Dr. R. A. Struble, previously with the Douglas Aircraft Company, Santa Monica, California, has been appointed to an assistant professorship; Dr. R. H. Oehmke, formerly a graduate student at the University of Chicago, has been appointed to an instructorship.

Michigan State College announces the following: Professor G. deB. Robinson, who is on leave of absence from the University of Toronto, has accepted a visiting professorship for the year 1953–54; Dr. J. F. Hannan of Catholic University of America has been appointed to an assistant professorship; Dr. J. W. Gaddum has resigned as mathematician with the United States Air Force to accept an instructorship at the College; Associate Professor B. M. Stewart has been promoted to a professorship; Assistant Professor L. M. Kelly has been promoted to an associate professorship; Associate Professor J. H. Bell has resigned to accept a position as supervisory mathematician with the Bendix Aviation Corporation Research Laboratories, Detroit.

Portland State Extension Center reports: Associate Professor T. S. Peterson of the University of Oregon has transferred to the Center as Associate Professor of Mathematics; Dr. W. O. Buschman has been promoted to an assistant professorship.

Princeton University announces: Professor Solomon Lefschetz, Henry Burchard Fine Professor since 1933 and Chairman of the Department of Mathematics since 1945, has retired with the title of Fine Professor Emeritus; Professor Emil Artin has been appointed Fine Professor; Associate Professor D. C. Spencer has been promoted to a professorship and Dr. J. T. Tate, Higgins Research Fellow, to an assistant professorship; Professor A. W. Tucker has been designated Chairman of the Department.

Swarthmore College announces the promotions of Dr. E. R. Mullins, Jr. and Dr. David Rosen to assistant professorships.

At Syracuse University: Assistant Professor Ruth W. Stokes has been promoted to an associate professorship; Instructor H. C. Bennett has been promoted to an assistant professorship.

The University of Texas announces that beginning September 1, 1953 there will be one mathematics department, to be known as the Department of Mathematics and Astronomy. This replaces the two former Departments of Applied Mathematics and Astronomy and of Pure Mathematics.

The University of Washington announces: Assistant Professor D. G. Chapman has been promoted to an associate professorship; Dr. M. G. Arsove has been promoted to an assistant professorship; Assistant Professor V. L. Klee of the University of Virginia has been appointed to an assistant professorship; Dr. E. A. Michael of the University of Chicago has been appointed to an assistant professorship; Dr. A. E. Livingston of the Institute for Advanced Study has

been appointed to an instructorship; Dr. R. F. Tate of the University of California has been appointed to an instructorship; Assistant Professor F. H. Brownell, Jr., has been awarded a Ford Foundation Fellowship and will spend the year 1953–54 at the Institute for Advanced Study.

Assistant Professor A. G. Anderson of Oberlin College has been appointed to an assistant professorship at Duquesne University.

Mr. R. E. Barlow, formerly a student at Knox College, has been appointed to an assistant instructorship at the University of Oregon.

Dr. J. D. Baum, previously a student at Yale University, has been appointed to an instructorship at Oberlin College.

Assistant Professor H. M. Beatty of Ohio State University has retired.

Professor E. T. Bell of California Institute of Technology has retired with the title of Professor Emeritus.

Dr. Eleazer Bromberg of the Office of Naval Research, Washington, D. C., has been appointed Assistant Chief of the Computing Center, Institute of Mathematical Sciences, New York University.

Dr. R. S. Burington, chief mathematician of the Bureau of Ordnance, Navy Department, and head of the Evaluation and Analysis Group of the Bureau of Ordnance, has been named Special Assistant to the Director of Research and Development, Bureau of Ordnance, Navy Department, Washington, D. C.

Mr. R. G. Buschman of McNeese State College has accepted an appointment as a fellow at the University of Colorado.

Mr. S. C. Cobb, formerly at Phillips Academy, Andover, Massachusetts, has accepted a position at St. John's School, Houston, Texas.

Associate Professor Rufus Crane of Ohio Wesleyan University has retired.

Mr. T. H. Dewey, previously a research assistant at Los Alamos Scientific Laboratory, New Mexico, has a position as a mathematician at the Radiation Laboratory of the University of California.

Mr. R. C. DiPrima, who has been a part-time instructor at Carnegie Institute of Technology, has received an appointment as a research associate at Massachusetts Institute of Technology.

Mr. M. E. Drummond, formerly a research assistant at the University of Oklahoma, has accepted a position as a mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Mr. Anderson Duggar, Jr., previously a student at the University of Detroit, is now President of the Equitable Engineering Company, Detroit, Michigan.

Mr. G. W. Fairchild, who has been associated with Northrop Aircraft Incorporated, Hawthorne, California, has a position as a research analyst with Bendix Computer Division, Los Angeles, California.

Dr. J. M. G. Fell has been appointed to an instructorship at California Institute of Technology.

Dr. A. M. Feyerherm of Iowa State College has been appointed to an assistant professorship at Kansas State College.

Assistant Professor N. J. Fine of the University of Pennsylvania has been

promoted to an associate professorship; he will spend the academic year 1953–54 at the Institute for Advanced Study on a National Science Foundation Postdoctoral Fellowship.

Dr. C. D. Firestone has accepted a position as a mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Dr. H. D. Friedman, formerly a graduate student at Pennsylvania State College is employed by the General Electric Company, Syracuse, New York.

Mr. E. P. Graney of the Willow Run Research Center, University of Michigan, has been promoted to the position of Research Associate.

Dr. E. J. Gumbel of Columbia University has been promoted to an adjunct professorship; during the summer term he served as Visiting Professor at the Free University of Berlin.

Professor Georgia M. Haswell of Kansas Wesleyan University has been appointed Dean of Women at Pfeiffer Junior College, Misenheimer, North Carolina.

Mr. R. E. Horton, who has been in the United States Air Force, has returned to his position as an instructor at Los Angeles City College.

Associate Professor H. T. Karnes of Louisiana State University has been promoted to a professorship.

Instructor M. S. Klamkin of Brooklyn Polytechnic Institute has been promoted to an assistant professorship.

Miss Rachael La Roe, head of the Department of Mathematics and Physics at Grand Canyon College, has been appointed Associate Professor of Mathematics and Physics at Mary Hardin-Baylor College.

Mrs. Helen H. Meek, formerly at the David Taylor Model Basin, Washington, D. C., has a position as a mathematician at the Institute for Numerical Analysis, National Bureau of Standards.

Mr. V. A. Miculka of Frank Phillips College, Borger, Texas, has accepted a graduate assistantship at the University of Oklahoma.

Assistant Professor G. W. Morgan, Graduate Division of Applied Mathematics, Brown University, has been promoted to an associate professorship.

Mr. H. W. Morrow, Jr. of the University of Florida is engaged as a mechanical engineer with the Texas Automatic Sprinkler Company, Houston, Texas.

Assistant Professor D. J. Myatt of Antioch College has accepted a position with the Atlantic Research Corporation, Alexandria, Virginia.

Dr. W. R. Orton of Oberlin College has been appointed to an assistant professorship at the University of Arkansas.

Assistant Professor F. D. Parker of St. Lawrence University has been appointed to an associate professorship at Clarkson College of Technology.

Assistant Professor W. D. Peeples, Jr., of Howard College has been promoted to an associate professorship.

Associate Professor P. I. Peters of Union College has accepted a position as an actuarial assistant with the Commonwealth Life Insurance Company, Louisville, Kentucky.

Miss Edna M. Norskog of Illinois State Normal University died on May 14, 1953.

Reverend D. A. Steele, professor of mathematics at Fordham University, died on June 18, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 78 persons have been elected to membership by the Board of Governors on applications duly certified.

N. E. ALBRECHT, M.A. (Minnesota) Asst. Professor, Hamline University.

R. F. ANASTASIO, B.S. (C.C.N.Y.) Aeronautical Research Scientist, Ames Aeronautical Laboratory, Moffett Field, Calif.

MRS. RUTH E. BERARD, B.S. (Medical Evangelists) Medical Technologist, St. Catherine's Hospital, East Chicago, Ind.

P. M. BERRY, Student, University of Oklahoma.

DAVID BLACKWELL, Ph.D. (Illinois) Professor, Howard University.

D. H. BLANKSTEEN, B.S. (C.C.N.Y.) Office Manager, E. & W. Blanksteen, New York City.

R. O. BLUMMER, JR., M.S. (N. Texas S. C.) Asso. Mathematician, Vitro Corporation of America, Silver Spring, Md.

ELEAZER BROMBERG, Ph.D. (N.Y.U.) Asst. Chief, Computing Center, Institute of Mathematical Sciences, New York University.

MORTON BROWN, Student, University of Wisconsin.

EUGENE BUTKOV, Student, University of British Columbia.

M. L. CLINNICK, Student, University of California.

R. F. DENNEMEYER, M.A. (U.C.L.A.) 715C West Queen Street, Inglewood, Calif.

ROLAND DEWIT, Room 914, Central Y.M.C.A., Dayton 2, Ohio.

V. J. DIGRICOLI, B.S. (Manhattan) Mathematician, International Business Machines Corporation, New York City.

CHESTER DOMORACKI, B.S. (Alliance) 25 Temple Street, Haverhill, Mass.

T. H. ELFERT, B.S. (St. Norbert) 317 14th Avenue, Green Bay, Wis.

M. P. EMERSON, Ph.D. (Illinois) Asst. Professor, Harpur College.

R. J. ENGERT, M.S. (Illinois) Grad. Student, University of Illinois.

P. C. FIFE, B.A. (California) Grad. Student, University of California.

EDWARD FLANDERS, B.A. (Oklahoma A. & M.) 1502 West Admiral Road, Stillwater, Okla.

F. A. FLORIO, B.A. (La Salle) Physicist, Frankford Arsenal, Philadelphia, Pa.

ALLEN FOX, Student, Brooklyn College.

RICHARD GABEL, M.A. (Columbia) Grad. Student, American University.

I. B. GOLDBERG, Student, New York University.

M. J. GOLDSTEIN, Student, Memphis State College.

A. J. GUNTHER, M.A. (Columbia) Teacher, Avon Old Farms, Avon, Conn.

W. J. HALLMARK, M.A. (Texas) Professor, San Antonio College.

W. J. HARDELL, M.S. (Michigan S. C.) Grad. Assistant, Michigan State College.

Miss Edna M. Norskog of Illinois State Normal University died on May 14, 1953.

Reverend D. A. Steele, professor of mathematics at Fordham University, died on June 18, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 78 persons have been elected to membership by the Board of Governors on applications duly certified.

N. E. ALBRECHT, M.A. (Minnesota) Asst. Professor, Hamline University.

R. F. ANASTASIO, B.S. (C.C.N.Y.) Aeronautical Research Scientist, Ames Aeronautical Laboratory, Moffett Field, Calif.

MRS. RUTH E. BERARD, B.S. (Medical Evangelists) Medical Technologist, St. Catherine's Hospital, East Chicago, Ind.

P. M. BERRY, Student, University of Oklahoma.

DAVID BLACKWELL, Ph.D. (Illinois) Professor, Howard University.

D. H. BLANKSTEEN, B.S. (C.C.N.Y.) Office Manager, E. & W. Blanksteen, New York City.

R. O. BLUMMER, JR., M.S. (N. Texas S. C.) Asso. Mathematician, Vitro Corporation of America, Silver Spring, Md.

ELEAZER BROMBERG, Ph.D. (N.Y.U.) Asst. Chief, Computing Center, Institute of Mathematical Sciences, New York University.

MORTON BROWN, Student, University of Wisconsin.

EUGENE BUTKOV, Student, University of British Columbia.

M. L. CLINNICK, Student, University of California.

R. F. DENNEMEYER, M.A. (U.C.L.A.) 715C West Queen Street, Inglewood, Calif.

ROLAND DEWIT, Room 914, Central Y.M.C.A., Dayton 2, Ohio.

V. J. DIGRICOLI, B.S. (Manhattan) Mathematician, International Business Machines Corporation, New York City.

CHESTER DOMORACKI, B.S. (Alliance) 25 Temple Street, Haverhill, Mass.

T. H. ELFERT, B.S. (St. Norbert) 317 14th Avenue, Green Bay, Wis.

M. P. EMERSON, Ph.D. (Illinois) Asst. Professor, Harpur College.

R. J. ENGERT, M.S. (Illinois) Grad. Student, University of Illinois.

P. C. FIFE, B.A. (California) Grad. Student, University of California.

EDWARD FLANDERS, B.A. (Oklahoma A. & M.) 1502 West Admiral Road, Stillwater, Okla.

F. A. FLORIO, B.A. (La Salle) Physicist, Frankford Arsenal, Philadelphia, Pa.

ALLEN FOX, Student, Brooklyn College.

RICHARD GABEL, M.A. (Columbia) Grad. Student, American University.

I. B. GOLDBERG, Student, New York University.

M. J. GOLDSTEIN, Student, Memphis State College.

A. J. GUNTHER, M.A. (Columbia) Teacher, Avon Old Farms, Avon, Conn.

W. J. HALLMARK, M.A. (Texas) Professor, San Antonio College.

W. J. HARDELL, M.S. (Michigan S. C.) Grad. Assistant, Michigan State College.

A. R. HARVEY, Ph.D. (Harvard) Assoc. Professor, San Diego State College.

JOY S. HELLER, B.A. (Brooklyn) 597 Crown Street, Brooklyn 13, N. Y.

D. R. HERSCHBACH, Student, Stanford University.

D. S. HOFFMAN, M.S. (Pittsburgh) Mathematician-Chief Programmer, Gulf Research and Development Company, Pittsburgh, Pa.

MARY L. HOLTON, B.A. (Wellesley) 88-30 80th Street, Woodhaven, N. Y.

LUC HUANG, Student, University of California.

S. Y. HUSSEINI, B.S. (Rutgers) Grad. Student, Rutgers University.

C. C. IRVING, B.S. (Memphis S. C.) 582-G Camilla, Memphis 4, Tenn.

R. V. JAMISON, M.A. (California) Instr., Oregon State College.

BERNARD KALMANOWICZ, Student, Brooklyn College.

ANTHONY KAMIENSKI, Student, Alliance College.

I. N. KATZ, B.A. (Yeshiva) Assistant, Yeshiva University.

O. L. KINGSLEY, B.S. in Chem. (Denver) Mathematician, White Sands Proving Grounds, Las Cruces, N. M.

REV. BROTHER T. B. KNEALE, B.S. (St. Mary's Coll. of California) Grad. Student, University of Notre Dame.

C. W. KOINER, 3405 Florecita Drive, Altadena, Calif.

L. H. KOOPMANS, B.A. (San Diego, S. C.) Grad. Student, University of California.

PEGGY J. KOSSOW, Student, University of Wisconsin.

ISRAEL KRONGOLD, Student, Brooklyn College.

J. F. LANAHAN, M.S. (Michigan) Instr., University of Detroit.

M. H. LANE, B.S. (Troy S. T. C.) Mathematical Statistician, Air Proving Grounds, Eglin Air Force Base, Fla.

B. R. LEVY, B.A. (N.Y.U.) Research Assistant, Mathematics Research Group, New York University.

L. F. LIBELO, JR., Student, Brooklyn College.

E. L. LIPPERT, B.S. in Chem. (Oklahoma) Grad. Student, University of Oklahoma.

SEVERIN LOW, Student, McGill University.

SAUL MANDEL, Student, University of Oklahoma.

GABRIEL MARGULIES, B.S. (U. of Washington) Research Assistant, Graduate Institute for Applied Mathematics, Indiana University.

ESSOR MASO, B.A. (U.C.L.A.) Supervisor Scientific Computer, Hughes Aircraft Company, Culver City, Calif.

J. C. MONTGOMERY, Ph.D. (Yale) Asso. Professor, University of Connecticut.

MARY P. MOSELEY, M.S. (Iowa) Teacher, Corpus Christi Independent School District, Texas.

K. R. MOUNT, Student, University of Illinois.

REV. FLORIAN MUGGLI, M.A. (Notre Dame) Instr., St. John's University.

G. W. O'SHAUGHNESSY, B.A. (C. of St. Thomas) Welch, Minn.

STANLEY PAUMER, Teacher, High School, Basin, Wyo.

S. R. PETRICK, Student, Iowa State College.

F. A. RAYMOND, B.A. (Syracuse) 469 Whittier Avenue, Syracuse 4, N. Y.

H. M. ROSENBAUM, M.A. (Buffalo) Assistant Research Mathematician, Cornell Aeronautical Laboratory, Buffalo, N. Y.

S. M. ROSENZWEIG, B.S. (C.C.N.Y.) Part-time Teaching Assistant, Massachusetts Institute of Technology.

D. W. SASSER, Student, State College of Washington.

H. S. SCHLOSS, B.A. (Minnesota) United States Army.

J. J. SCIARRA, B.S. (Ursinus) 5104 Springfield Avenue, Philadelphia 43, Pa.

R. E. SECHLER, Student, Albion College.

J. M. SHAHEEN, M.A. (Cincinnati) Taft Teaching Fellow, University of Cincinnati.

SISTER MARY CARMEL, M.A. (Ohio State) Instr., Albertus Magnus College.

R. K. SMITH, B.A. (California) Engineer, American Cyanamid Company, Idaho Falls, Idaho.

C. M. SOMMERFIELD, Student, Brooklyn College.

JEAN STREMP, B.A. (Seton Hill) Student, Seton Hill College.

J. P. THOMSEN, Student, American University.

HAROLD WEITZNER, Student, University of California.

N. A. WHEELER, Student, Reed College.

L. G. WOODBY, Ph.D. (Michigan) Asso. Professor, State Teachers College, Mankato, Minn.

THE MARCH MEETING OF THE METROPOLITAN NEW YORK SECTION

The twelfth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Teachers College, Columbia University, New York City, on March 28, 1953. Professor W. H. Fagerstrom, Collegiate Vice-Chairman, presided at the morning session, and Professor L. F. Ollmann, Chairman of the Section, presided at the afternoon session.

One hundred thirty-two persons attended the meeting, including the following eighty-four members of the Association:

R. G. Archibald, Winifred A. Asprey, L. F. Babcock, C. Y. Bartholomew, J. S. Bergen, Samuel Borofsky, C. B. Boyer, A. D. Bradley, A. B. Brown, Jewell H. Bushey, A. J. Carlan, Audrey M. Carlan, P. L. Chessin, Charles Clos, P. J. Cocuzza, H. R. Cooley, T. F. Cope, W. H. H. Cowles, Mary P. Dolciani, Jesse Douglas, Brother E. I. Duggan, J. N. Eastham, Samuel Eilenberg, W. H. Fagerstrom, H. F. Fehr, William Forman, R. M. Foster, E. T. Frankel, Leona Freeman, Brother Bernard Gerald, B. P. Gill, Bernard Greenspan, G. C. Helme, L. J. Herbach, Arthur Herskowitz, E. Marie Hove, T. R. Humphreys, O. J. Karst, E. R. Kiely, H. S. Kieval, A. E. Kinney, M. S. Klamkin, Morris Kline, Charles Koren, David Kotler, H. C. Kranzer, R. A. C. Lane, C. H. Lehmann, M. E. Levenson, D. M. MacEwen, J. R. Macy, J. H. Manheim, May H. Maria, K. A. McGown, Brother J. G. McKenna, F. H. Miller, Jack Minker, Morris Morduchow, Eugene Odin, L. F. Ollmann, C. F. Pinzka, J. J. Quinn, J. K. Reckzeh, M. R. Reeks, Selby Robinson, I. H. Rose, N. J. Rose, M. F. Rosskopf, H. D. Ruderman, J. P. Russell, D. A. Russo, John Salerno, Charles Salkind, Aaron Shapiro, E. I. Shapiro, James Singer, Sister M. Anita, E. R. Stabler, R. L. Swain, N. Y. Tang, H. E. Wahlert, Etta A. Waite, M. E. White, R. C. Yates.

The following officers were elected for the year 1953–54: Chairman, Professor W. H. Fagerstrom, City College of the City of New York; Collegiate Vice-Chairman, Professor H. F. Fehr, Teachers College, Columbia University; High School Vice-Chairman, Mr. Charles Salkind, Samuel Tilden High School, Brooklyn; Secretary, Professor E. Marie Hove, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting, the following report on the activities of the Committee on Contests and Awards was given by its chairman, Professor W. H. Fagerstrom:

The Committee on Contests and Awards of the Metropolitan New York Section reported that 339 schools had registered for the fourth annual contest and that these schools had requested 9,672 copies of the contest booklets. In addition to these schools, there are three units operating independently using the Metropolitan New York Section's questions. These units are centered at 1) The University of Oregon, 2) The University of British Columbia, and 3) The Board of Education of Buffalo, N. Y. These three units have requested a total of 2,300 booklets. The 339 schools listed above are from 29 states and 2 Canadian provinces. The contest will be held on May 14, 1953.

Dr. Paul Bulger, Assistant Provost of Columbia University, welcomed the people at the meeting, and then the following papers were presented:

1. *Mathematics and the liberal arts student*, by Professor Morris Kline, New York University.

The practice in American colleges and universities of offering college algebra and trigo-

nometry to students who do not intend to use mathematics in later life was severely criticized. These courses have no cultural content and consist of dry, unmotivated, and purely technical procedures which students learn to repeat in parrot-like fashion. Instead of such courses, the speaker described and recommended a course which presents basic mathematical concepts in the cultural context in which these concepts arose and which includes a presentation of the significance of these concepts for modern civilization and culture. Such a course is being tried at the Washington Square College of New York University.

2. *Theory and methods of electronic digital computation*, by Mr. J. H. White, Jr., International Business Machines, New York City, introduced by the Secretary.

This paper states what one can expect of computing machines and particularly why they are important for scientific and engineering problems. Each of IBM's three types of electronic calculators is described in terms of the five elements of logical design: input, output, storage, arithmetic unit, and control unit. They are used in industry, government, and education. Twenty-six colleges and universities now have IBM computing installations. Some of the types of mathematical and physical problems readily solved by these machines are mentioned with emphasis on a problem of aircraft propeller design involving solution of two simultaneous fourth-order differential equations reduced to a set of forty simultaneous linear equations in forty unknowns.

3. *Trends in mathematics: the elementary school*, by Dr. Laura K. Eads, Board of Education of the City of New York, introduced by the Secretary.

The elementary school mathematics program in New York City, designated as *Developmental Mathematics*, is moving toward helping children to think mathematically: to discover mathematical relationships, to derive their own generalizations, to develop independence in problem-solving through estimation and computation without the use of paper and pencil. This program emphasizes the development of mathematical concepts, facts, and processes through levels of learning, engaging in experience, using representative materials, thinking through mathematical relationships, written computation. The mathematics is learned sequentially from the first grade. Understanding of the mathematical meaning or structure of numbers and processes, and of measurement is developed before skill in computation is emphasized.

4. *Curriculum trends in high school mathematics*, by Mr. Max Peters, Long Island City High School, New York City, introduced by the Secretary.

The most significant factor affecting the mathematics curriculum in the high schools is the fundamental change in student composition. There is grave danger that the traditional academic mathematics courses will be seriously weakened in the attempt to reach the mediocre and poor student. The development of second track courses can help to check this tendency. The two major movements in the academic mathematics high school curriculum are greater emphasis upon meaning and more systematic attempts at integration. Complicated manipulation in algebra has been de-emphasized in favor of rationalization of operations. The postulational structure of plane geometry has been stressed. Units of coordinate geometry have been included in plane geometry courses and progress has been made in developing a fused course in the eleventh year combining intermediate algebra and trigonometry.

5. *Topology: Its relation to other branches of mathematics*, by Professor Samuel Eilenberg, Columbia University.

H. S. KIEVAL, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-sixth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, Colorado, on April 17 and 18, 1953. Professor B. W. Jones, Chairman of the Section, presided at all the sessions.

Of the eighty-five persons who registered, the following fifty-five were members of the Association:

C. F. Barr, D. L. Barrick, B. C. Bellamy, W. E. Briggs, J. R. Britton, R. K. Butz, F. M Carpenter, Sarvadaman Chowla, G. S. Cook, F. W. Donaldson, W. E. Dorgan, F. N. Fisch, C. A. Grimm, Arnold Grudin, H. T. Guard, R. R. Gutzman, Marian S. Gysland, C. L. Harbison, Leota C. Hayward, I. L. Hebel, Ruth I. Hoffman, LeRoy Holubar, P. F. Hultquist, J. A. Hurry, C. A. Hutchinson, B. W. Jones, A. J. Kemper, Claribel Kendall, J. S. Leech, D. C. B. Marsh, Jr., Garner McCrossen, H. C. McKenzie, E. B. McLeod, Jr., W. E. Mientka, W. K. Nelson, Greta Neubauer, D. K. Parks, Lily B. Powell, O. M. Rasmussen, O. H. Rechard, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, K. H. Stahl, P. O. Steen, E. L. Swanson, C. W. Thomson, E. P. Tovani, E. L. Vanderburgh, V. J. Varineau, W. W. Varner, J. F. Wagner.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor M. L. Madison, Colorado Agricultural and Mechanical College; Vice-Chairman, Professor Nathan Schwid, University of Wyoming; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The program of papers for the meetings was as follows:

1. *Some results in number theory using a partial summation method*, by Mr. W. E. Briggs, University of Colorado.

In the classical proofs of theorems concerning the representation of primes by binary quadratic forms, it is necessary to use facts about the continuity, differentiability, and behavior as $s\to 1^+$ of the series $\sum(ax^2+2bxy+cy^2)^{-s}$ and other series similar to it. The summation is extended over all x, y which make the form prime to $2D$, where $D=b^2-ac$, and which satisfy certain other conditions if $D>0$. These facts can all be proved simply by estimating the sum as $\sum_{n=1}^{\infty}n^{-s}[T(n)-T(n-1)]$, where $T(n)$ is the number of lattice points within $ax^2+2bxy+cy^2=n$ which make the form prime to $2D$ and satisfy the other conditions if $D>0$.

2. *Polynomials associated with matrices*, by Professor R. K. Butz, Colorado Agricultural and Mechanical College.

Notation was developed to handle the matric equation $AX=XB$, where A and B are specified matrices of order n and m, respectively, defined over an arbitrary field F, and X is to be determined in terms of parameters using only those operations with respect to which F is closed. The approach to the problem was that given by W. V. Parker (*The matrix equation $AX=XB$*, Duke Mathematical Journal, vol. 17, no. 1, 1950, p. 43).

3. *Some topics in the theory of numbers*, Professor Sarvadaman Chowla, University of Colorado.

4. *An approximation method in certain nonlinear boundary value problems*, by Professor Nathan Schwid, University of Wyoming.

In the differential equation of heat conduction,

$$c\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right),$$

8. *The content and method for a general mathematics course for adults*, by Miss Ruth I. Hoffman, Byers Junior High School, Denver, Colorado.

9. *The rapid growth of numerical analysis since 1943 and the challenge it offers to the university teacher*, by Mr. W. W. Varner, University of Colorado.

The mushrooming of numerical analysis and its inherent problem of error consideration has greatly increased the need of all teachers of mathematics, engineering, and the sciences to be meticulous in demanding that problem solutions be written in such a manner that the error or uncertainty in every result be clearly and unmistakably given. Certain aspects of this problem were discussed.

10. *On the improvement of service courses in freshman mathematics*, by Professor I. L. Hebel, Colorado School of Mines.

An outline is presented of a revised approach to the teaching of freshman mathematics adopted at Colorado School of Mines. The traditional sequence of topics is replaced by a unification that stresses the analytic geometry viewpoint throughout and that maintains the necessary rigor of a pre-engineering mathematics course. Principal results of the initial trial of the plan include increased student interest, improved faculty instruction, and a better prepared student for sophomore courses. The author and his staff feel that such a curriculum is a forward step in the solution of the perplexing freshman teaching problem.

11. *Mathematics used in university departments other than mathematics or engineering*, by Professor O. M. Rasmussen, University of Denver.

A report was presented on a part of a study using the methods of textbook analysis, interviews, and questionnaires to determine those mathematical skills and concepts that are desirable as preparation for non-mathematics course work for university students not majoring in mathematics or the physical sciences. The mathematics needed for a vast majority of these students is quite elementary and many of the students do not possess sufficient arithmetical maturity to enable them to gain maximum benefit from a large number of university courses. An understanding of elementary statistics is needed in many courses throughout the university.

J. R. Britton, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Texas Christian University, Fort Worth, Texas, on April 24–25, 1953. Professor C. B. Wright, Chairman of the Section, presided at the sessions. Professor L. R. Ford, who was an invited guest, contributed much to the success of the meeting.

There were one hundred eight persons in attendance, including the following sixty-two members of the Association:

T. A. Abouhalkah, R. C. Ailara, A. W. Ashburn, A. V. Banes, Ina M. Bramblett, H. E. Bray, Myrtle C. Brown, M. L. Coffman, L. A. Colquitt, J. V. Cooke, Don Cude, F. W. Donaldson, G. H. Dubay, L. K. Durst, Terrell Ellis, L. R. Ford, Gordon Fuller, R. L. Glass, Blanche B. Grover, W. T. Guy, Jr., E. H. Hanson, E. A. Hazelwood, E. R. Heineman, Fay H. Johnson, Ruth Kissel, E. C. Klipple, H. A. Luther, Hazel L. Mason, Lida B. May, Dorothy McCoy, W. K. McNabb, V. A. Miculka, Harlan C. Miller, B. C. Moore, E. D. Mouzon, Jr., C. A. Murray, Albert Newhouse, Bob Parker, H. C. Parrish, C. J. Pipes, C. B. Rader, Sr., L. W. Ramsey, Dorothy L. Rees, C. L. Riggs, Virginia E. Roberts, C. A. Rogers, R. Q. Seale, C. R. Sherer, D. P. Shore, Sister Mary of Perpetual Help, D. W. Starr, W. G. Stokes, W. W. Taylor, Earl Thomas, F. E. Ulrich, R. S.

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J. R. BRITTON, *Secretary*

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Underwood, Margaret M. Welch, Mabel Williams, H. A. Wood, A. W. Wortham, C. B. Wright, Martin Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Professor L. A. Colquitt, Texas Christian University; Vice-Chairman, Professor E. A. Hazelwood, Texas Technological College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The program consisted of the following papers:

1. *The summation of certain trigonometrical series by undergraduate methods*, by Professor E. C. Klipple, Agricultural and Mechanical College of Texas.

Certain trigonometric series can be summed without the use of complex numbers or general convergence theorems. In particular, the series $\cos x+\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+\cdots$ can be summed by elementary methods.

2. *Partial difference equations of elliptic type*, by Mr. Horace Flatt, Rice Institute, introduced by the Secretary.

Several properties of partial difference equations of elliptic type are discussed in this paper and in particular analogies between the theory of differential and difference equations are pointed out. The existence and uniqueness of a solution for the first boundary value problem for the general elliptic equation of second order is shown.

3. *The first two weeks of calculus*, by Professor L. R. Ford, University of Mississippi.

The speaker proposed the early introduction of the notions of upper and lower bounds. These lead naturally to the concepts of area, length of a curve, and the like. Further, they provide a reasonably simple approach to the ideas of continuity and of limit.

4. *A technique for simultaneous quadratics*, by Professor R. S. Underwood, Texas Technological College.

The usual elimination methods of algebra are almost hopelessly inadequate to find common real solutions of simultaneous quadratic equations when the number of unknowns exceeds two or three. However, many problems of this nature are solved easily by algebraic methods which are guided by the graphs of extended analytic geometry. For simplicity of presentation the examples chosen to illustrate the method involve only four or five unknowns, but this restriction is not necessary.

5. *The analytic and geometric definitions of the trigonometric functions*, by Professor E. R. Keown, Agricultural and Mechanical College of Texas, introduced by the Secretary.

The purpose of this paper is to derive the Euler formula $e^{ix}=\cos x+i\sin x$ and to show explicitly the connections between the analytic definition of the trigonometric functions as power series and their geometric definitions as ratios of line segments. The argument consist in showing that e^x as defined by the series is really a power and that e^{ix} is a complex number of modulus one for every real x whose real and imaginary parts are the $\cos x$ and the $\sin x$ when a proper choice of angular measure is made.

6. *Certain polynomial approximations*, by Mr. R. M. McLeod, Rice Institute, introduced by the Secretary.

Two theorems dealing with approximation of functions holomorphic and never zero in a

ment which must be met to enable a professional engineer to digest in a sufficient manner the statements and solutions of many problems of new books and articles written concerning the physical and chemical properties of oil and gas. Many mathematical formulae and equations are shown and solved, all of which are pertaining to petroleum engineering problems. The author has in mind an effective course of mathematics combining the working essence of formulae and their manipulation compiled, digested, and taught for the particular need of engineers. A lack of mathematics is the fault of the school, not the students. Engineering has become too exact to depend upon sketchy memory and doubt and manuals solicited from large manufacturers.

13. *The content of business mathematics*, by Professor Bob Parker, Texas Technological College.

There are three courses that ordinarily come under the heading of business mathematics: a first semester course, mathematics of finance, and statistics. The first course should be college algebra and not a version of business arithmetic since the student will need algebra in his later courses. In the mathematics of finance the line diagram and equation of value should be stressed. Statistics, other than an introductory course, should not be a business subject because the students lack the mathematical background for understanding the subject.

14. *Recent developments in the certification of teachers in Texas*, by Professor E. A. Hazlewood, Texas Technological College.

This paper deals with general developments in the teacher training field in Texas subsequent to the enactment of the Gilmer-Aikin Bills in 1949. Results of the statewide study of certification sponsored by the Texas Education Agency and other interested groups are reviewed. The provisions of the Certification Bill (H. B. 496) now before the Texas Legislature are discussed.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twenty-seventh meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Carnegie Institute of Technology, Pittsburgh, Pennsylvania, on May 2, 1953. Dr. Morris Ostrofsky, Chairman of the Section, presided at the morning session, while Professor David Moskovitz of the host institution presided in the afternoon.

There were seventy-five persons present including the following forty-five members of the Association:

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It was voted to support an autumn meeting designed to interest high school teachers particularly.

Officers elected to serve from June 1953 to June 1955 were: Chairman, Professor F. H. Steen, Allegheny College; Secretary-Treasurer, Dean L. T. Moston, Waynesburg College; Executive Council, President B. C. Patterson,

ment which must be met to enable a professional engineer to digest in a sufficient manner the statements and solutions of many problems of new books and articles written concerning the physical and chemical properties of oil and gas. Many mathematical formulae and equations are shown and solved, all of which are pertaining to petroleum engineering problems. The author has in mind an effective course of mathematics combining the working essence of formulae and their manipulation compiled, digested, and taught for the particular need of engineers. A lack of mathematics is the fault of the school, not the students. Engineering has become too exact to depend upon sketchy memory and doubt and manuals solicited from large manufacturers.

13. *The content of business mathematics*, by Professor Bob Parker, Texas Technological College.

There are three courses that ordinarily come under the heading of business mathematics: a first semester course, mathematics of finance, and statistics. The first course should be college algebra and not a version of business arithmetic since the student will need algebra in his later courses. In the mathematics of finance the line diagram and equation of value should be stressed. Statistics, other than an introductory course, should not be a business subject because the students lack the mathematical background for understanding the subject.

14. *Recent developments in the certification of teachers in Texas*, by Professor E. A. Hazlewood, Texas Technological College.

This paper deals with general developments in the teacher training field in Texas subsequent to the enactment of the Gilmer-Aikin Bills in 1949. Results of the statewide study of certification sponsored by the Texas Education Agency and other interested groups are reviewed. The provisions of the Certification Bill (H. B. 496) now before the Texas Legislature are discussed.

C. R. SHERER, *Secretary*

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which affords a means for obtaining the complementary function of a second order linear differential equation with constant coefficients and imaginary "roots" without resorting to Euler's expression for $e^{i\theta}$.

4. *Concerning metamathematics and Gödel's theorem*, by Mr. F. C. DeSua, University of Pittsburgh, introduced by the Secretary.

An exposition of some methods and results in metamathematics was presented. Results were stated informally without proof although heuristic accounts were given of Gödel's theorem and of Post's proof of the consistency of the propositional calculus.

5. *Some functions related to harmonic functions*, by Professor I. M. Sheffer, The Pennsylvania State College.

Let $u(x, y)$ be an analytic function of the real variables x, y and such that the power series for u and all its derivatives converge in and on a fixed circle C. There exist such functions u that are non-harmonic and such that u and all its derivatives satisfy the Gauss mean value theorem for this one circle C. Properties of these functions are examined. In particular, they satisfy a linear, homogeneous partial differential equation of infinite order in the Laplacian Δu, with constant coefficients.

6. *Exterior forms in Hilbert space*, by Professor T. T. Tanimoto, Allegheny College.

A Grassmann algebra of multilinear mapping in Hilbert space is constructed and a few applications are given.

7. *On finding the characteristic equation of a square matrix*, by Mr. W. Berger and Professor E. A. Saibel, Carnegie Institute of Technology, presented by Mr. Berger.

Using the elementary operations which leave the characteristic equation of the matrix unchanged, it is always possible to transform the matrix such that there is an array of zeros in the lower left-hand corner below the sub-diagonal. Using the Cayley-Hamilton theorem and post-multiplying by the column matrix $\{1, 0, 0, \cdots, 0\}$, a triangular set of equations in the coefficients of the characteristic equation results.

8. *The minimum essentials of a required course in mathematics for business majors*, by Professor Mary A. Goins, Marshall College.

The need for an adequate course in mathematics for students in business curricula prompted this study. A questionnaire containing twenty topics thought to be fairly representative of the minimum content of a reasonably adequate required course in mathematics for business students, with space provided for other suggested topics, was sent to the deans of forty-five professional schools of business in Colorado, Florida, Louisiana, Maryland, Michigan, Oregon, and Texas. The replies were tabulated and analyzed with reference to each curriculum. The final analysis showed that the topics mentioned more than fifty percent of times in all curricula might be taught adequately in a three semester-hour course. If six semester-hours were given to the subject, the topics which were required by all of the schools in all curricula could be taught thoroughly.

F. H. STEEN, *Secretary*

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3. *Is undergraduate mathematics part of general education*?, by Sister Gertrude Marie, Marian College.

Discrepancy between general education values of mathematics and the place accorded to mathematics in undergraduate curricula of representative colleges and universities is shown. Resources of mathematics capable of contributing significantly to realization of objectives of general education are balanced against the trend to relegate the subject to the status of "not required" or "alternative with science or philosophy." Recommendations include re-examination of the relationship of mathematics to other disciplines, revaluation of course materials on the basis of general importance, adoption of improved methods and newer instruction media, logical combination of branches of mathematics, and integration of subject matter with the student's total experience.

4. *The School and College Study of Admission with Advanced Standing*, by Professor J. C. Polley, Wabash College.

A brief description of a study, financed by grants from the Fund for the Advancement of Education, in which twelve colleges and twelve secondary schools are collaborating to examine the feasibility of, and set up standards for, granting college credit for courses taken in secondary school. Considered in particular are recommendations of the sub-committee on mathematics concerning a course sequence for the tenth, eleventh, and twelfth years, designed to replace present offerings and cover materials frequently included in the first year of college mathematics.

5. *Visualization in the integral calculus*, by Professor S. H. Gould, Purdue University.

On the principle that "ontogeny recapitulates phylogeny" the speaker discussed the advantage that can be gained in present day classrooms by visualizing the early attempts in the history of mathematics to find the volume of various solids.

6. *Sequential limit spaces*, by Professor J. L. Lawrence, Wabash College, introduced by Professor Polley.

Necessary and sufficient conditions on the collection of open sets are obtained in order that a subset having x as a limit point will contain a sequence of distinct points convergent to x. Such spaces are shown to be completely determined by a knowledge of the convergent sequences.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Louisville, Louisville, Kentucky, on May 9, 1953. Professor W. L. Moore, Chairman of the Section, presided at the morning and afternoon sessions.

Fifty-seven persons were present, including the following thirty-eight members of the Association:

H. H. Berry, J. M. Boswell, M. C. Brown, W. M. Bullitt, Esther A. Compton, J. B. Cornelison, H. H. Downing, R. I. Fields, Clarence Ford, A. W. Goodman, Reverend H. H. Gottbrath, Beulah Graham, Charles Hatfield, Aughtum S. Howard, G. B. Huff, Tadeusz Leser, A. G. McGlasson, D. G. Miller, W. L. Moore, R. S. Park, W. H. Pell, Sallie E. Pence, D. W. Pugsley, V. Elise Qualls, G. G. Roberts, W. J. Robinson, F. E. Ross, J. H. Simester, Sister M. Rosalin, Sister Mary Charlotte, R. H. Sprague, Guy Stevenson, R. P. Tapscott, J. T. Vallandingham, J. A. Ward, R. H. Wilson, T. M. Wright, W. M. Zaring.

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6. *Some first editions of famous mathematicians*, by Mr. W. M. Bullitt, Louisville, Kentucky.

Mr. Bullitt gave a very interesting account of the history of several first editions of famous mathematical papers in his collection. Since the originals are now on display at Harvard, he showed photographs of them.

7. *What should we teach in freshman mathematics?* A panel discussion with the following members: President J. M. Boswell, Cumberland College; Mr. J. W. Huss, auditor with Lybrand-Ross Bros., and Montgomery; Professor W. H. Pell, University of Kentucky; Professor W. J. Robinson, Centre College; Professor J. H. Simester, Speed Scientific School, University of Louisville.

The members of the panel deplored the high school preparation of freshmen and gave various methods of meeting that difficulty. Each member of the panel showed how he handled the problem.

8. *On introducing arguments into freshman mathematics*, by Professor G. B. Huff, University of Georgia. (By invitation.)

Professor Huff began by suggesting that the widespread dissatisfaction with freshman mathematics may be relieved by introducing mathematical arguments at every opportunity. If the fundamental laws and the number systems are mentioned early and by name, it was shown that arguments may be made in connection with most of the usual subject matter. In this way freshmen have the opportunity to do mathematics themselves and to draw logical conclusions from the results of algebraic operations.

9. *Gauss-Lucas theorem*, by Professor A. W. Goodman, University of Kentucky.

An expository talk covering the Gauss-Lucas theorem and Jensen's extension.

10. *Elementary methods of forecasting as used in a specific utility rate case*, by Mr. F. E. Ross, Lybrand-Ross Bros., and Montgomery.

The forecasts were made in the hearings of a state regulatory agency in the application of a city bus company for increased fares. Historical data as to actual riding experience were normalized to adjust for strikes and forecasts were made by methods of finite differences, least squares, and more simple percentage relationships. Sampling methods were not considered as feasible or conclusive for such forecasts.

J. A. WARD, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the United States Naval Proving Ground, Dahlgren, Virginia, on May 2, 1953. Professor Marian M. Torrey, Chairman of the Section, presided at the morning and afternoon sessions.

There were one hundred eleven persons in attendance, including the following sixty-five members of the Association:

J. C. Abbott, D. F. Atkins, R. P. Bailey, N. H. Ball, J. E. Barker, W. E. Barnes, J. W. Blincoe, T. A. Botts, C. C. Bramble, Mary P. Burkhart, H. H. Campaigne, J. F. Canu, S. H.

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If f is real-valued and continuous on a 2-sphere, S^2, there exist three points p_1, p_2, p_3 on S^2 such that

$$f(p_1) = f(p_2) = f(p_3),$$

and there exists a point r contained in $f(S^2)$ such that some component of $f^{-1}(r)$ contains a pair of antipodal points of S^2. These results give a generalization of a result obtained by Yamabe Yujobo, and include as a special case two related theorems of Kakutani and de Mira Fenandes. The theorems proved yield corollaries pertaining to properties of a two closed set covering of S^2.

5. *The logical design of the Dahlgren calculators*, by Mr. R. A. Niemann, United States Naval Proving Ground, introduced by the Chairman.

The designs of the Aiken Relay Calculator (Mark II) and of the Mark III Electronic Calculator were discussed. The storage capacity of the Mark II is 100 ten digit numbers, that of the Mark III is 4350 sixteen digit numbers. The output of the Mark II can be recorded on punched paper tape or printed directly, that of the Mark III is given on magnetic tape which can be fed into electric typewriters for printing. The Mark II can do twenty-four operations per second, the Mark III can do one hundred forty. Trends in the design of future computers were discussed.

6. *Certain problems of approximation using Bernoulli and Euler polynomials*, by Professor D. C. Lewis, Jr., The Johns Hopkins University.

This was the invited lecture. The problem of expanding a function in terms of its even order derivatives at two distinct points leads to series of Bernoulli polynomials. To expand it in terms of even derivatives at one point and odd derivatives at the other point leads to series of Euler polynomials. Such expansions are possible only for entire functions of exponential type, the l.u.b. of the permitted type being inversely proportional to the distance between the two points. Results of this kind were obtained in part by H. Poritsky and N. Bielson. Certain determinant formulas were obtained for both Bernoulli and Euler polynomials.

C. H. FRICK, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The May meeting of the Minnesota Section of the Mathematical Association of America was held at St. Olaf College in Northfield, Minnesota, on May 9, 1953. Sessions were held in the forenoon, at luncheon and in the afternoon. Professor C. S. Carlson, Mr. J. S. Hill and Professor A. G. Swanson, Chairman of the Section, presided at the respective sessions.

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At the business meeting, the following officers were elected for the coming

is finite for all x such that $V<+\infty$, and such that $\lim_{x\to\infty}\log V(x)/x=0$. This formulation applies to quantum as well as to classical mechanics.

4. *A diagrammatic interpretation of the formal definition of the derivative of $f(x)$*, by Professor A. L. Duquette, St. John's University, introduced by Reverend W. C. Kalinowski.

Given any ϵ, one can construct an ϵ-neighborhood containing the endpoint of a line segment representing the derivative $f'(x_0)$, such that for a corresponding δ-neighborhood about x_0, all of the chords in the δ-neighborhood passing through x_0, when extended, pass through the ϵ-neighborhood.

5. *On the sum of the face angles and the sum of the angular deficiencies of a polyhedron*, by Mr. N. W. Johnson, Jr., Carleton College.

In terms of a new definition of the density d of a polygon, it is found that the sum of the exterior angles of any polygon is $2\pi d$. The sum of the face angles of a polyhedron of genus p with V vertices and f_d faces of density d is given by

$$\sum \beta = 2\pi\left[V - 2 + 2p - \sum_d f_d(d-1)\right].$$

The amount δ by which 2π exceeds the sum of the face angles at a vertex of a polyhedron is defined to be the angular deficiency at that vertex. It is shown that

$$\sum \delta = 2\pi\left[2 - 2p + \sum_d f_d(d-1)\right].$$

Examples are given to illustrate both formulas.

6. *Report on the two latest electronic computers: the IBM 701 and the ERA 1103*, by Mr. J. Stanley Hill, Actuary, Minnesota Mutual Life Insurance Company.

A very brief description of each machine was given and their likenesses and dissimilarities were noted. The remarkable improvement in speeds over earlier electronic calculators was noted. Emphasis was placed upon the fact that these are "scientific" calculators, capable of handling problems with tremendous calculational magnitude, rather than "commercial" calculators whose ratio of calculating capacity to input and output capacity is much lower.

F. C. SMITH, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The ninth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the United States Military Academy, West Point, New York, on May 8 and 9, 1953. An informal reception was held on May 8; on May 9 the Chairman of the Section, Professor J. F. Randolph of the University of Rochester, presided at the morning session, and the Vice-Chairman, Professor Harriet F. Montague of the University of Buffalo, presided at the afternoon session.

One hundred three persons attended the meeting, including the following fifty-nine members of the Association:

R. A. Beaver, R. L. Beinert, W. W. Bessell, L. H. Bunyan, F. J. H. Burkett, E. A. Butler, W. B. Carver, Mary P. Dolciani, E. J. Downie, Howard Eves, A. B. Farnell, O. J. Farrell, Jean Feidner, J. E. Freund, E. I. Gale, H. M. Gehman, B. H. Gere, Lillian Gough, N. G. Gunderson,

is finite for all x such that $V<+\infty$, and such that $\lim_{x\to\infty}\log V(x)/x=0$. This formulation applies to quantum as well as to classical mechanics.

4. *A diagrammatic interpretation of the formal definition of the derivative of* $f(x)$, by Professor A. L. Duquette, St. John's University, introduced by Reverend W. C. Kalinowski.

Given any ϵ, one can construct an ϵ-neighborhood containing the endpoint of a line segment representing the derivative $f'(x_0)$, such that for a corresponding δ-neighborhood about x_0, all of the chords in the δ-neighborhood passing through x_0, when extended, pass through the ϵ-neighborhood.

5. *On the sum of the face angles and the sum of the angular deficiencies of a polyhedron*, by Mr. N. W. Johnson, Jr., Carleton College.

In terms of a new definition of the density d of a polygon, it is found that the sum of the exterior angles of any polygon is $2\pi d$. The sum of the face angles of a polyhedron of genus p with V vertices and f_d faces of density d is given by

$$\sum \beta = 2\pi\left[V - 2 + 2p - \sum_d f_d(d-1)\right].$$

The amount δ by which 2π exceeds the sum of the face angles at a vertex of a polyhedron is defined to be the angular deficiency at that vertex. It is shown that

$$\sum \delta = 2\pi\left[2 - 2p + \sum_d f_d(d-1)\right].$$

Examples are given to illustrate both formulas.

6. *Report on the two latest electronic computers: the IBM 701 and the ERA 1103*, by Mr. J. Stanley Hill, Actuary, Minnesota Mutual Life Insurance Company.

A very brief description of each machine was given and their likenesses and dissimilarities were noted. The remarkable improvement in speeds over earlier electronic calculators was noted. Emphasis was placed upon the fact that these are "scientific" calculators, capable of handling problems with tremendous calculational magnitude, rather than "commercial" calculators whose ratio of calculating capacity to input and output capacity is much lower.

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5. *The Mangler transformation*, by Mr. Irwin Stoner, Bell Aircraft Corporation.

The speaker derived the transformation of the flow equations about bodies of revolution to the flow equations about 2-dimensional bodies which was stated by Mangler in 1945 and 1946. The derivation considered the equations of motion written in terms of the stream function. The special cases of a conical body and an ogive were then considered.

6. *Teaching a course in the history of elementary mathematics*, by Professor Howard Eves, Champlain College.

In teaching a course in History of Elementary Mathematics the speaker wished to make the students' participation something more than carrying out reading assignments capped with a term paper. A solution was found by forming a catalogue of problem studies, arranged to fit the lecture sequence. Weekly assignments from these problem studies serve to stimulate student interest, give a concrete aspect to the course, crystallize many historical methods and processes, furnish valuable classroom material for the future teacher, and introduce the mathematics major to a number of things of mathematical interest often not encountered in his other courses. This paper concerned itself with an explanation and discussion of these problem studies.

N. G. GUNDERSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

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MINNESOTA, Bemidji State Teachers College, October 10, 1953.

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PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

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SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.

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THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 9

CONTENTS

NOVEMBER 1953

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

THE MATHEMATICAL WORK OF JACQUES HADAMARD*

S. MANDELBROJT, Rice Institute

It is not my intention to give here the biography of Jacques Hadamard; I shall not even try to outline it. Very fortunately he is living, living vigorously and still creating, and I hope he will give us mathematicians a good example of a very long life, even if we are unable to make ours as fertile as his. This month he had his 87th anniversary; this should give some of you a hint of the year when he was born.

Few mathematicians have ever worked in so many fields as did Hadamard, and when I say few, I mean almost none. When he retired as Professor of the Collège de France in 1937 after forty years of service, he had either taught almost every mathematical subject or participated very efficiently in discussions about it in his famous "Séminaires" at the Collège. I have never seen or heard about weekly mathematical events as important, as passionately interesting, as those "Séminaires."

Great mathematicians, and lesser ones too, from countries all over the world, mathematical countries, I mean, considered it a very great honor to be invited by Hadamard to speak about their own work, or about a paper recently published in the subject they knew best. I mean the subject they knew better than any other subject; but usually that same subject, whatever it was, and whoever spoke, was still better known, or at least better understood, by the Master, Professor Hadamard.

How often we youngsters (I speak of the twenties, when even some of *us* were young!), how often, I say, we youngsters felt embarrassed and blushed, and felt much pity for men with great names, sometimes very great names, who triumphantly announced fine and deep theorems, in fields which were their fortresses, when these theorems were, after their exposition, "explained" to the audience *and* to the lecturer, the specialist, by Hadamard—explained and related to other fields in mathematics, and very often simplified. In those seminars, ideas, techniques, knowledge were braced together, and from them, a few weeks later, sometimes on the spot, came new results. The great inspirer was Hadamard.

We shall only speak about ideas and results published in his own papers and books, ideas which often have completely changed the aspect of a field, and, even more often, created new fields.

I remember Hardy introducing Hadamard to the London Mathematical Society in 1944, where he was asked to speak about his most striking discoveries. Hardy called Hadamard the "living legend" in mathematics.

I shall do my best to bring before you some aspects of this legendary mathematical career.

* Presented as an invited address before the annual meeting of the Mathematical Association of America, at St. Louis, Missouri, December 30, 1952.

I said I will not even outline Hadamard's biography, but you may be curious to know that he was not a mathematical "*Wunderkind.*" As he says himself, until the age of eleven or twelve he was the worst, or almost the worst, student in arithmetic. He once asked his father, himself a Professor, if mathematics were studied at the Ecole Normale. "Yes, of course," his father said, "in the Science Section." "Oh, well," was young Hadamard's answer, "then it is not for me."

His first publications appeared in 1884 when he was nineteen. But not until 1888 were Notes published in the Comptes Rendus de l'Académie des Sciences which began to make him famous. Historically I have thus to begin with the Theory of Functions of a Complex Variable.

The Notes of 1888 and 1889 were an introduction to Hadamard's Thesis which was published in 1892, and which rapidly became classical.

That year of 1892 is one of the richest in the history of modern Function Theory, since, in that year, there not only appeared Hadamard's Thesis devoted mostly to the research on singularities of an analytic function by means of its Taylor coefficients at a point, but there also appeared his famous work on entire functions. His function theoretic results of 1892 enabled him also, a few years later, to solve one of the ancient and most important problems in the Theory of Numbers.

Let us now speak about ideas, theorems, results; we shall rarely mention years, data, titles. . . .

It had been known since the work of Abel and Cauchy that a Taylor series defines completely an analytic function, and, defining precisely the meaning of analytical continuation, Weierstrass, and Méray in France, took that series as the starting point for the definition of an analytic function. In other words, when the sequence of coefficients of an entire series is given, the corresponding function is unique. But it is merely an uniqueness theorem, and the problem arises of indicating the properties of the function so defined by the coefficients, for instance when the radius of convergence is finite. Few results of the inverse problem were known. For instance, the arithmetical nature of coefficients of an algebraical function were indicated by Eisenstein and Tchebycheff. Darboux has shown some properties of growth of coefficients when the singularities, supposed simple in nature, are known. An interesting, but false, result on the affixes of singularities, when the coefficients are known, was given by Lecornu.

But Hadamard is the real creator of the theory of detection of singularities of the analytical continuation of a Taylor series.

He, first, gave the value of the radius of convergence of a Taylor series, in introducing the notion of limit superior of the $|a_n|^{1/n}(f(z)=\sum a_n z^n)$. Until then only when the limit of this expression exists could one find the radius.

He gave then a necessary and sufficient condition bearing on the a_n in order that a given point on the circle of convergence be a singularity. This condition, given by expressions each of which contains a finite number of coefficients, has

since furnished, under its original form, or correspondingly adapted forms, a wealth of important results. And first of all, the famous "Hadamard's gap theorem": $\sum a_n z^{\lambda_n}$, if $\lambda_{n+1}/\lambda_n > \lambda > 1$, admits its circle of convergence (if finite) as a cut—is only a particular case of a result which follows from the criterion on singularities.

He also proved the infinitely elegant and important theorem on polar singularities related to the deep study of determinants formed by the coefficients, and discussed the classification of singularities on the circle of convergence by their "order," a notion rich in content, but which, to my understanding, was not fully used by mathematicians of later generations. It is in relationship with this notion of order that Hadamard introduced functions of "*écart fini.*"

These are briefly, too briefly, I know, the results of Hadamard's Thesis. He got his degree! Some years later he proved his no less famous theorem on composition of singularities. Stated rapidly it reads: $\sum a_n b_n z^n$ has no other singularities than those which can be expressed as products of the form $\alpha\beta$ where α is a singularity of $\sum a_n z^n$, and where β is a singularity of $\sum b_n z^n$.

For sixty years these results have served as inspiration or as tools for a very great number of mathematicians. Few mathematical publications which have appeared since have had an influence on the minds of young mathematicians comparable to that of the two Hadamard papers I have just mentioned.

But let us come back to 1892. In that year Hadamard presented to the Academy a paper, on entire functions and on Riemann's ζ-Function, for the Grand Prix de l'Académie des Sciences. The mathematical world in Paris was sure that the great mathematician Stieltjes would get the prize. Stieltjes thought, indeed, that he had proved the famous "*Riemannische Vermutung*": the zeros of $\zeta(s)$ lie, all, on the line $\frac{1}{2}+it$. Let me quote a passage of Hadamard's famous paper (1896) on the Theory of Numbers: "Stieltjes has proved, as predicted by Riemann, that those zeros (of $\zeta(s)$) are all of the form $\frac{1}{2}+it$, but his proof was never published, and it was not even established that the function ζ has no zeros on the line $R(s)=1$."

But, it was Hadamard who won the prize, and I believe that you would have given it to him too. For, by means of the results of his thesis, he was able in his paper presented to the Academy, to show the relationship between the coefficients of the Taylor series of an entire function and its zeros; he could then also evaluate the genus of the entire function. These results had a dramatic bearing on the Theory of Numbers, since if ρ_n denote the zeros of $\zeta(s)$, $\sum 1/|\rho_n| = \infty$, and $\sum 1/|\rho_n|^2 < \infty$. In the paper, from which I quoted a passage a few minutes ago, he also proved that ζ does not take the value zero on $\sigma=1$, and thus he had now everything he needed to prove the most important proposition on the behavior of primes.

Here, I believe, we should give some older historical facts.

At the beginning of the last century Legendre made the assumption that, on denoting by $\pi(x)$ the number of primes smaller than x (I use, of course,

modern notations), then

$$\pi(x) = \frac{x}{\log x - A(x)}$$

with $A(x)$ tending to a finite limit.

From this it should follow that $\pi(x) \sim x/\log x$. This remained the great unsolved assumption for almost exactly a century.

It is true that Tchebycheff, in the middle of the nineteenth century, by very fine and difficult technical means, had shown that

$$.92129 \leqq \underline{\lim} \frac{\pi(x) \log x}{x} \leqq \overline{\lim} \frac{\pi(x) \log x}{x} \leqq 1.10555 \cdots,$$

but he did not prove that $\pi(x) \log x/x$ tends to a limit, and there was no hope that his method would be able to furnish such a proof. Many mathematicians were able, it is true, to tighten the limits of indetermination indicated by Tchebycheff, on using the same methods, perhaps somewhat improved. Sylvester was among them. But there was nothing fundamentally new in these improvements. Let us quote Sylvester (1881) on this matter: "But to pronounce with certainty upon the existence of such possibility ($\lim \pi(x) \log x/x = 1$) we shall probably have to wait until someone is born into the world as far surpassing Tchebycheff in insight and penetration as Tchebycheff proved himself superior in these qualities to the ordinary run of mankind."

And, as Landau says, when Sylvester wrote these words Hadamard was already born (I imagine that by now you have calculated the year of his birth!)

Riemann introduced his ζ function for the study of the behavior of primes. But he did not prove Legendre's assumption; he did, of course, prove important properties of the function; he made important assumptions on it which were proved to be right and which helped a lot. (I do not speak here about *the* assumption: the "*Vermutung.*")

Hadamard, based on his own result on the genus of an entire function related to ζ, a result made possible only because of his fundamental papers on analytic functions, quoted a little while ago, and based also on his proof that $\zeta(s) \neq 0$ for $\sigma = 1$, has given the complete proof of the great assumption on the distribution of primes.

He also proved analogous theorems on the distribution of primes belonging to a given arithmetical progression since by his methods he was able to study Dirichlet series which, with respect to those primes, play the same role as ζ plays with respect to all the primes.

I must not leave the Theory of Functions without mentioning at least some facts which play such an important role in modern mathematics; let us mention only the titles: The three circle theorem which showed the importance of convexity in the study of analytic functions; the statement of the problem of quasi-analyticity, as distinct from the problem of generalizing the notion of analytic

same feature as that of the geodesics of surfaces studied by Hadamard, one could then foresee the philosophical difficulties he will have to solve.

Hadamard's work on equations with partial derivatives furnished one of the greatest contributions to this field during the first part of the century: the study of the possibility of a solution of a problem in relation to its physical character, the fundamental difference between Cauchy's and Dirichlet's problems from that point of view.

In a general way the entire approach to the question involved depends on the nature of the characteristics and the choice of initial conditions. His greatest achievement was perhaps the systematic introduction of convenient fundamental solutions.

Hadamard applied these ideas to dynamics in the motion of waves, having first made a general study of the kinematics of continuous media. One of the most inspiring pieces of work is his study of the principle of Huyghens; he could point out the cases where the principle really appears. This is for instance the case when the given system comes into rest after the passage of a wave; when the principle does not hold it corresponds to the case where after the passage of the wave a residual wave remains.

I will only mention by name his work on integro-differential equations, first introduced by him, his classical formula on the variation of Green's function corresponding to a small variation of the corresponding region, his formula, expressed by a limit, of a linear functional. He was the first man to envisage the Calculus of Variations by means of functionals of Volterra.

THE CONVERGENCE OF NUMERICAL ITERATION*

H. A. ANTOSIEWICZ† and J. M. HAMMERSLEY‡

Iteration arises frequently in the numerical solution of applied problems. Textbook statements on its convergence are often cursory. We hope this note will put students on their guard.

We shall deal with solving

$$x = f(x) \tag{1}$$

by means of the iteration

$$x_n = f(x_{n-1}) \tag{2}$$

* This work was performed under a National Bureau of Standards contract with the American University.

† Montana State College; and the American University, Washington, D. C.

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starting with a trial value x_0. For simplicity we shall suppose throughout that $f(x)$ is a real function of a real variable x, that (1) possesses a unique solution, and that this solution is $x=0$. From the point of view of theory, there is no real loss of generality involved in this last assumption; for if $x=a\neq 0$ were the solution of $x=f(x)$, then $x=0$ would be the solution of $x=g(x)=f(x+a)-a$. To avoid triviality, we shall also suppose throughout that the initial value x_0 is not zero.

We shall confine our attention to equations in a single unknown x; the difficulties in the case of several unknowns are more severe.

Under certain conditions upon the function $f(x)$ the sequence $x_0, x_1, x_2, \cdots$ will converge to the desired root $x=0$. We now invite the reader, who cares to test his appreciation of such conditions, to answer the following questions (with the aid of a textbook if he so desires).

Question 1: Is it sufficient for convergence that, for some given $k<1$, $|f'(\xi)| \leqq k$ for every ξ in a (sufficiently small) neighborhood of the root $x=0$, and that x_0 shall belong to this neighborhood? (In this question and subsequently, $f'(x)$ denotes the derivative of $f(x)$.)

Question 2: Is it sufficient for divergence that, for some given $k>1$, $|f'(\xi)| \geqq k$ for every ξ in a (sufficiently small) neighborhood of the root $x=0$, and that x_0 shall belong to this neighborhood?

Question 3: Consider two functions $f_1(x)$ and $f_2(x)$ which both satisfy the conditions of Question 1 in a *common* (sufficiently small) neighborhood of the root $x=0$; and suppose that k_1 and k_2 are respectively the smallest possible values of k for which these conditions hold [*i.e.* there exist no constants $k_1'<k_1$, $k_2'<k_2$ such that $|f_1'(\xi)| \leqq k_1'$, $|f_2'(\xi)| \leqq k_2'$ for all ξ in this neighborhood]. If $k_1<k_2$ and if both iterative processes converge, will the convergence for $f_1(x)$ be more rapid than that for $f_2(x)$?

Question 4: Is it necessary for convergence that all conditions stated in Question 1 shall hold (a) if we restrict ourselves to the class of functions $f(x)$ which are *everywhere* differentiable or (b) if we make no such restrictions?

Question 5: Can a condition bearing on the derivative $f'(0)$ or on a Lipschitz condition solely at the root $x=0$ be sufficient for convergence?

Question 6: Are there any functions $f(x)$, satisfying $f(0)=0$ and being discontinuous both to the left and to the right of $x=0$, for which the iteration (2) converges whenever x_0 lies within some (sufficiently small) neighborhood of $x=0$?

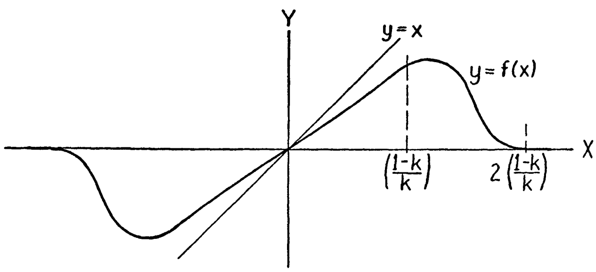
The answer to Question 1 is "Yes." This is a corollary to our answer to Question 5 (see below).

The answer to Question 2 is "No" as shown by the example

$$f(x) = \begin{cases} 2x & |x| \leqq 1 \\ 0 & |x| > 1 \end{cases} \tag{3}$$

for which the process (2) converges.

The answer to Question 3 is "No." The example



$$(4)\qquad f(x) = \begin{cases} (\frac{3}{4})^2 kx & |x| \leqq (1-k)/k \\ (\frac{3}{4})^2(kx+2k-2)^2(3kx+2k-2)/(1-k)^2 & \\ \qquad\qquad (1-k)/k < |x| < 2(1-k)/k & \\ 0 & |x| \geqq 2(1-k)/k \end{cases}$$

satisfies the conditions of Question 1; and yet just one step of the iteration will yield the desired root $x=0$ if

$$k \geqq 1/(1+\tfrac{1}{2}|x_0|);$$

whereas an infinite number of steps is needed if

$$k < 1/(1+\tfrac{1}{2}|x_0|).$$

The convergence is slowest in the case

$$k = 1/\left(1+\frac{9}{10}|x_0|\right).$$

The smallness of k, therefore, is not always a guarantee for rapid convergence as stated in textbooks. In practice, however, it will be a useful (though not wholly reliable) guide for the rapidity of convergence.

The answer to Question 4(a) is "No." The process (2) converges (very rapidly) for the function

$$(5)\qquad f(x) = \begin{cases} |x|^{3/2}e^{-x^2}\sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

whatever the initial value x_0; although $f'(x)$ is unbounded in every neighborhood of the root $x=0$. *A fortiori*, the answer to Question 4(b) is "No." In one of the standard textbooks the conditions of Question 1 are falsely stated as both *necessary* and sufficient for convergence.

The answer to Question 5 is "Yes." It is sufficient for convergence that, solely at the root $x=0$, $f(x)$ shall satisfy a Lipschitz condition (of order unity) with an implied constant k less than unity: that is to say

$$\limsup_{x\to 0}\left|\frac{f(x)}{x}\right| \leqq k < 1. \tag{6}$$

For, if (6) holds, there exists a number $\delta>0$ such that

$$|x_n| = |f(x_{n-1})| \leqq K|x_{n-1}|, \qquad k < K < 1$$

whenever $|x_{n-1}|\leqq\delta$. Therefore, if x_0 belongs to the neighborhood $|x|\leqq\delta$, $|x_n|\leqq K^n|x_0|\to 0$ and $n\to\infty$. The reader will see that $f(x)$ can satisfy (6) even though it may not be differentiable; if, however, $f(x)$ is differentiable, we may replace (6) by the condition

$$|f'(0)| < 1. \tag{7}$$

Condition (7) is a weaker condition than the condition of Question 1. The reader will also notice that the foregoing proof of convergence is very much shorter than the corresponding proofs of weaker versions found in some textbooks. Finally, a condition sufficient for convergence need hold only at a single point, as we have just shown; but an analogous condition sufficient for divergence would, it seems, have to hold for all points; see for instance example (3) above.

The answer to Question 6 is "Yes." A simple example is

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Here at most two steps yield the root $x=0$, although the function is everywhere discontinuous. A more elaborate example shows that convergence is still possible even though $f(x)$ is everywhere discontinuous, everywhere unbounded, and non-measurable. For let I denote the set of all irrational numbers which are not rational multiples of π; and let S be a non-measurable subset of I. Put $f(x)=0$ if x is rational, $f(x)=\frac{1}{2}$ if x belongs to S, $f(x)=q$ if $x=p\pi/q$ where p/q is a non-zero rational number in its lowest terms, and $f(x)=1$ otherwise. Again at most two steps yield the root $x=0$. This example also suggests that the discovery of conditions necessary for convergence may be difficult or impossible.

It is only fair to add that the foregoing discussion is theoretical and academic. In practice, it will not be known that the root lies at $x=0$; it may merely be that we know the root lies between 0 and 1, say. In such a case, the comforting condition for convergence will be that $|f'(x)|\leqq k<1$ for all x in $0\leqq x\leqq 1$.

UPPER LIMITS TO THE REAL ROOTS OF POLYNOMIAL EQUATIONS

LOUISA S. GRINSTEIN, University of Buffalo

1. Introduction. In the solution of a numerical polynomial equation with real coefficients, it is important that the approximate location of the real roots be known. In particular, we should like to know beyond what limits no roots may exist. To some extent, calculations involved in the application of Horner's or Newton's process or the method of iteration may be shortened by utilizing these bounds.

This problem first received attention in the seventeenth century. In his *De Limitibus Aequationum*, Florimond de Beaune (1601–1652), a distinguished French mathematician and successor of Descartes, endeavored to show that the limits of positive roots might be found from the coefficients in equations up to the fourth degree. Unfortunately, he merely considered specific cases and did not venture to solve the problem for the n-th degree. As early as 1683, John Wallis, in commenting on de Beaune's and also on E. Bartholinus's rules for finding bounds, stated that this "subject is yet capable of further development" [17]. The first general methods for obtaining limits date from Newton (1722) and Maclaurin (1748). Since then, researches of many mathematicians have resulted in numerous methods for obtaining more useful bounds. The labor involved in applying some of these procedures, however, causes them to be of theoretical interest only. In this paper, the more important rules will be described and compared as to relative accuracy and efficiency.

Throughout, theorems have been translated to conform with the following notation. Let:

(1) $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,\ a_0 > 0$;

(2) $x_i =$ any positive real root of $f(x) = 0$;

(3) $s =$ the largest x_i ($s = 0$, if no x_i exists);

(4) $k =$ the number of non-negative coefficients preceding the first negative one;

(5) $b =$ the least a_i, $i = 1, \cdots, k-1$;

(6) $d =$ the greatest a_i, $i = 0, \cdots, k-1$;

(7) $g =$ the absolute value of the greatest negative coefficient;

(8) $G =$ the absolute value of the second greatest negative coefficient;

(9) $c =$ the number of negative coefficients. (If $c = 0$, then $s = 0$, and g and G need not be defined.)

Each of the cited methods provides a rule for locating a non-negative value of x satisfying $f(x) \geqq 0$, together with the added requirement that the substitution for x of all greater positive numbers yields this same inequality.

For greater clarity, the various procedures are grouped under separate subdivisions. Observed likenesses of form and approach led to these arbitrary classes. Thus the proofs of all tests in type I involve the given function and its

derivatives. The rules listed under II all seem to be very flexible in numerical situations. All formulae in III reduce to the same "set" form under certain conditions. Similarity of form led to IV, while Sylvester's process was placed in a different category.

2. Type I. The basic theorem here is Newton's rule:

I:A. *If*

$$f(x_0), f'(x_0), f''(x_0), \cdots, f^{(n)}(x_0)$$

are non-negative, then $s \leqq x_0$. [15]

Very close answers are obtained when the equation, $f(x)=0$, has only real roots. Presence of any complex roots may tend to augment the calculated upper limit. Practically speaking, however, the theorem is too cumbersome.

Designed to cut down on the labor entailed in using I:A is Laguerre's procedure:

I:B. *If* x_0 *is non-negative, and if all the coefficients in the quotient and remainder of* $f(x)$ *synthetically divided by* $(x-x_0)$ *are non-negative, then* $s \leqq x_0$. [16]

While this test does reduce the numerical work in I:A, it can come no closer to the greatest x_i. Efforts at improving I:B usually increase the computations.

The following two processes are more useful when dealing with tabular functions rather than with polynomial equations.

I:C. *If for some positive* $\Delta x = h$,

$$f(x_0),\ \Delta f(x_0),\ \Delta^2 f(x_0),\ \cdots,\ \Delta^n f(x_0),$$

are all positive, then

$$s < x_0 + (n-1)h. \ [14]$$

I:D. *If*

$$f(x_0),\ \Delta f(x_0 - h),\ \Delta^2 f(x_0 - 2h),\ \cdots,\ \Delta^n f(x_0 - nh)$$

are all positive, then $s < x_0$. [14]

The connection between the last cited theorems and Newton's rule can be seen by applying the fundamental relationship between finite differences and derivatives to Newton's formula for forward interpolation. The Taylor expansion and consequently I:A will result.

3. Type II. Among the different procedures here classified is the following elementary one:

II:A. *Using the inequality* $f(x) \geqq 0$, *if the negative terms of the polynomial are transposed to the right of the inequality sign and the entire inequality is divided by* x^n, *then* $s \leqq x_0$, *where* x_0 *is any non-negative value of* x, *for which, and for all higher values, the assumed inequality holds.* [18]

Even though this is but a rewording of the definition for upper limit, it may yield a closer bound than some more formalized test.

Another rule to be noted is this fundamental one:

II:B. *If the terms of an equation are arranged in groups wherein the coefficient of the highest power of x appearing is positive, s will equal or be less than that non-negative value of x for which each group is non-negative.* [2]

Ingenuity is needed to get a reasonably useful answer. For example, consider the equation

$$x^4 - 4x^3 + 33x^2 - 2x + 18 = 0.$$

Here, a regrouping such as

$$x^3(x - 4) + x(33x - 2) + 18 = 0,$$

gives $s<4$. Yet a rearrangement such as

$$x^2(x^2 - 4x + 5) + (28x^2 - 2x + 18) = 0$$

yields $s=0$ since both quadratic groups have only complex roots. Thus, insight seems to be a major factor in success.

A third method which may supply varying results with regard to one equation is as follows:

II:C. *To simplify solution, modify the given equation in any way provided that the coefficient of the highest degree term remaining be positive and that the value of $f(x)$ for $x \geqq 1$ be not increased. If s^* denotes the largest x_i of the new equation,*

$$s < \max\,[1, s^*]. \text{ [7]}$$

Typifying one objection against this rule's use is the case of

$$x^5 - x^4 - x^3 - 2x^2 - 3x - 256 = 0.$$

The effort spent in manipulation might more profitably be expended employing another procedure or even approximating to the roots.

4. **Type III.** At the other extreme of flexibility are the "set" formulas which rely chiefly on the magnitude of coefficients present. The most commonly cited seem to be:

III:A. $s<1+(g/a_0)^{1/k}$. [5]

III:B. *If each negative coefficient be taken positively and divided by the sum of all preceding positive coefficients, s will be less than the greatest quotient thus formed increased by unity.* [2]

The usefulness of these theorems stems from the fact that both give relatively close bounds without much computation.

The remaining rules in this class are basically similar since all attempt to

improve on III:A and III:B either by allowing the other coefficients to have weight or by lessening the complexity involved. Usually, progress made in one direction is achieved at the other's expense.

Among the simpler tests, we have Maclaurin's rule:

III:B:1. $s<1+g/a_0$. [13]

The sole advantage which this theorem has over the preceding ones is its ease in application. Although sometimes both III:A and III:B reduce to III:B:1, in general method III:B:1 yields useless results.

A modification proposed by Vene in 1822 is:

III:B:2. $s<1+g/d$. [18]

This procedure can be of use only when there are several relatively large positive coefficients preceding the first negative one.

Some other methods embodying but a minor variation in form are:

III:B:3. $s<1+g/\sum_{j=0}^{k-1}a_j$. [8]

III:B:4. $s<1+(g+G)/(2a_0+2a_1+\cdots+2a_{k-2}+a_{k-1})$. [8]

III:C:1. $s<(1+g/b)^{1/(k-1)}$, $b>0$. [13]

III:C:2. $s<(1+g/a_0)^{1/k}$. [13]

III:D. $s<(1+g/a_p)^{1/(k-p)}$, where $a_p(\neq 0)$ *is the coefficient of the* $(p+1)$*th term, and* $p<k$. [10; also §8A below]

Let us now consider some "set" rules of greater complexity, the first being:

III:C. *If an arbitrary number of such pairs of terms* $a_{n-j}x^j$ *and* $a_{n-j+A}x^{j-A}$ *with* $a_{n-j}\geqq|a_{n-j+A}|$, $a_{n-j+A}<0$, $A<j$, *are dropped and if, in the resulting equation* $a_{n-m}x^m$, *where* $a_{n-m}\neq 0$, *is any term preceding all negative ones and* $(m-K)$ *is the power of the first negative term, then in sequential form*:

$$s \leqq 1 + (g/a_{n-m})^{1/K};\ \phi[1 + (g/a_{n-m})^{1/K}];\ \cdots$$

where $\phi(x)\equiv 1+[(1-x^{(-m+K-1)})g/a_{n-m}]^{1/K}$; $x>1$. [7]

When no significant changes can be made in the equation, the first term of the bounding sequence will equal III:A. By continuing the iterative process, one may approach somewhat closer to s. Sometimes the sequence may converge on s, thus providing a very limited process for approximating to an incommensurable root. Wherever the equation can be modified extensively, this procedure may prove more useful.

The following, a very complicated rule, reduces to simpler forms in certain cases:

III:F. *Given any number* h *satisfying the inequality*

$$g - (h - 1)\sum_{j=0}^{k-1} a_j h^{(k-1)-j} > 0,$$

then

$$s < \max\left\{h,\ 1 + \left[\left(g + a_0(h - 1)^k - (h - 1)\sum_{j=0}^{k-1} a_j h^{(k-1)-j}\right)\Big/ a_0\right]^{1/k}\right\}. \quad [4]$$

Taking $k=1$, one obtains Maclaurin's rule. If $h=1$, method III:A results. Best use is made of the theorem when a relatively large difference exists between

$$a_0(h - 1)^k \quad \text{and} \quad (h - 1)\sum_{j=0}^{k-1} a_j h^{(k-1)-j}.$$

From a theorem stated incorrectly by Corliss [4; also §8C below] two rules can be derived:

III:G:1. $s<\max\left\{2,\ 1+\left[\left(g-\sum_{j=0}^{k-1} a_j\right)\Big/ a_0\right]^{1/k}\right\}$ *whenever the radicand* >1 *and* $k>1$.

III:G:2. $s<\max\left\{2,\ 1+\left[\left(g+1-\sum_{j=0}^{k-1} a_j\right)\Big/ a_0\right]^{1/k}\right\}$ *whenever the radicand* >1.

Of these two variations, the first, where applicable, will supply a somewhat more useful result.

Still greater formality is found in Mourgues's inaccurately described theorem [9, §8D, below]:

III:H. *If the negative terms of an equation are so taken that the exponents of the variable* x:

$$n - k,\ n - q,\ \cdots,\ n - r,\ n - t$$

form a decreasing sequence of positive integral numbers and the corresponding coefficients are so modified that

$$a_k/a_0 = A_k;\ a_q/a_0 \leqq Q;\ \cdots;\ a_r/a_0 \leqq R;\ a_t/a_0 \leqq T$$

where

$$A_k \leqq Q \leqq \cdots \leqq R \leqq T,$$

then, given any number $H \geqq 2$,

$$s < \max\left\{1 + \left[(A_kH^{t-k} + QH^{t-q} + \cdots + RH^{t-r} + T)/(H^{t-k} + H^{t-q} + \cdots + H^{t-r} + 1) - \sum_{j=1}^{k-1} (a_jH^{k-j-1})/a_0\right]^{1/k},\ H\right\}.$$

Unlike other procedures of this class, the above will reduce to III:B:1 when $k=1$ only if no negative coefficient exceeds a_1. Thus the method, because of its cumbersome form, may provide a very useful result.

In general, most of the tests in this section are much too inflexible. It seems best to rely on the two standard formulae, III:A and III:B, since many of the refinements are often nullified in numerical situations.

5. Type IV. The tests in this section are closely allied with those of preceding subdivisions. Nevertheless, in form, they lie in a separate grouping.

IV:A. $s \leqq \max\left[\{(c|a_i|)/a_0\}^{1/i}\right]$, *where a_i denotes negative coefficients only.* [3]

IV:B. $s < \max\left[\{|a_i|/a_0\}^{1/i} + \{|a_j|/a_0\}^{1/j}\right]$, $i \neq j$, *where a_i and a_j both denote negative coefficients only.* [11]

IV:C. $s < \max\left[2\{|a_i|/a_0\}^{1/i}\right]$, $i=1, \cdots, n$. [1]

Usually IV:C gives the poorest values since it is not only dependent on the various negative coefficients but also on the positive ones. If n is relatively large and the negative terms occur quite near the beginning of the equation, it is better to use IV:B.

6. Type V. Sylvester's approach involves a simple continued fraction expansion of the quotient of two polynomials:

V. *Let* $$\frac{f'(x)}{f(x)} = 0 + \cfrac{1}{X_1 + \cfrac{1}{X_2 + \cdots}} = (0, X_1, \cdots, X_m), \text{ where } X_i = A_i x - B_i; \quad i = 1, \cdots, m.$$

Then

$$s < \max\left[(B_1 \pm 1)/A_1;\ (B_2 \pm 2)/A_2;\ \cdots;\ (B_{m-1} \pm 2)/A_{m-1};\ (B_m \pm 1)/A_m\right]. \quad [12]$$

This method has never attained any but theoretical popularity because of the calculational difficulties arising in numerical cases. So far, any effort at modification to lessen the computation has only succeeded in increasing the limit obtained.

7. Summary. No startingly new conclusions can be drawn as to the relative merits of all these tests. It seems clear that generally some method of type I, III or IV should give a satisfactory result. In numerical examples, the possibilities of the more flexible forms in II should never be forgotten.

Thus, one ought to be aware of the major approaches. In a given situation, use should be made of the most applicable rule.

8. Appendix on Erroneous Statements. In the course of this study, the following erroneous or misleading remarks were noted:

A. By neglecting to note the significance of discarding a positive quantity, Myers [10] was led to assert that s might sometimes be achieved from III:D. In brief, he states:

"With regard to $x>0$,

$$f(x) \geqq [a_p(x-1)x^{n-p} - gx^{n-k+1} + g]/(x-1). \tag{1}$$

Dropping g from the right-hand member does not affect the inequality when $x>1$. Thus,

$$f(x) \geqq \{x^{n-p}[a_p(x-1) - gx^{p-k+1}]\}/(x-1), \textit{ etc.}" \tag{2}$$

As the remainder of the proof is correct, it need not concern us here. The equality sign holds in (1) if $k=1$ and $a_i=g$ $(i=1, \cdots, n)$. It is tacitly assumed that $g>0$ since otherwise $s=0$. By dropping the positive g, we find that the equality will be affected. Therefore s can never be thus attained.

B. Here, too, should be cited Corliss's attempt [4] at adapting III:B to a special situation:

"If $k=1$, it is more effective to let $x=my$ and then to use III:B on the y-equation. Choose an m (either by inspection or by application of the theory involving maxima and minima) which will give a satisfactory upper limit."

Misleading results may arise, however, if one tries to apply this method to various equations with $a_1<0$. From a graphical standpoint, the most unfavorable case actually reduces to finding the minimum point on the curve:

$$x = \max\,[m + |a_1|/a_0; \cdots ; m + |a_n|/a_0m^{n-1}],$$

where a_i $(i=1, \cdots, n)$ denote all negative coefficients. Unless a_1/a_0 is the greatest negative coefficient, one can not state definitely that the minimum point of the greatest x_i-curve $[x_i=m+|a_i|/a_0m^{i-1}]$ will coincide with that of the x-curve. Often, there is no quick way of determining the desired point. Thus, this method can be utilized to advantage only if after obtaining a value for m, that value is reinserted in the fractions a_i/a_0m^i, to check whether the original relationships still persist. If the originally picked fraction is at least one of the greatest (there being one or more having the same value), then only will the limit be correct.

C. At times erroneous is theorem III:G as given originally by Corliss [4]:

$$\text{“}s < \max\left\{2,\ 1 + \left[\left(g - \sum_{j=0}^{k-1} a_j\right)\Big/a_0\right]^{1/k}\right\}, \textit{ whenever the radicand } > 1.\text{”}$$

When $k=1$, the test may yield a number less than s. To discover why such an error can occur, it is necessary to examine the proof more closely:

"No $x>1$ will be a root of $f(x)$ if

$$(a_0x^{k-1} + a_1x^{k-2} + \cdots + a_{k-1})(x-1) - g > 0. \tag{1}$$

Make use of the relationship

$$x^{k-1} \geqq h^{k-1} - (h-1)^{k-1} + (x-1)^{k-1} \tag{2}$$

which is true when $x \geqq h \geqq 1$. Substituting for x^{k-1} and for x in (1), obtain:

$$[(a_0 h^{k-1} + \cdots + a_{k-1}) - a_0(h-1)^{k-1}](h-1) + a_0(x-1)^k - g > 0. \tag{3}$$

Putting $h=1$ in the bracket of (3) and $h=2$ in the multiplier, $(h-1)$, gives theorem III:G."

Now if $k=1$, expression (2) reduces to the triviality: $1=1$. Therefore, from (1) we have

$$s < 1 + g/a_0$$

which is just III:B:1. To alter III:G so that it can be used in all instances, give the value 1 to the term $a_0(h-1)^{k-1}$ occurring in (3). This is sufficient since 1 is the largest value which this term can achieve. So, the two modified forms, III:G:1 and III:G:2, result.

D. In discussing the use of III:G, Mourgues asserts [9] that one should begin by taking $H=2$, or 3. Then, if the limit obtained so be too great, one should replace H by one of the numbers 4, 5, $\cdots$. Unfortunately, he fails to mention that though H might be increased without limit, s is less than the greater of the two numbers, H and the calculated bound.

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MATHEMATICAL NOTES

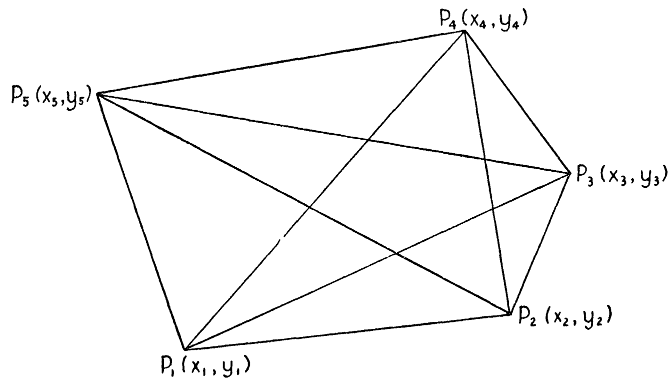
EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A PENTAGON THEOREM

W. L. WILLIAMS, University of South Carolina

Let $P_i(x_i, y_i)$, $(i=1, \cdots, 5)$ be any five points in a plane, referred to a system of rectangular axes, no three of which lie in a straight line. Connect them as shown in the figure below. Consider the quadrilateral $p_1p_2p_3p_4$. Let $L_i=0$, $(i=1, \cdots, 4)$,



be the equations of its sides, where (x_i, y_i) is the initial point of the side, reading in a counter-clockwise direction. Then,

$$\frac{L_1\cdot L_3}{L_2\cdot L_4} = \lambda \tag{1}$$

is the equation of a system of conics through P_1, P_2, P_3, and P_4. If the equations of the sides $L_i=0$ are expressed in determinant form and if (x_5, y_5) is regarded as the variable point, (1) may be written in the form

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}\begin{vmatrix} x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}}{\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}} = \lambda_5 \tag{2}$$

where the subscript on λ indicates the variable point.

Each determinant in (2) is twice the area of a certain one of the four triangles of the pentagon formed by $P_1, \cdots, P_5$. Let these triangular areas be represented by D with subscripts indicating the points involved. Equation (2), then, takes the form

$$\frac{D_{125}D_{345}}{D_{235}D_{145}} = \lambda_5.$$

Four additional similar relations may be formed from the other four quadrilaterals of the pentagon, all of which may be expressed by the equations,

$$\lambda_i = \frac{D_{i,i+1,i+2}D_{i,i+3,i+4}}{D_{i,i+1,i+4}D_{i,i+2,i+3}}, \qquad (i = 1, \cdots, 5), \tag{3}$$

where it is understood here and hereafter that the subscripts 6, 7, 8 and 9 are to be replaced, respectively, by 1, 2, 3, and 4.

In a paper by the author, *Permanent configurations in the problem of five bodies*,* it was shown that if the D's are eliminated from (3), the λ's satisfy a system of equations of the form

$$\lambda_i - \lambda_{i+2}\lambda_{i+3} - \lambda_i\lambda_{i+2}\lambda_{i+3} = 0, \qquad (i = 1, \cdots, 5),$$

which, by using (3), may be expressed in the equivalent form

$$D_{i,i+1,i+3}D_{i,i+2,i+4} = D_{i,i+1,i+2}D_{i,i+3,i+4} + D_{i,i+1,i+4}D_{i,i+2,i+3}, \qquad (i = 1, \cdots, 5), \tag{4}$$

a system of five equations which are identically satisfied by any pentagon whatever, either concave or convex.

Consider any vertex P_i, $(i=1, \cdots, 5)$, of the pentagon and the quadrilateral formed by the remaining vertices. Call this quadrilateral the *associated* quadrilateral of P_i. Let the triangles of the pentagon having P_i as one vertex and a diagonal of the associated quadrilateral as a side be called a *diagonal* triangle and call the other triangles of the pentagon formed by P_i and the sides of the associated quadrilateral *side* triangles. Then by equations (4) we have the

THEOREM: *In any pentagon the product of twice the areas of any two diagonal triangles belonging to the same quadrilateral is equal to the sum of the products of twice the areas of the opposite side triangles, where in forming the triangular areas the vertices of the pentagon are to be taken in the order P_1, P_2, P_3, P_4, and P_5.* It may be of interest to observe that this theorem bears some resemblance to the theorem of Ptolemy which states that if a quadrilateral is inscribed in a circle the product of its diagonals is equal to the sum of the products of the opposite sides. It could be regarded, perhaps, as the companion theorem to Ptolemy's, for a pentagon inscribed in a conic.

* Trans. Amer. Math. Soc., vol. 44, No. 3, November 1938, pp. 563–579.

ON A THEOREM OF BÉLA SZ.-NAGY

V. L. KLEE, JR.,* Institute for Advanced Study and University of Virginia

If M and N are subsets of a metric space and f maps M onto N, then f will be called "Lipschitzian" provided

$$\sup \{\rho(fx, fy)/\rho(x, y): x, y \in M, x \neq y\} < \infty.$$

Béla Sz.-Nagy observed that if M is the unit cell $\{x: \|x\| \leqq 1\}$ in Euclidean k-space, N is a bounded closed convex body whose interior includes the origin $\emptyset$, and f is the radial projection (defined below) of M onto N, then f is Lipschitzian (his term: "dehnungsbeschränkt"). A proof has been given by Vincze and Szüsz [3]. Now in terms of the gauge functionals of the sets, the radial projection can be defined quite simply (see below). The purpose of this note is to show how the result of Sz.-Nagy follows (in an arbitrary normed linear space) from elementary properties of gauge functionals.

By a "gauge" on a real linear space L is meant a real-valued function g on L which satisfies the following conditions: (1) $g(tx)=t(gx)$ whenever $t \geqq 0$; (2) $g(x+y) \leqq gx+gy$; (3) $gx>0$ whenever $x \neq \emptyset$. The fundamental relationship between gauges and convex sets is well-known [1; pp. 21–22]: If g is a gauge, then the set $C_g=\{x: gx \leqq 1\}$ is convex and intersects each line through $\emptyset$ in a closed segment having $\emptyset$ as an inner point. Conversely, to each such set C corresponds a unique gauge $g=g_c$ such that $C_g=C$. This gauge can be defined by: $g_c x=\inf \{t: t>0, t^{-1}x \in C\}$.

Suppose B and C are bounded closed convex bodies in a normed linear space, each including $\emptyset$ in its interior. Let FB and FC be the boundaries of B and C. For each ray ρ emanating from $\emptyset$, let f map $\rho \cap FB$ (a single point) onto $\rho \cap FC$ (a single point), and for each $x \in FB$ let f map the segment $[\emptyset, x]$ linearly onto $[\emptyset, fx]$. Then f is the "radial projection" of B onto C. In terms of the gauges $b=g_b$ and $c=g_c$, f can be defined as follows: $f\emptyset=\emptyset$ and $fx=(bx/cx)x$ for $\emptyset \neq x \in B$.

For each gauge g on L and point x of L, let $g^*x=\max [gx, g(-x)]$. It is not hard to verify that g^* is a gauge satisfying

$$(1') \qquad g^*(tx) = |t| (g^*x).$$

(The gauges g^* are characterized by having $C_{g^*}=-C_{g^*}$). From (1′) and (2) follow

$$(A) \qquad g^*(rx+sy) \leqq |r| (g^*x) + |s| (g^*y)$$

and

$$(B) \qquad |gx - gy| \leqq g^*(x-y).$$

We can now prove the

* National Research Fellow.

THEOREM. *Suppose B and C are bounded closed convex bodies in a normed linear space E, each including $\emptyset$ in its interior. Then the radial projection of B onto C is Lipschitzian.*

Proof: Let $b=g_B$ and $c=g_C$. Let a be the norm of E, so that $a=a^*=g_U$, where U is the unit cell of E. Since B and C are both bounded and each includes $\emptyset$ in its interior there is a number $r>1$ such that $r^{-1}U\subset B\cap C\subset B\cup C\subset rU$, and then (D) $r^{-1}a\leqq b,c \leqq ra$. To prove the theorem, it suffices to find a number $N<\infty$ such that

$$a\left[\frac{bu}{cu}u-\frac{bv}{cv}v\right]\leqq Na(u-v)$$

whenever $u, v\in B$ and $u\neq\emptyset\neq v$. Consider an arbitrary such pair of points with, say, $cv\leqq cu$. Let $t=(cu)^{-1}$, $x=tu$, and $y=tv$. Then $cy\leqq cx=1$ and $y, x\in C$. We have

$$\begin{aligned} ta\left[\frac{bu}{cu}u-\frac{bv}{cv}v\right] &= a^*\left[\frac{bx}{cx}(x-y)+\left(\frac{bx}{cx}-\frac{by}{cy}\right)y\right] \\ &\leqq \frac{bx}{cx}a^*(x-y)+\left|(bx-by)+\frac{by}{cy}(cy-cx)\right|a^*y \\ &\leqq \frac{bx}{cx}a(x-y)+|bx-by|\,ay+\frac{by}{cy}|cy-cx|\,ay \\ &\leqq \left[\frac{bx}{cx}+ay\frac{b^*(x-y)}{a(x-y)}+ay\frac{by}{cy}\frac{c^*(x-y)}{a(x-y)}\right]a(x-y) \\ &\leqq Ma(x-y) = tMa(u-v), \end{aligned}$$

where $M=(\sup_C b/c)+(\sup_C a)(\sup_{C-C} b^*/a)+(\sup_C a)(\sup_C b/c)(\sup_{C-C} c^*/a)$. (Here $C-C$ is the set $\{x-y:x, y\in C\}$.) In establishing these inequalities, (3) is used throughout. The first inequality uses (A) for $a=a^*$ and the fact that $cx=1$; the third uses (B) for b and c. From (D) it follows that $M<\infty$, completing the proof.

In closing, we note that the results of Vincze and Szüsz [3] have been extended by Haupt [2], and that some of Haupt's results can be obtained by the method used here.

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2. Otto Haupt, Bemerkung zu einem Abbildungssatz von Herrn Béla Sz.-Nagy, S.-B. Math. Nat. Kl. Bayer. Akad. Wiss. 1951, pp. 147–161 (1952).

3. St. Vincze and P. Szüsz, Beweis eines Abbildungssatzes von Béla Sz.-Nagy, Acta. Sci. Math. Szeged, vol. 14 (1951) pp. 96–100.

ON A THEOREM OF KIRZBRAUN AND VALENTINE

I. J. SCHOENBERG, University of Pennsylvania

THEOREM 1. *In the euclidean space E_n we are given a set of m spheres*

$$S_i: \quad |x - x_i| \leqq r_i, \qquad (i = 1, \cdots, m),$$

which have a point in common:

$$\bigcap_{i=1}^{m} S_i \neq 0.$$

If the spheres

$$S_i': \quad |x - x_i'| \leqq r_i, \qquad (i = 1, \cdots, m),$$

are such that

$$|x_i' - x_j'| \leqq |x_i - x_j|, \qquad (i, j = 1, \cdots, m), \tag{1}$$

then

$$\bigcap_{i=1}^{m} S_i' \neq 0. \tag{2}$$

In other words: If a set of m closed spheres in E_n have a common point then again they will have a common point if we displace their centers so as not to be further apart than they were before. This theorem is seen to be equivalent to the following:

THEOREM 2. *Let x_i, x_i' $(i=1, \cdots, m)$ be points in E_n satisfying* (1). *Let also the point p be given. Then there exists a point p' such that*

$$|p' - x_i'| \leqq |p - x_i|, \qquad (i = 1, \cdots, m). \tag{3}$$

Theorem 2 implies Theorem 1, for if $p \in \cap S_i$ then, by (3), $|p'-x_i'| \leqq |p-x_i| \leqq r_i$ or $p' \in \cap S_i'$, which proves (2). It is equally obvious that Theorem 1 implies Theorem 2. This result is due to Kirzbraun [1]. It was rediscovered by Valentine [3], [4]. Mickle [2] gave a new proof. Valentine uses in his proof a theorem of Knaster, Kuratowski and Mazurkiewicz. Mickle uses certain quadratic forms which were used by the present writer to characterize E_n metrically. The following is a blend of the proofs of Valentine and Mickle which avoids the use of the auxiliary tools just mentioned.

Proof of THEOREM 2: We lose no generality by assuming that $p \neq x_i$ $(i=1, \cdots, m)$, for if $p = x_j$, then $p' = x_j'$ will satisfy (3). Consider the function

$$f(x) = \max_i \frac{|x - x_i'|}{|p - x_i|},$$

which is defined and continuous throughout E_n. Since $f(x) \to +\infty$ as $|x| \to \infty$,

But then

$$0 = \sum c_i x_i' - p' = \sum c_i x_i' - \sum c_i p'$$

or

$$\sum_{i=1}^{k} c_i R_i' = 0, \qquad c_i \geqq 0, \ \sum c_i = 1. \tag{12}$$

A contradiction is now easily reached: on multiplying (11) by $c_i c_j$ and summing over i and j we obtain the inequality

$$\left(\sum_1^k c_i R_i'\right)^2 > \left(\sum_1^k c_i R_i\right)^2$$

which is obviously impossible for its left side vanishes by (12) while its right side is non-negative.

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1. M. D. Kirzbraun, Über die zusammenziehende und Lipschitzsche Transformationen, Fund. Math., vol. 22, 1934, pp. 77–108.

2. E. J. Mickle, On the extension of a transformation, Bull. Amer. Math. Soc., vol. 55, 1949, pp. 160–164.

3. F. A. Valentine, On the extension of a vector function so as to preserve a Lipschitz condition, Bull. Amer. Math. Soc., vol. 49, 1943, pp. 100–108.

4. F. A. Valentine, A Lipschitz preserving extension for a vector function, Amer. J. Math., vol. 67, 1945, pp. 83–93.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A NOTE ON THE BASE OF NATURAL LOGARITHMS

ERNEST LEACH, Massachusetts Institute of Technology

A quick approach to the theory of natural logarithms starts from the defining formula

$$\ln x = \int_1^x t^{-1} dt. \tag{1}$$

The base e is then defined implicitly by:

But then

$$0 = \sum c_i x_i' - p' = \sum c_i x_i' - \sum c_i p'$$

or

$$\sum_{i=1}^{k} c_i R_i' = 0, \qquad c_i \geqq 0, \ \sum c_i = 1. \tag{12}$$

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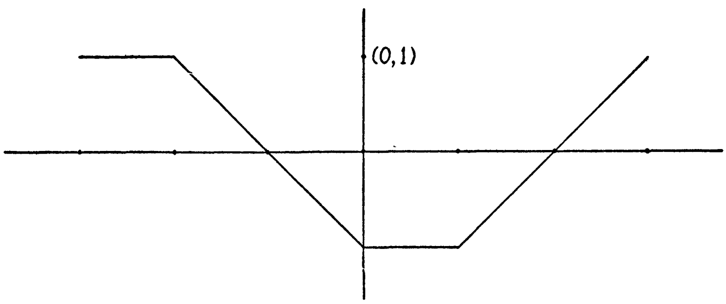
A KIND OF PROBLEM THAT EFFECTIVELY TESTS FAMILIARITY WITH FUNCTIONAL RELATIONS

K. O. MAY, Carleton College

The kind of problem described here probes the student's understanding of the functional concept and notation, tests his familiarity with the functions being studied, and is easy to run off on a duplicator and to correct.

Present the student with the graph of a rather simple function, perhaps piecewise constant or at least piecewise linear. Label this function $f(x)$ and then require the student to graph $-f(x), f(-x), |f(x)|, f(|x|), e^{f(x)}, \sin f(x)$ or whatever functions involve the ideas whose understanding you wish to test. The correct graphs are often decorative and always easily recognized. The student's deviations indicate clearly in many cases the nature of his difficulties. While the resulting graphs are easy to check, it is not hard to construct examples that require considerable thought and challenge the best students. On the other hand, the simpler examples, such as $-f(x)$ or the inverse function of $f(x)$, appeal to the manipulative students. Obvious modifications, such as the use of two functions, will occur to any teacher.

As an example suppose that $y=f(x)$ has the graph:



Sketch: $y=|f(x)|$, $y=f(|x|)$, $y=-f(x)$, $y=e^{f(x)}$, $y=\text{Arc}\sin f(x)$.

SOME REMARKS ON CENTROIDS

R. B. DEAL and W. N. HUFF, The University of Oklahoma

Recently it was pointed out by two students in our integral calculus classes that the ordinate of the centroid of area of one arch of the cycloid and that of the centroid of the volume of revolution of the arch about the Y-axis were equal. The question was raised as to the reason for this and the present note concerns this problem.

Let us consider an area, A, in the first quadrant, bounded by the curves $y=f(x)$ above and $y=g(x)$ below, the curves intersecting at (a, b) and (c, d). Let $(\bar{x}, \bar{y})$ be the centroid of area A, $(\bar{x}_1, 0)$ the centroid of V_1, the volume of rotation about the x-axis, and $(0, \bar{y}_2)$ the centroid of V_2, the volume of rotation about the y-axis.

Then

$$A\bar{y} = \frac{1}{2}\int_a^c (f^2 - g^2)dx = \frac{V_1}{2\pi},$$

$$V_2\bar{y}_2 = \pi\int_a^c (f^2 - g^2)xdx.$$

Since by the theorem of Pappus $V_2 = 2\pi\bar{x}A$, it follows that

$$2\pi\bar{x}A\bar{y}_2 = \pi\int_a^c (f^2 - g^2)xdx = \bar{x}_1V_1 = 2\pi A\bar{x}_1\bar{y}.$$

Hence $\bar{x}\bar{y}_2 = \bar{x}_1\bar{y}$.

This can also be done by double integrals as follows:

$$2\pi A\bar{y}\bar{x}_1 = V_1\bar{x}_1 = \int_a^c\int_g^f 2\pi xydydx = V_2\bar{y}_2 = 2\pi A\bar{x}\bar{y}_2.$$

The above result then follows.

If the area in question possesses an axis of symmetry $x = k$, then $\bar{x}_1 = \bar{x} = k$ and $\bar{y}_2 = \bar{y}$. For the arch of the cycloid $k = \pi a$.

ON THE CONVERGENCE OF THE p-SERIES

R. M. Foster and M. S. Klamkin, Polytechnic Institute of Brooklyn

In quite a few textbooks, we find the statement that the p-series,

$$\sum_{n=1}^{\infty} n^{-p}, \tag{1}$$

converges for $p > 1$, and diverges for $p \leqq 1$. This is not quite correct. It should rather be stated, that the series converges for p(constant) > 1, and diverges for $p \leqq 1$. An example illustrating this point is given by the series (1) with $p = 1 + (1/n)$. This series diverges since

$$\lim_{n\to\infty} \frac{n^{1+(1/n)}}{n} = 1, \tag{2}$$

and $\sum_{n=1}^{\infty} n^{-1}$ diverges. This example was given as a problem in this Monthly (Nov. 1948, p. 584). One of the solutions gave the following generalization of this problem:

If $\lim_{n\to\infty}\phi(n) = 0$, then $\sum_{n=1}^{\infty} n^{-[1+\phi(n)]}$ diverges. This latter statement, however, is not correct. For, let r be constant and let $n^{\phi(n)} = \psi(n) = (\ln n)^r$. Then

$$\lim_{n\to\infty}\phi(n) = \lim_{n\to\infty}\frac{\ln\psi(n)}{\ln n} = 0, \tag{3}$$

but the series

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)} = \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^r} \tag{4}$$

converges for r(constant) >1, and diverges for $r \leqq 1$. Similarly, other functions $\psi(n)$ can be chosen such that the series converges or diverges.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1086. *Proposed by T. A. Bickerstaff, University of Mississippi*

"That was a good lunch; now for a good cigar and then I must catch the one o'clock train. Let's see—my watch says exactly nine o'clock but that can't be right. It's still running and well wound. Now I remember I wound it and set it just this morning by the radio. Maybe I carelessly set the hands in reverse position. If so, exactly what time is it?"

E 1087. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Let a be an arbitrary positive integer and let

$$S_m = \sum_{j=1}^{a} j^m.$$

We have

$$a(a+1)^2 = S_1 + 3S_2, \qquad a^2(a+1)^3 = S_2 + 2S_3 + 5S_4,$$
$$a^3(a+1)^4 = S_3 + 5S_4 + 3S_5 + 7S_6, \cdots .$$

Determine coefficients c_k such that

$$a^n(a+1)^{n+1} = \sum_{k=n}^{2n} c_k S_k.$$

E 1088. *Proposed by M. J. Pascual, Siena College*

A and B agree to play n games and he who goes first has a chance of $a/(a+b)$ of winning that game. If the winner of any game goes first in the following game, then find (1) the probability of A winning the nth game if he goes first in the opening game, (2) the expected value of A's winnings in n games if each game is worth d dollars.

E 1089. *Proposed by W. R. Utz, University of Missouri*

Show that if $f(x)$ is a real function bounded below on $I=[a, b]$ and $g(x)$ is real and either (i) monotone increasing on I or (ii) continuous with $g(b)>g(x)$ for $a\leqq x\leqq b$, then given $\epsilon>0$ there is a constant $\lambda(\epsilon)>0$ such that $f(x)-\lambda g(x)$ cannot attain $\inf_{x\epsilon I}[f(x)-\lambda g(x)]$ on $[a, b-\epsilon]$. (The special case $g(x)=x$ is used by E. Baiada, *Ann. Scuola Norm. Super. Pisa*, vol. 15 (1950), p. 111.)

E 1090. *Proposed by B. M. Stewart, Michigan State College*

From one vertex of a triangle lines are to be drawn dividing the triangle into a set S of n triangles having equal inscribed circles.

(1) Show that in general the set S may be constructed by ruler and compass if and only if $n=2^s$.

(2) Show that the $n-k+1$ triangles formed by taking sets of k adjacent triangles of the set S have equal inscribed circles ($k=2, 3, \cdots, n-1$).

(3) Find a neat construction when $n=2$. (*Note.* From (2) it follows that repeated application of (3) will solve the problem when $n=2^s$.)

SOLUTIONS

The Spider and the Fly

E 1056 [1953, 188]. *Proposed by W. R. Ransom, Tufts College*

A rectangular room has a spider one foot down from the ceiling at the middle of one end, and a fly one foot up from the floor at the middle of the other end. There are three paths by which the spider can crawl to the fly, and which become straight lines when the sides of the room are properly developed. What dimensions of the room will make these three paths equal in length?

Editorial Note. There seems to be no easily located treatment of the spider and fly problem which considers the maximum number n of possible geodesic routes from the spider to the fly, and the number d of these routes which have distinct lengths, in terms of the length l, width w, and height h of the room, and the positions of the spider and fly on the end walls. What is the smallest possible value for n, and, for given n, what are the greatest and least possible values for d? What is the shortest route? Kasner and Newman in *Mathematics and the Imagination*, p. 182, indicate that $n\geqq 4$. Kraitchik in *Mathematical Recreations*, pp. 17–21, indicates that $n\geqq 8$, and shows that for $l:w:h=390:240:260$ we may have all eight routes equal in length. Boblétt constructed a model

with $l:w:h=5:2:2$ on which 16 routes are indicated, these routes being equal in symmetrical pairs. It is easy to show that there exist rooms (very flat ones) having n as large as may be desired.

Contributions by A. P. Boblétt, Julian Braun, Mary Dean Clement and J. D. Haggard (jointly), J. R. Hatcher and C. E. Jones (jointly), Frank Herlihy, R. E. Jackson, M. S. Klamkin, F. A. Lee, Jr., C. S. Ogilvy, Bart Park, M. J. Pascual, and the proposer.

A Series Involving Secants

E 1057 [1953, 188]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find the sum of the first n terms of the series

$$\sec\theta + (\sec\theta\sec 2\theta)/2 + (\sec\theta\sec 2\theta\sec 4\theta)/4 + \cdots.$$

Solution by J. R. Hatcher, Fisk University. Using the identities $\sec\alpha = 2\sin\alpha/\sin 2\alpha$ and $\csc\alpha = \cot\alpha/2 - \cot\alpha$, we conclude that the sum $S_n(\theta)$ of the first n terms is

$$\begin{aligned} S_n(\theta) &= 2\sin\theta(\csc 2\theta + \csc 4\theta + \cdots + \csc 2^n\theta) \\ &= 2\sin\theta(\cot\theta - \cot 2^n\theta) \\ &= [2\sin(2^n-1)\theta]/[\sin 2^n\theta]. \end{aligned}$$

Also solved by H. M. Feldman, N. J. Fine, H. W. Gould, Eugene Malek, Robert Muguercia, S. Parameswaran, L. A. Ringenberg, Francis Sévier, F. Underwood, Chih-yi Wang, J. V. Whittaker, and the proposer.

Underwood pointed out that the problem is essentially Ex. 10, p. 125 of *Plane Trigonometry, Part II*, by S. L. Loney (1908).

Two Properties of the Divisors of an Integer

E 1058 [1953, 188]. *Proposed by M. Perisastri, M. R. College, Vizianagram, India*

If $d_1, d_2, \cdots, d_k$ are the divisors of n, show that

$$(d_1 d_2 \cdots d_k)^2 = n^k, \tag{1}$$

$$d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k} = n^{d_1+d_2+\cdots+d_k} e^{-nrk}, \qquad 0 \leqq r \leqq (\log 3)/3. \tag{2}$$

Solution by W. E. Briggs, University of Colorado. If $d_1=1, d_2, \cdots, d_k=n$ is a complete set of divisors of n, then $n/d_1, n/d_2, \cdots, n/d_k$ is another complete set. Therefore

$$d_1 d_2 \cdots d_k = (n/d_1)(n/d_2)\cdots(n/d_k) = n^k/d_1 d_2 \cdots d_k,$$

whence

$$(d_1 d_2 \cdots d_k)^2 = n^k.$$

Next let $d_i = n/d_{k-i+1}$, $i=1, 2, \cdots, k$. Then

$$d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k} = (n/d_k)^{d_1}(n/d_{k-1})^{d_2} \cdots (n/d_1)^{d_k} = n^{d_1+d_2+\cdots+d_k}/d_1^{d_k} d_2^{d_{k-1}} \cdots d_k^{d_1},$$

and

$$1/d_1^{d_k} \cdots d_k^{d_1} = 1/d_1^{n/d_1} \cdots d_k^{n/d_k} = (d_1^{1/d_1} \cdots d_k^{1/d_k})^{-n}.$$

But it is well known that for all positive integers m, $1 \leqq m^{1/m} \leqq 3^{1/3}$. Therefore

$$1 \leqq d_1^{1/d_1} \cdots d_k^{1/d_k} \leqq 3^{k/3} = e^{\log 3^{k/3}} = e^{k(\log 3)/3}$$

and result (2) follows.

Also solved by N. J. Fine, Vern Hoggatt, M. S. Klamkin, A. E. Livingston, Azriel Rosenfeld, J. V. Whittaker, and the proposer.

A Polygon Theorem

E 1059 [1953, 188]. *Proposed by Chih-yi Wang, Hampton Institute*

Let a circle and an inscribed closed polygon of n sides be given. Show that the product of the distances of a point on the circumference of the circle from the sides of the polygon is equal to the product of the distances of the same point from the sides of the tangential polygon (*i.e.*, the polygon formed by the tangents to the circle at the vertices) of the given polygon.

I. *Solution by L. J. Lander, University of California.* Take the circle as the unit circle, and the point on the circumference as (1, 0). Let $V_1, V_2, \cdots, V_n$ be the vertices of the inscribed polygon, and let OV_k make an angle of $2\theta_k$ with the x-axis (where O is the origin). If d_k is the distance of (1, 0) from the side V_kV_{k+1}, and D_k is its distance from the side of the tangential polygon passing through V_k, we have

$$d_k = \cos(\theta_{k+1} - \theta_k) - \cos(\theta_{k+1} + \theta_k) = 2\sin\theta_k \sin\theta_{k+1},$$
$$D_k = 1 - \cos 2\theta_k = 2\sin^2\theta_k.$$

It is now clear that $\Pi d_k = \Pi D_k$.

II. *Solution by C. W. Trigg, Los Angeles City College.* This is a generalization of 3726 [1937, 58] wherein is proven the lemma: the perpendicular from a point on a circle to a chord is the mean proportional between the perpendiculars to the tangents at the extremities of the chord. Let x_i, y_i denote the lengths of the perpendiculars from the point to the chord and to a tangent, $(i=1, 2, \cdots, n)$. Then $x_i = \sqrt{y_iy_{i+1}}$, $x_{i+1} = \sqrt{y_{i+1}y_{i+2}}$, $\cdots$. It follows immediately that $\Pi x_i = \Pi y_i$.

Also solved by W. B. Carver, Russell Godard and Vern Hoggatt (jointly), J. R. Hatcher, C. S. Ogilvy, S. Parameswaran, Michael Skaliskyj, Sister M. Stephenie, Roscoe Woods, and the proposer.

Carver and Sister Stephanie used complex coordinates. Carver remarked that a set of n vertices leads to $(n-1)!/2$ different inscribed n-gons, all having

the same tangential polygon. When $n=4$ there are three inscribed polygons. Applying the result of the problem to these three inscribed polygons one can readily show that *in a complete cyclic quadrangle the product of the distances of any point on the circumcircle from any pair of opposite sides of the quadrangle is equal to the product of the distances of the point from any other pair of opposite sides of the quadrangle.*

Woods pointed out that the problem is stated as a theorem on p. 72 of Johnson's *Modern Geometry.*

The feet of the perpendiculars dropped on the sides of the tangential polygon lie on a cardioid (see, *e.g.*, Yates' *Curves and Their Properties*, 4 (c), p. 5).

Multinomial Coefficients

E 1060 [1953, 188]. *Proposed by H. K. Crowder, Case Institute of Technology*

Let D be a determinant of order n whose ith row, $i=1, \cdots, n$, is

$$a_i, a_{i-1}, \cdots, a_1, 1, 0, \cdots, 0.$$

Show that

$$D = (-1)^n \sum \frac{(-1)(-2)\cdots(-\sum \alpha_i)}{\alpha_1!\alpha_2!\cdots\alpha_n!} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n},$$

where the sum is taken over all non-negative integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n.$$

Solution by the Proposer. The sum is the coefficient, for x sufficiently small, of x^n in the expansion of

$$(1 + a_1x + a_2x^2 + \cdots + a_nx^n)^{-1}$$

(see, *e.g.*, Chrystal, *Textbook of Algebra, Part II*, 2nd ed. (1900), p. 214). Let

$$(1 + a_1x + a_2x^2 + \cdots + a_nx^n)^{-1} = 1 + b_1x + b_2x^2 + \cdots.$$

Then

$$(1 + a_1x + a_2x^2 + \cdots + a_nx^n)(1 + b_1x + b_2x^2 + \cdots) = 1,$$

and collecting coefficients we have

$$\begin{aligned}
&a_1 + b_1 = 0,\\
&a_2 + a_1b_1 + b_2 = 0,\\
&a_3 + a_2b_1 + a_1b_2 + b_3 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\\
&a_n + a_{n-1}b_1 + a_{n-2}b_2 + \cdots + b_n = 0.
\end{aligned}$$

Solving this system of linear equations in the b_i for b_n we easily find $b_n = (-1)^nD$.

A sidelight is that the number of solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$$

is the number of partitions $P(n)$ of the integer n, so that the number of terms in the expansion of the determinant D is $P(n)$.

Also solved by N. J. Fine and F. D. Parker.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4558. *Proposed by V. F. Ivanoff, San Francisco*

If an octagon, $l_1l_2 \cdots l_8$ is inscribed in a conic, then the eight points of intersection of the sides l_i and $l_j (j \equiv i+3, \bmod 8)$ lie on another conic.

4559. *Proposed by Harry Pollard, Institute for Advanced Study, and Paul Erdös, American University*

Let $0 < a_1 \leqq a_2 \leqq \cdots$ and put

$$F(y) = \prod_{k=1}^{\infty}\left(1 + \frac{y^2}{a_k^2}\right).$$

The necessary and sufficient condition for the convergence of

$$\int_1^{\infty} \frac{\log F(y)}{y^2}\, dy$$

is that $\sum_{k=1}^{\infty} 1/a_k$ converges.

4560. *Proposed by L. J. Mordell, St. John's College, Cambridge, England*

Prove that there are exactly $(\lambda+1)^{n+1} - \lambda^{n+1}$ sets of n integers, $x_1, x_2, \cdots, x_n$, which satisfy the inequalities

$$|x_r| \leqq \lambda, \qquad |x_r - x_s| \leqq \lambda, \qquad (r, s = 1, 2, \cdots, n),$$

where λ is a positive integer.

4561. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Prove

$$S_{2,p}+\sum_{n=1}^{r}\left[n^2\binom{n+p}{p}\right]^{-1}=S_{2,r}+\sum_{n=1}^{p}\left[n^2\binom{n+r}{r}\right]^{-1}$$

where $S_{2,p}=\sum_{n=1}^{p}1/n^2$.

4562. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, let P be a point having normal coördinates (x, y, z) and consider the points A', B', C' with normal coördinates $(-x/2, y, z)$, $(x, -y/2, z)$, $(x, y, -z/2)$. (1) The points A, B, C, A', B', C' lie on one conic S, and there is a conic with respect to which the triangles ABC and $A'B'C'$ are self polar. (2) If AP, BP, CP cut BC, CA, AB in A_1, B_1, C_1 and if $A'P$, $B'P$, $C'P$ cut $B'C'$, $C'A'$, $A'B'$ in A_1', B_1', C_1', the triangles ABC, $A'B'C'$ are circumscribed about a conic Σ, the points of tangency being A_1, B_1, C_1 and A_1', B_1', C_1'. (3) The conics S and Σ have double contact along the common polar of P with respect to these conics.

SOLUTIONS

A Set of Irrational Numbers

4494 [1952, 412]. *Proposed by D. J. Newman, Harvard University*

Let $a_1<a_2<\cdots$ be a sequence of positive integers, and let ξ be the unending decimal fraction formed by juxtaposing the a's (e.g., if the a's are the primes, ξ would be $0.23571113\cdots$). Prove that if $\sum 1/a_n=\infty$, then ξ is irrational.

Solution by Robert Breusch, Amherst College. Assume ξ is rational, with a period $b_1b_2\cdots b_q$, $0\leqq b_j<10$. From a certain index n_0 on, the a_n will be formed from the b's; if $q+1$ of them should contain the same number of digits, then the last number of this sequence would necessarily be equal to the first one, contrary to the assumption that the a_n increase.

Thus

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}<q\cdot\sum_{n=0}^{\infty}\frac{1}{10^n}<11q.$$

But this is contrary to the divergence of $\sum 1/a_n$. Therefore ξ must be irrational.

Discussion by Fritz Herzog, Michigan State College. The above result should be considered in connection with the following theorem of Copeland and Erdös: If $a_1, a_2, \cdots$ *is an increasing sequence of integers such that for every* $\theta<1$ *the number of* a's *up to* N *exceeds* N^θ *provided* N *is sufficiently large, then the infinite decimal* $\xi=0.a_1a_2a_3\cdots$ *is normal with respect to the base in which these in-*

*tegers are expressed.** A number is normal provided each of the digits 0, 1, 2, $\cdots$, 9 occurs with a limiting relative frequency of 1/10 and each of the 10^k sequences of k digits occurs with the frequency 10^{-k}.

To the question as to whether the numbers of problem 4494 are necessarily normal a negative answer is provided by the following example. We select the numbers a_n in groups which will be referred to alternately as "good" groups and "bad" groups. The first good group is to consist of the numbers 1, 2, $\cdots$, 9. These are followed by the first bad group, viz. the numbers 99, 999, $\cdots$, $10^{m_1}-1$, where m_1 is so large that the relative frequency of the digit 9 among the numbers selected so far exceeds $\frac{1}{2}$. Next we take for the second good group all numbers from 10^{m_1} to $10^{m_1+1}-1$, *i.e.*, all numbers with m_1+1 digits. For the second bad group we take the numbers $10^\mu-1$, $\mu=m_1+2, m_1+3, \cdots, m_2$, where again m_2 is so large that the relative frequency of the digit 9 among all the numbers selected up to this point is greater than $\frac{1}{2}$. Continuing in this manner we obtain a sequence $\{a_n\}$ and a number ξ. The latter is obviously not normal since the relative frequency of the digit 9 is greater than $\frac{1}{2}$ at the end of every bad group. On the other hand $\sum_n 1/a_n=\infty$, since the sum of the reciprocals of the a_n of any good group is greater than log 10.

Also solved by N. J. Fine, Harry Furstenberg, Fritz Herzog, M. S. Klamkin, Norman Miller, Leo Moser, M. Pearl, L. L. Pennisi, Louis Weiner, and the Proposer.

A Restricted Mean Value Theorem for Complex Variables

4495 [1952, 412]. *Proposed by M. S. Wertheim, Ithaca, New York*

Find all regular functions of a complex variable for which the mean value theorem holds for all a, b, in the form

$$\frac{F(b)-F(a)}{b-a}=F'(c)$$

where c is on the straight line segment ab.

Solution by N. J. Fine, University of Pennsylvania. Let a be a point such that $F''(a)\neq 0$ (leaving aside the trivial case where F is linear). We may assume that $a=0$ and $F(a)=0$. Since $F''(0)\neq 0$, the equation

$$F'(w)=\frac{F(z)}{z}$$

defines w as an analytic function of z in the neighborhood of the origin, with $w(0)=0$. By assumption, however, $w(z)=z\cdot\theta(z)$ where $\theta(z)$ assumes only real values. So $\theta(z)$ is constant, whence F satisfies

$$F'(\theta z)=\frac{F(z)}{z}\cdot$$

* Bull. Amer. Math. Society, v. 52 (1946), pp. 857–860.

It is now easy to see, by equating coefficients in the power series expansions of both sides, that $\theta=\frac{1}{2}$ and $F^{(n)}(0)=0$ for $n>2$. Thus $F(z)$ is quadratic.

Also solved by O. E. Stanaitis and the Proposer.

Entire Functions with Prescribed Complex Roots

4498 [1952, 469]. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Chatham, New Jersey*

Prove that every complex number is a zero of some entire function whose power series has rational coefficients.

Solution by Fritz Herzog, Michigan State College. Let $A(z)=\sum a_n z^n$ be any entire function. If $A^*(z)=\sum \bar{a}_n z^n$ then $A(z)\cdot A^*(z)$ is an entire function whose power series has real coefficients.

Let $B(z)=\sum b_n z^n$ be any entire function with real b_n. Then there exists an entire function $C(z)=\sum c_n z^n$ (with $C(z)\not\equiv 0$), such that the power series of $B(z)\cdot C(z)$ has rational coefficients. Indeed, if b_n is the first non-vanishing coefficient among the b's, we merely have to choose $c_0, c_1, c_2, \cdots$, successively in such a way that $-1/m!<c_m<1/m!$ and that the number

$$b_h c_m + b_{h+1}c_{m-1} + \cdots + b_{h+m}c_0$$

becomes rational. (To ensure $C(z)\not\equiv 0$, choose $c_0\neq 0$.)

The statement of the proposed problem is proved by application of the above procedure to $A(z)=\omega-z$, where ω is the given complex number. Moreover, by the use of Weierstrass' Product Theorem, we can extend that statement from a single point ω to any given point-set in the complex plane which has no finite limit-point.

Also solved by P. J. Cohen, A. M. Gleason, Melvin Henriksen, V. Ganapathy Iyer, Joseph Lehner, L. L. Pennisi, and the Proposers.

Editorial Note. As pointed out by Henriksen, this is a known result. See A. Hurwitz (*Acta Mathematica*, 14 (1890–91), pp. 211–215), O. Helmer (*Duke Math. Journal*, 6 (1940), pp. 345–356), and J. Lehner (*Journ. London Math. Society*, 25 (1950), pp. 279–282).

Probability that a Congruence Has No Solution

4499 [1952, 469]. *Proposed by P. G. Federbush, Student, Massachusetts Institute of Technology*

The coefficients of the congruence

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \equiv 0 \pmod{p}$$

are subject to the restriction $0\leqq a_j\leqq p-1$, $a_0\neq 0$, but otherwise are taken at random. p is a prime. What is the probability that the congruence has no solution?

Solution by Leonard Carlitz, Duke University. Consider the number $\phi_n(M)$ of primary polynomials

$$x^n + a_1 x^{n-1} + \cdots + a_n \pmod{p}$$

of degree n, prime to a given polynomial $M(x)$ (mod p). Then by the argument used in elementary number-theory to evaluate the Euler ϕ-function we can show that

$$\phi_n(M) = p^n \prod_{P|M} \left(1 - \frac{1}{|P|}\right) \qquad (|P| = p^{\deg R}), \tag{1}$$

where the product is over all irreducible P dividing M, and it is understood that we retain only those terms of the expanded product which are of non-negative degree in p. (For a different proof of (1), see *American Journal of Mathematics*, vol. 54 (1932), p. 44.)

Now for the problem in hand we take $M = x^p - x \equiv x(x-1) \cdots (x-p+1)$ (mod p). Hence (1) becomes

$$\phi_n(x^p - x) = p^n \left(1 - \frac{1}{p}\right)^p,$$

with the same understanding as before about which terms to retain. Since the total number of primary polynomials of degree n(mod p) is p^n, we see that the probability that the stated congruence have no solution is given by

$$p^{-p}(p-1)^p \qquad (n \geqq p)$$

$$\sum_{r=0}^{n} (-1)^r \binom{p}{r} p^{-r} \qquad (n < p).$$

Also solved by A. M. Gleason and the Proposer.

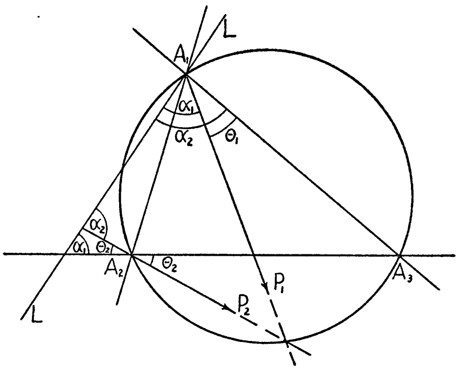
Editorial Note. Fritz Herzog refers to a paper by K. Zsigmondy, "Ueber die Anzahl derjenigen ganzzahligen Functionen n^{ten} Grades von x, welche in Bezug auf einen gegebenen Primzahlmodul eine vorgeschriebene Anzahl von Wurzeln besitzen," *Sitzungsber. Ak. Wiss. Wien, Math.-naturw. Kl.*, vol. 103 (1894), pp. 135–144. (See also L. E. Dickson, *History of the Theory of Numbers*, vol. I, p. 230.) The present problem is a special case of the one considered by Zsigmondy and the result may be obtained directly from his.

Concurrent Lines

4500 [1952, 469]. *Proposed by J. R. Musselman, Western Reserve University*

Given three points $A_i (i=1, 2, 3)$ and a line L cut by the lines $A_j A_k$ making angles α_i in the positive sense. Show that the lines drawn through A_i making angles $\pi - \alpha_i$, in the positive sense, with L are concurrent at a point P on the circumcircle of $A_1A_2A_3$. Further, if the altitudes with A_i make angles β_i with L

then the lines through A_i making angles $\pi-\beta_i$ with L are concurrent on the circumcircle of $A_1A_2A_3$ at a point Q diametrically opposite to P.



Solution by W. J. Robinson, Centre College, Kentucky. The line L can be replaced by any line parallel to L without changing the problem. Hence, take L through A_i so that it remains outside the triangle $A_1A_2A_3$, as in the drawing (where A_i is taken as A_1). The hypothesis of the problem is satisfied if the pairs of angles labeled α_1 and α_2 are equal. Let the lines drawn through A_1 and A_2 meet the circumcircle again in P_1 and P_2. Continuity and symmetry now complete the proof if we can show that A_1P_1 and A_2P_2 intersect on the circle.

But the angles labeled θ_1 and θ_2 are equal since each is $\alpha_2-\alpha_1$. Also θ_1 and θ_2 are measured respectively by half arc P_1A_3 and half arc P_2A_3. Hence the two arcs are equal and $P_1=P_2=P$.

Since β_i is the complement of α_i, the α-line through A_i is perpendicular to the β-line. But such lines meet at Q, the point diametrically opposite to P, since all inscribed angles are in semicircles.

Also solved by A. P. Boblétt, A. M. Gleason, Joseph Langr, and S. V. Venkataramana Rao.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

Introduction to Measure and Integration. By M. E. Munroe. Cambridge, Mass., Addison-Wesley Publishing Co., Inc., 1953. 10+310 pages, \$7.50.

This book has been written primarily as a textbook for a first course in measure and integration theory. It presupposes that the reader has a basic

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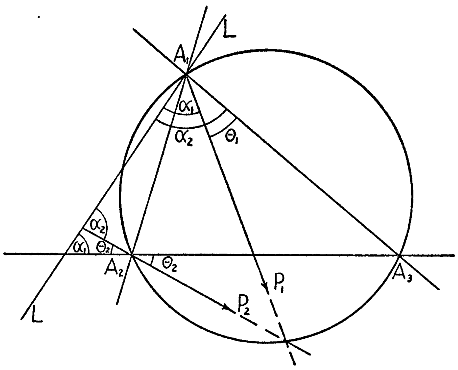

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knowledge of real function theory and in particular, that he is familiar with "ϵ, δ" method of proof. The author's approach is from the postulational point of view with emphasis given to a number of the important interpretations of the theory.

The first chapter is a review of that material of basic mathematics and real function theory which will be used in later sections, for example, algebra of sets, cardinal numbers, metric spaces, limits, and function spaces. Chapter II is concerned with the theory of additive set functions and measures. General procedures are given for constructing outer measures and metric outer measures.

Chapter III deals primarily with specific examples of measures, *i.e.*, Lebesgue-Stieltjes, Hausdorff, and Haar. It concludes with a discussion of non-measurable sets. Chapter IV opens with a discussion of measurable functions, and their basic properties. Approximation theorems are then developed. It concludes with a short section on stochastic variables. In Chapter V the integral is defined *via* simple functions and its basic properties developed. Absolute continuity, Fubini's theory, and the expectation of a stochastic variable are treated in the latter sections. Chapter VI deals with convergence theorems, L^p spaces, linear normed spaces, the mean ergodic theorem, and orthogonal expansions. Chapter VII is concerned with differentiation. It contains sections on Vitali coverings, differentiation of additive set functions, the Lebesgue decomposition, metric density and approximate continuity, and differentiation with respect to nets.

The format and typography of the book are pleasing. The author's choice of terminology and notation is modern. Pertinent references are listed at the end of each chapter. In addition to the usual index, an index of symbols and an index of postulates are provided.

The author's style is clear and concise. Motivation is provided and possible misunderstandings pointed out. The book was written with flexibility in mind. A number of the sections mentioned above are starred so that they may be taken up or omitted depending upon an instructor's conception of what topics should constitute a course in measure and integration. Exercises are abundant and play an important role. Besides being used to familiarize the reader with the terminology and to embellish the text material, they are used extensively to emphasize the importance of counter examples and to develop different approaches to the theory. It should be pointed out the topology used is of a metric nature and that in general the ultimate in generality has been avoided when it appears to unduly complicate the discussion.

This is an excellent textbook and in addition should prove to be a good reference for the non-specialist in integration.

E. H. CRISLER
Oberlin College

Introduction to the Foundations of Mathematics. By R. L. Wilder. New York, John Wiley and Sons, Inc., 1952. xiv+305 pages. $5.75.

Professor Wilder's course on the foundations of mathematics has for over twenty years been a source of intellectual stimulation for students from many departments of the University of Michigan. The publication of his book now affords others an opportunity to benefit from his experience as a student and teacher in this field. Although the nature of the subject requires of the reader an ability to handle abstract concepts, he will need very little in the way of mathematical background in order to appreciate the author's exposition.

The book is divided into two parts, the first, comprising nearly two thirds of the text, on fundamental concepts and the second on foundations. Part I begins with a discussion (two chapters) on the axiomatic method. This is illustrated by the development of a set of postulates for a portion of Euclidean geometry and of some of their consequences. Here, as throughout, historical comments and numerous problems add much to the value of the text. The next three chapters deal with set theory, including such topics as the Russell paradox, infinite sets, the Choice Axiom, and cardinal and ordinal numbers. A chapter on the real number system, and one on groups and related algebraic structures complete this section. Part II, after brief discussions of the views of Kronecker, Cantor, Boole, Frege, Peano, and Poincaré, and an outline of Zermelo's axiomatisation of set theory, contains chapters on logicism, intuitionism, and formalism, and closes with one, entitled "The Cultural Setting of Mathematics," in which the author expresses his own convictions concerning the nature of mathematics. There is an eleven page bibliography and the book is well indexed.

The latter part of the book is particularly valuable since there is little available in English for the student who wishes to learn something of the subjects treated therein. This is especially true of intuitionism, and the author's chapter on this, most of which is devoted to the very complicated intuitionist set theory, furnishes an excellent introduction to the subject. The chapter on formalism includes proofs of Gödel's theorems on incompleteness and consistency of formal systems (omitting, of course, the proofs of recursiveness of various number-theoretic functions).* In the final chapter the author argues that the scope of mathematics varies enormously with the culture, thus rendering impossible a definition of "mathematics."

Professor Wilder's book is without question a real addition to the still quite scanty collection of texts on the foundations of mathematics. Many students will certainly find the reading of it a stimulating experience and, through the excellent system of bibliographical references, will be encouraged to continue further study of the subject.

H. E. VAUGHAN
University of Illinois

* The example given on p. 258 of translating a statement about formulas into one about their Gödel numbers is, unfortunately, incorrect. Consider, for instance, the Gödel numbers of $F_1(x)$ and $F_2(x)$.

Numerical Solutions of Differential Equations. By W. E. Milne. New York, John Wiley and Sons, Inc., 1953. 11+275 pp. $6.50.

The author has succeeded admirably in presenting some of the principal techniques available for the numerical solution of ordinary and partial differential equations. This is the most thorough, practical and comprehensive treatise which has appeared in English. By reading carefully about twenty pages suggested in the preface the practical computer can handle almost any "decent" problem in ordinary differential equations.

Part I includes seven chapters: (1) Introduction; (2) Elementary Numerical Solution; (3) Analytical Foundations; (4) Methods Based on Numerical Integration; (5) Methods of Runge-Kutta. Methods Based on Higher Derivatives; (6) Systems of Equations. Higher Order Equations; (7) Two-Point Boundary Conditions. Part II, Partial Equations, contains the headings: (8) Explicit Methods. Parabolic and Hyperbolic Equations; (9) Linear Equations and Matrices; (10) Implicit Methods. Elliptic Equations; (11) Characteristic Numbers. Three Appendixes, (A) Round-off errors; (B) Large-scale computing machines, (C) The Monte Carlo method, are followed by fourteen pages of Bibliography and two Indexes.

One of the important features of the book is the careful analysis of error, both step by step, and total. For instance in section 41 one formula is used to predict or integrate ahead and another to correct, thus providing a measure of the truncation error. Moreover certain formulas give a bound for the maximum accumulated error over a range.

Chapter 3 starts with Taylor's series and ends with numerous Quadrature Formulas in terms of Ordinates, Backward Differences, and Central Differences. Considerable use is made of difference equations and their solution. In chapter 6 the D, D^2 are replaced by central difference operators and Method XIII involves the solution of a truncated central difference equation and its eventual correction.

In Part II one treats only some simpler equations for which less general methods have been devised. These include either an approximate difference equation in two variables or a system of algebraic equations. Other topics treated are general boundary conditions, the point pattern, variable coefficients, curved boundaries, relaxation, iteration, accelerating convergence, orthogonality, and removal of singularities on the boundary.

There are fifty formal computations and many exercises. Answers to a number of these have been computed. Several typographical errors were found but on the whole the printing, figures and stencils are clear and the binding excellent.

C. C. Camp
University of Nebraska

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

NOTE ON MAGIC CIRCLES

G. E. RAYNOR, Lehigh University

In a recent paper* S. W. McInnis raised the following two questions: Is the number of perfect magic circles finite? If finite, can an example of each different set be illustrated?

That the answer to the first question is in the negative can be seen at once from the following two observations.

(1) Any magic square can be deformed into a perfect magic circle so that the columns of the square become the radii of the circle and the rows of the square become the rings of the circle.

(2) It is well known that magic squares of all orders, except of order two, exist.

As a matter of fact, the number of perfect magic circles with n radii is even greater than the number of magic squares of order n since in forming the circle as described above the diagonal property of the magic square is not needed. Thus a perfect magic circle can be formed from any semi-magic square.

A THEOREM FOR THE THEORY OF NUMBERS

ALBERTO CORAZAO, Escuela Militar de Chorrillos, Lima, Peru

If N is a composite integer whose positive factors are p and q, then N can be expressed as the sum of three addends, p, q and t, all of which are connected intimately to some arithmetical progressions. To this end, we state the following theorem.

THEOREM. *If p and q are two positive integers and if t is an integer representing the pth term of an arithmetical progression whose first term is -1 and whose common difference is $q-1$, then*

$$pq = p + q + t. \tag{1}$$

Proof: The formula for the pth term, t, of an arithmetical progression whose first term is -1 and whose common difference is $q-1$ is

$$t = -1 + (p-1)(q-1)$$

whence

$$t = -1 + pq - p - q + 1 = pq - p - q,$$

which shows the validity of (1).

* This MONTHLY, vol. 60, No. 5, 1953, pp. 347–351.

$$6 \times 8 = 6 + 8 + 34 = 48.$$

By (2),

$$qp = q + p + t_{pq}.$$

Since $qp = pq$, it follows that

$$t_{qp} = t_{pq}.$$

But the values of t are based on two different progressions. To illustrate, consider

$$8 \times 5 = 8 + 5 + t.$$

Here t, from the array, comes from the progression

$$-1, 3, 7, 11, 15, 19, 23, 27, \cdots$$

and

$$8 \times 5 = 8 + 5 + 27 = 40.$$

On the other hand, in the product

$$5 \times 8 = 5 + 8 + t,$$

t, from the array, comes from the progression

$$-1, 6, 13, 20, 27, \cdots$$

and

$$5 \times 8 = 5 + 8 + 27 = 40.$$

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

College of St. Thomas announces: Professor F. C. Smith has been named Chairman of the Department of Mathematics; Dr. Kenneth McMillin has been appointed to an assistant professorship; Mr. Roy Dowling has been appointed to an instructorship.

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At the University of South Carolina: Dr. Stephen Kulik of Claremont Men's

College has been appointed to an associate professorship; Dr. H. S. Collins of the University of Maryland has been appointed to an assistant professorship; Professor D. H. Clanton of Allen Military Academy and Mr. D. T. Walker, previously at the University of Georgia, have been appointed to instructorships.

Wayne University announces: Associate Professor Max Coral has been promoted to a professorship; Dr. Fred Brafman has been promoted to an assistant professorship; Professor A. F. Stevenson, formerly of the University of Toronto, has been appointed Associate Professor of Mathematics and Physics; Professor J. W. Baldwin, who retired in June, 1953, has been given the title of Professor Emeritus; Mr. Walter Hoffman of the University of Michigan and Mr. H. T. Slaby, University of Wisconsin, have been appointed to instructorships; Dr. S. T. C. Moy of the University of Illinois has been appointed to a part-time instructorship.

Mr. M. J. Aegerter, previously a graduate student at the University of Wisconsin, has accepted a position as a mathematician at the United States Naval Ordnance Test Station, China Lake, California.

Mr. J. T. Ahlin, who has been a research engineer with Douglas Aircraft Company, Santa Monica, California, is now Applied Science Representative with International Business Machines Corporation, Santa Monica.

Mr. J. E. Alman of Boston University has been appointed Director of the Office of Statistical and Research Services of the University.

Miss Florence R. Anderson, formerly a research laboratory analyst with Northrop Aircraft, Hawthorne, California, has accepted a position as a research engineer with North American Aviation, Los Angeles, California.

Assistant Professor H. M. Anderson of Gustavus Adolphus College has been promoted to an associate professorship.

Mr. J. M. Anderson, previously a student at the University of South Dakota, has been appointed to a graduate assistantship at Iowa State College.

Mr. D. L. Arenson, who has been Senior Research Engineer at the Cook Research Laboratory, Skokie, Illinois, is now Technical Director of the Aerophysics Section.

Assistant Professor Grace E. Bates of Mount Holyoke College has been promoted to an associate professorship.

Assistant Professor J. W. Beach of the University of New Mexico has been appointed to an associate professorship at Northern Illinois State Teachers College.

Assistant Professor Helen P. Beard of Newcomb College, Tulane University, is on leave of absence for the year 1953–54 and is at the Statistical Laboratory, University of California.

Assistant Instructor Imogene C. Beckemeyer of Southern Illinois University has been promoted to an instructorship.

Dr. V. N. Behrns of the University of Buffalo has accepted a position as Senior Operations Analyst with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

At the University of Dayton, Reverend William Bellmer, professor of mathematics and head of the Department of Mathematics, has been named Associate Dean of Science.

Mr. K. S. Bergman, formerly a field assistant with the United States Geological Survey, Spokane, Washington, is now Assistant Attorney General of the State of Washington.

Professor R. V. Blair of Vanderbilt University has been given the title of Professor Emeritus.

Assistant Professor G. M. Bloom of Miami University has been promoted to an associate professorship.

Mr. G. F. Bradfield has a position as Acting Educational Director at the Andersen Air Force Base, Guam.

Miss Mary P. Burkart, previously an instructor at Trinity College, Washington, D. C., has a position as a mathematician in the Navy Department, David Taylor Model Basin, Washington, D. C.

Assistant Professor H. N. Carter of the University of Tulsa has been promoted to an associate professorship.

Miss Jo Ann Cipolla, who has been a student at Elmira College, has accepted a position as a mathematician with E. I. duPont de Nemours and Company, Wilmington, Delaware.

Associate Professor Paul Civin of the University of Oregon is at the Institute for Advanced Study for the academic year 1953–54.

Miss Helen E. Core, formerly a research assistant at the University of Michigan, has been appointed Head of the Department of Mathematics of Northwestern Michigan College, Traverse City.

Mr. D. W. Crowe, previously a teaching fellow at the University of Michigan, has been appointed to a research assistantship.

Dr. R. B. Deal of the University of Oklahoma has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

Assistant Professor S. P. Diliberto of the University of California has been appointed Assistant Dean of the College of Letters and Science.

Professor W. E. Edington, formerly head of the Department of Mathematics of DePauw University, has been given the title of Professor Emeritus.

Professor Howard Eves of Champlain College has been appointed to a professorship at Harpur College.

Mr. E. A. Fay, formerly a graduate student at the University of California, has a position as a statistician with the United States Naval Ordnance Test Station, China Lake, California.

Dr. J. F. Foster of the University of Arizona has been promoted to an assistant professorship.

Associate Professor J. E. Freund of Alfred University has been promoted to a professorship.

Mr. A. R. Friedenheit, who has been a stress analyst with North American

Aviation, Columbus, Ohio, has accepted a position as a mathematician with Douglas Aircraft Company, Santa Monica, California.

Mr. D. A. Gorsline of Equitable Life Assurance Society, Albany, New York, has been promoted to the position of District Manager.

Mr. J. S. Griffin, Jr., of Tulane University has been promoted to an instructorship.

Dr. W. N. Hallett is now Training Director of Goodwill Industries of Pittsburgh.

Associate Professor H. J. Hamilton of Pomona College has been promoted to a professorship.

Mr. C. V. Hannan, III, who has been a radio engineer with the Civil Aeronautics Administration, Oklahoma City, has been promoted to the position of Supervisory Electronics Engineer.

Mr. R. C. Haseltine has been promoted to the position of Research Engineer, Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Miss Joy S. Heller, previously a student at Brooklyn College, has been appointed to a graduate assistantship at the University of Oregon.

Mr. John Hilzman, who has been a student at the University of Rhode Island, has been appointed to a graduate assistantship at Oregon State College.

Assistant Professor Jessie M. Hoag of Southwestern Louisiana Institute has been promoted to an associate professorship.

Instructor S. B. Hobbs of the University of New Hampshire has accepted a position as an assistant project engineer with Sperry Gyroscope Company, Great Neck, New York.

Assistant Professor P. G. Hodge of the University of California at Los Angeles has been appointed to an associate professorship at the Polytechnic Institute of Brooklyn.

Mr. R. R. Hohl, formerly a graduate assistant at Lehigh University, has been appointed to a graduate assistantship at Albright College.

Mr. F. X. Holzhauer, formerly a teaching fellow at the University of Detroit, is now a research engineer with Ford Motor Company, Dearborn, Michigan.

Mr. R. E. Horton of Los Angeles City College has replaced Mr. C. W. Trigg, Los Angeles City College, as editor of the Problems and Questions Department of *Mathematics Magazine.*

Mr. David Horwitz has been promoted to the position of Associate Engineer, Armour Research Foundation, Chicago, Illinois.

Miss Ruth A. Huffman, who has been engaged as a statistician at Tinker Air Force Base, Oklahoma City, has accepted a position as an engineer with Chance Vought Aircraft, Dallas, Texas.

Mrs. Elizabeth M. Hutcheson, previously a student at the University of Cincinnati, is now with the Operations Research Group of Arthur D. Little, Incorporated, Cambridge, Massachusetts.

Mr. W. R. Hydeman, formerly with the Department of Defense, Washington, D. C., is engaged as a mathematician with the Engineering Research Associates, Division of Remington Rand, Arlington, Virginia.

Mr. R. E. Jackson, previously a student at the University of California, has a position as a mathematician with International Business Machines Corporation, Los Angeles, California.

Dr. H. G. Jacob, who has been an assistant at Yale University, has been appointed to an assistant professorship at Louisiana State University.

Dr. T. A. Jeeves of the University of California has been promoted to an assistant professorship.

Mr. Sidney Kaplan has left his position as Research Mathematician, Planning Research Branch, Office Comptroller of the Army, and has joined the staff of the Computing Systems Engineering Section, Victor Division, Radio Corporation of America, Camden, New Jersey.

Mr. C. E. Kelley of the University of Missouri has been appointed to an instructorship at Wentworth Military Academy, Lexington, Missouri.

Mr. C. G. Koch, who has been a student at Marquette University, has accepted a position with Illinois Bell Telephone Company, Chicago, Illinois.

Mr. S. B. Kramer, formerly a research assistant at the Polytechnic Institute of Brooklyn, has a position as an assistant project engineer with Sperry Gyroscope Corporation, Great Neck, New York.

Mr. H. C. Kranzer, previously a student at New York University, has been appointed to a research assistantship at the University.

Assistant Professor C. E. Langenhop of Iowa State College has been promoted to an associate professorship.

Dr. J. R. Lee, who has been an instructor at the University of Michigan, has been appointed to an associate professorship at the College of William and Mary.

Mr. Walter Littman, formerly a student at New York University, has been appointed to a research assistantship at the Institute for Mathematics of the University.

Professor E. R. Lorch of Columbia University has been appointed a Fulbright lecturer at the University of Rome.

Mr. H. J. Miller, who has been teaching at St. Patricks High School, Elizabeth, New Jersey, is teaching now at Linden High School, New Jersey.

Assistant Professor J. T. Moore of Georgia Institute of Technology has been appointed to an associate professorship at the University of Florida.

Professor Emeritus E. E. Moots of Cornell College has been appointed Lecturer at Occidental College for the academic year 1953–54.

Mr. S. I. Neuwirth, who has been affiliated with the Research Division of Schering Corporation, Bloomfield, New Jersey, has accepted the position of Biometrician with the Committee on Research of the American Medical Association, Chicago, Illinois.

Professor M. J. Norris of the College of St. Thomas is now a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor T. E. Rine of Illinois State Normal University has been promoted to an associate professorship.

Professor R. A. Rosenbaum of Reed College has been appointed to a professorship at Wesleyan University.

Assistant Professor H. L. Royden of Stanford University has been promoted to an associate professorship.

Professor L. P. Siceloff of Columbia University has been given the title of Professor Emeritus.

Mr. E. C. Smith, Jr., has been appointed to an instructorship at the University of Oregon.

Mr. J. L. Spenceley, previously of Alpena Junior College, Michigan, is teaching at Grand Haven High School, Michigan.

Mr. James Stone, who has been employed at the Oak Ridge National Laboratory, Tennessee, has joined the staff of Battelle Institute, Columbus, Ohio.

Mrs. Barbara F. Turner, formerly a student at Agnes Scott College, is teaching at Griffin High School, Georgia.

Professor J. A. Ward was on leave of absence from the University of Kentucky during the summer of 1953 and served as consultant at Holloman Air Force Base, New Mexico.

Associate Professor R. L. Westhafer of New Mexico College of Agriculture and Mechanic Arts has been promoted to a professorship.

Lt. G. C. Zader of the United States Naval Reserve has been appointed to an assistant professorship at The Citadel.

Dr. Henry Zatzkis of the University of Connecticut has been appointed to an assistant professorship at Newark College of Engineering.

Associate Professor E. A. Goodhue of the Missouri School of Mines and Metallurgy died on June 9, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FOURTH SUMMER MEETING OF THE ASSOCIATION

The thirty-fourth summer meeting of the Mathematical Association of America was held at Queen's University and the Royal Military College, Kingston, Ontario, Canada, on Monday and Tuesday, August 31 and September 1, 1953, in conjunction with the summer meetings of the American Mathematical Society, the Canadian Mathematical Congress, the Institute of Mathematical Statistics, and the Econometric Society. A total of six hundred and thirty-seven

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adults were registered, including the following two hundred and ninety-three members of the Association:

C. R. Adams, M. I. Aissen, Bess E. Allen, E. B. Allen, C. B. Allendoerfer, A. G. Anderson, R. D. Anderson, R. G. Archibald, Helen C. Arens, E. L. Arnoff, H. E. Arnold, Nachman Aronszajn, M. G. Arsove, H. T. R. Aude, Miriam C. Ayer, R. E. Barlow, I. A. Barnett, C. F. Barr, J. H. Barrett, R. C. F. Bartels, P. T. Bateman, W. R. Baum, L. D. Baumert, Samuel Beatty, P. R. Beesack, E. G. Begle, R. L. Beinert, Theodore Bennett, J. S. Bergen, R. H. Bing, D. W. Blackett, Shirley A. Blackett, David Blackwell, M. Isobel Blyth, H. F. Bohnenblust, Evelyn Boyd, C. B. Boyer, A. T. Brauer, G. U. Brauer, A. M. Bryson, F. J. H. Burkett, G. C. Bush, G. H. Butcher, James William Butler, E. A. Cameron, J. W. Campbell, K. H. Carlson, A. B. Carson, F. L. Celauro, Jeremiah Certaine, Abraham Charnes, Elsie T. Church, Randolph Church, R. V. Churchill, Paul Civin, Willie E. Clark, Helen E. Clarkson, D. E. Coffey, A. C. Cohen, Jr., L. W. Cohen, R. H. Cole, A. J. Coleman, J. B. Coleman, Esther Comegys, R. M. Conkling, T. F. Cope, A. H. Copeland, H. S. M. Coxeter, C. C. Craig, A. B. Cunningham, H. B. Curry, A. E. Danese, J. R. Davis, Jr., R. Y. Dean, Douglas Derry, W. A. Dolid, E. J. Downie, W. L. Duren, Jr., W. F. Eberlein, P. D. Edwards, M. P. Emerson, H. P. Evans, T. G. Evans, H. F. Fehr, H. H. Ferns, J. V. Finch, D. T. Finkbeiner, C. H. Fischer, G. E. Forsythe, J. S. Frame, Evelyn Frank, H. D. Friedman, Orrin Frink, Jr., Ruth M. Frisch, R. E. Fullerton, Ilse N. Gal, S. I. Gal, A. E. Gault, H. M. Gehman, Irving Gerst, B. C. Getchell, K. G. Getman, K. S. Ghent, Seymour Ginsburg, Sidney Glusman, Herbert Goertzel, Casper Goffman, Michael Goldberg, S. H. Gould, J. W. Green, Simon Green, F. L. Griffin, V. G. Grove, P. R. Halmos, H. W. Handsfield, W. C. Hansen, Bertha I. Hart, G. E. Hay, C. E. Heilman, E. R. Heineman, M. J. Hellman, Melvin Henriksen, J. G. Herriot, I. N. Herstein, Max Herzberger, Edwin Hewitt, T. H. Hildebrandt, R. V. Hogg, F. E. Hohn, D. L. Holl, T. R. Hollcroft, Aughtum S. Howard, W. A. Hurwitz, W. R. Hutcherson, Jack Indritz, R. L. Jeffery, L. W. Johnson, R. E. Johnson, F. E. Johnston, B. W. Jones, P. S. Jones, G. K. Kalisch, L. H. Kanter, Leo Katz, Hyman Kaufman, Dora E. Kearney, J. B. Kelly, J. R. F. Kent, D. E. Kibbey, E. C. Kiefer, F. T. Kocher, Jr., H. L. Krall, Max Kramer, Saul Kravetz, Solomon Kullback, O. E. Lancaster, N. D. Lane, C. E. Langenhop, E. H. Larguier, R. D. Larsson, H. L. Lee, F. C. Leone, Caroline A. Lester, W. J. LeVeque, F. A. Lewis, Florence Long, M. M. Lotkin, R. C. Lyndon, R. W. MacDowell, H. M. MacNeille, M. S. MacPhail, W. G. Madow, V. S. Mallory, Beckham Martin, Ceslovas Masaitis, K. O. May, N. H. McCoy, S. W. McCuskey, W. H. McEwen, S. W. McInnis, J. E. McLaughlin, E. J. McShane, Paul Meier, B. E. Meserve, C. E. Miller, E. B. Miller, Norman Miller, E. B. Mode, Harriet F. Montague, Mabel D. Montgomery, J. T. Moore, T. W. Moore, J. S. Morrel, D. C. Morrow, Thirza Mossman, C. W. Munshower, W. L. Murdock, Zeev Nehari, C. R. Newell, C. V. Newsom, Abba V. Newton, C. P. Nicholas, C. O. Oakley, E. G. Olds, E. M. Olson, J. C. Oxtoby, F. D. Parker, Sallie E. Pence, F. W. Perkins, G. M. Petersen, H. P. Pettit, C. G. Phipps, C. F. Pinzka, George Piranian, Everett Pitcher, J. C. Polley, I. R. Pounder, G. B. Price, Tibor Rado, J. F. Randolph, Ruth B. Rasmusen, G. E. Raynor, H. W. Reddick, R. M. Redheffer, P. K. Rees, R. F. Reeves, Eric Reissner, C. E. Rhodes, Audrey I. Richards, J. D. Riley, E. K. Ritter, H. E. Robbins, G. deB. Robinson, Louis Robinson, Arthur Rosenthal, Edward Rosenthall, M. F. Rosskopf, S. G. Roth, E. H. Rothe, J. P. Russell, Arthur Saastad, Charles Saltzer, Hans Samelson, R. D. Schafer, J. A. Schatz, Edith R. Schneckenburger, B. L. Schwartz, W. R. Scott, Edward Silverman, Annette Sinclair, Aubrey Henderson Smith, Robert Elijah Smith, D. O. Snow, W. S. Snyder, Vivian E. Spencer, Marion E. Stark, F. H. Steen, C. F. Stephens, B. M. Stewart, Irving Sussman, R. L. Swain, William Clare Taylor, G. H. M. Thomas, D. L. Thomsen, Jr., R. M. Thrall, H. S. Thurston, H. E. Tinnappel, Marian M. Torrey, A. W. Tucker, J. W. Tukey, H. L. Turrittin, J. L. Ullman, J. P. Van Alstyne, Helen E. Van Sant, J. E. Vollmer, T. L. Wade, Jr., R. W. Wagner, Eleanor B. Walters, Susie L. Ward, J. F. Wardwell, W. G. Warnock, J. V. Wehausen, C. P. Wells, J. G. Wendel, F. J. Weyl, A. L. Whiteman, G. T. Whyburn, W. L. G. Williams, Clement Winston, L. M. Winer, H. M. Zerbe, A. D. Ziebur, Antoni Zygmund.

Sessions of the Association were held on Monday afternoon, on Tuesday morning, and on Tuesday evening in Convocation Hall of Queen's University. President E. J. McShane presided at all sessions, except the joint session on Tuesday morning when the presiding officer was Professor W. L. G. Williams representing the Canadian Mathematical Congress. The second series of Earle Raymond Hedrick Lectures was delivered by Professor P. R. Halmos. The Program Committee for the meeting consisted of A. W. Tucker, Chairman, H. S. M. Coxeter, and Abel Gauthier.

FIRST SESSION OF THE ASSOCIATION

"Recent Developments of Mathematics in Canada," by Professor R. L. Jeffery, Queen's University.

"Some Mathematical Concepts," by Professor M. H. A. Newman, University of Manchester.

The Earle Raymond Hedrick Lectures: "Axiomatic Set Theory," Lecture I, by Professor P. R. Halmos, University of Chicago.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Axiomatic Set Theory," Lecture II, by Professor P. R. Halmos, University of Chicago.

Joint session with the Canadian Mathematical Congress:

"Integration in Archimedes," by Professor S. H. Gould, Purdue University.

"Analysis: Notes on the Evolution of a Subject and a Name," by Professor C. B. Boyer, Brooklyn College.

"Mathematics in Russian Universities between the Two World Wars," by Professor G. G. Lorentz, University of Toronto. (Presented by title.)

"The History of the Golden Section," by Professor H. S. M. Coxeter, University of Toronto.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Axiomatic Set Theory," Lecture III, by Professor P. R. Halmos, University of Chicago.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Reading Room of the Students' Memorial Union of Queen's University. Twenty-five members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the following dates for future meetings of the Association: December 31, 1953, at Johns Hopkins University, Baltimore, Maryland; August 30–31, 1954, at the University of Wyoming, Laramie, Wyoming; December 1954, at a place not yet determined; August 29–30, 1955, at the University of Michigan, Ann Arbor, Michigan; December 1955, at Rice Institute, Houston, Texas.

until Saturday evening, September 5. Meals were served at Ban Righ Hall, the Royal Military College, and the Students' Union.

A welcoming reception was held on Monday evening in Wallace Hall of the Students' Union. A conducted tour of the city of Kingston and of Old Fort Henry was held on Tuesday afternoon. On Wednesday afternoon there was a choice between a motorboat trip from Gananoque among the Thousand Islands or a trip via a scenic route to the Sand Banks for swimming and a picnic supper. On Wednesday evening there was a showing of Canadian films in Grant Hall. A tea for the ladies was held on Thursday afternoon in Ban Righ Hall. On Thursday evening the members of the mathematical organizations were the guests of Queen's University and the Canadian Mathematical Congress at a theatre party. The International Players of Kingston presented the comedy "Goodbye Again" for the amusement of a large and enthusiastic audience.

At the theatre party, Professor J. W. Tukey presented a resolution of thanks on behalf of the visiting mathematicians, expressing gratitude for the felicitous arrangements and generous hospitality that marked this Kingston meeting as a successful and enjoyable one, long to be remembered by those present. Thanks were given to the administrations of Queen's University and of the Royal Military College, and especially to Professor Norman Miller as chairman of the Committee on Arrangements. The resolution also added an admiring tribute to the Canadian Mathematical Congress which has set such a brilliant record of achievements in its first eight years, and thanked the officers of the Congress for the important part they have had in holding this meeting in Canada.

HARRY M. GEHMAN, *Secretary-Treasurer*

REPORT TO THE ILLINOIS SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA OF ITS COMMITTEE ON THE STRENGTHENING OF MATHEMATICS TEACHING*

A. ANALYSIS OF THE PROBLEM AND STATEMENT OF PRINCIPLES

1. The nature of the problem and the task of the committee

The Problem as the Committee Sees It

On the basis of observation by its members and by others, the committee believes that there exists a powerful and dangerous trend against solid content in education, a trend which extends from the elementary schools through the high schools and which involves all subject-matter fields. It is important therefore to encourage teachers of mathematics, and of other subjects as well, to cooperate in a united effort to improve the quality of teaching and learning, and to improve the significance of the subject-matter taught in the schools.

* Presented to the Association at the Navy Pier, Chicago, Illinois, May 9, 1953.

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The Brighter Side of the Picture

The committee recognizes, of course, that many teachers, administrators, school board members and other citizens are in fact doing splendid work on behalf of good education, frequently in the face of serious difficulties. To these it pledges all the support and cooperation it can give. Indeed, it hopes that its conclusions and recommendations will give voice to many of their problems and many of their aspirations.

The Committee's Task

Apart from a recognition of the seriousness and the scope of the problem, some aspects of which are detailed below, and apart from a recognition of the strength of the forces working against a sound educational program, the committee believes that its function is not primarily to join with the mounting roll of critics, but rather that its obligation is to develop—and, so far as possible, to implement—concrete, workable proposals. In short, the committee believes that its function is to help begin the task of reconstruction.

2. Problems relating primarily to the teacher

The Influence of Early Training of the Prospective Teacher

The importance of good teaching of mathematics, beginning at the elementary level and continuing through the high school, is emphasized by the fact that students learning poor mathematical methods in school will some day make poor teachers unless teacher training programs correct these unfortunate early learnings. It is a familiar experience of those connected with such programs that this is often a well-nigh impossible task. It is also a matter of experience that the teachers whose early training in mathematics was faulty frequently prefer teaching by rote to emphasizing principles and understanding. They will naturally place as little emphasis as possible on the subject they dislike. Many of them may not realize either the importance or the difficulty of teaching mathematics well. In fact, they may even believe that they have no mathematics teaching problems.

Influence of Educators and Administrators

A problem complementary to the one just described is posed by the type of professional educator or public school administrator who declares that penmanship, or the alphabet, or grammar, or mathematics is not important in modern education. The influence of such educators is demoralizing to the already weak teacher and is discouraging to the good teacher who has to fight for his subject against those whose authority is greater. Their influence can only serve to promote inferior teaching or outright neglect of the more vital but also more difficult subject matters. Again, it is a matter of observation that the "core" and "life adjustment" programs have played into the hands of such individuals,

with the result that basic disciplines, such as mathematical training for example, are neglected or, in some cases, are omitted entirely. The consequence of such neglect is disastrous for many an able student and unless corrected, it will inevitably be disastrous for the nation. The influence of the educator and the administrator ought to be such as to eliminate vague and ill-defined aims and to encourage the thorough teaching of fundamentals. In this way and only in this way can the educators, the teachers and the schools best serve the interests of the students and of the nation.

The Professional Training of Teachers

In regard to the preparation of teachers, the committee believes that, on the undergraduate level, the segregation of prospective teachers into colleges of education, with a consequent emphasis on courses in "education," is a mistake. The prospective teacher, as much as other students if not more, needs a well-rounded, liberal education. Moreover, experienced elementary and secondary teachers should, we believe, play the maximum possible part in the professional training of the prospective teacher. We believe that specialization in "education" as a major subject should be reserved to the graduate levels.

Abdication of the Scholars

It must be admitted, however, that the subject matter fields have lost the prospective teachers to the colleges of education largely by default. In past decades we pretended, in our scholarly isolation, that a genuine concern with the problems and the preparation of the elementary and secondary teacher was beneath us. But it is one of the laws of society that when those who by rights should assume leadership refuse to accept the responsibility, others step into assume their tasks. In the current and mounting cry over the state of American education, one thing is conspicuous: there is too little self-examination. That many who do not comprehend the true nature and purpose of education are in the position of directing it is the result of the scholars' abdication of one of their most precious responsibilities. Now, when it is almost too late, we perceive the magnitude of the catastrophe.

Graduate Mathematics Courses for Teachers

The loss by default penetrates even into the graduate level, where there is a definite need for special, non-Ph.D. courses providing material that is interesting, broadening, helpful and especially selected for teachers. It is important that such courses be offered in summer sessions where they could be far more profitable to the returning teacher than are the too commonly repetitious courses offered by colleges of education, courses which are taken mainly because they can be passed satisfactorily by someone out of contact with the Ph.D.-type of graduate work. The experiment for college teachers beginning at Boulder, Colorado, this summer may well provide suggestions for similar programs for high

school and elementary teachers, programs which would enable them to meet certification requirements while learning something both inspirational and valuable.

Workshop for Teachers

The committee also believes that there is a need for more workshops for teachers at which teachers themselves would be present and discuss the problems and difficulties of teaching elementary and high school mathematics. Adequate provision should always be made for discussion, and the time allotted for this should not be usurped by the discussion leader, as is all too frequently the case. In large measure, these workshops should be small, localized meetings of short duration, located within easy access of the teachers for whom they are intended. Such meetings would provide an opportunity for beginning teachers to contact more experienced teachers for advice and encouragement. At the present time, many a promising young teacher abandons the profession because of discouragement during the initial years. The orientation and aid obtainable from other teachers at workshops such as are here proposed could serve to prevent many of these losses of teachers to other fields of activity. Indeed, this advisory function of experienced teachers ought to be organized into a systematic scheme for getting aid to the beginning teachers who need it.

An important part of such workshops could be concurrent meetings for high school students, designed to give them orientation and inspiration in mathematics.

These local workshops would of course be intended to supplement rather than to supplant the major meetings drawing on wider areas, such as are being held from time to time by various colleges and organizations.

Fellowships, Scholarships and Stipends

As a variation of the workshop plan, the committee believes that some foundation, or even state departments of education, would do well to finance a system of traveling fellowships for outstanding teachers, who would thus be enabled to visit many schools to offer advice and encouragement, and through broad observation and experience, to make concrete proposals for the improvement of the teaching situation as a whole.

Another method of improving the performance of teachers is through scholarships and stipends which aid underpaid members of the profession to avail themselves of the benefits of various special workshops and training programs in existence throughout the country. The *Mathematics Teacher* is to be commended for its policy of circulating information about such programs and stipends. Certainly the number of such stipends ought to be increased as a means of recognizing promise and of improving the standards of the profession. We believe that school boards should recognize this and, whenever possible, finance such opportunities for their teachers.

Salaries and Promotions

Finally, the committee recognizes the need for higher salaries in the teaching profession, salaries adequate to draw competent people to the profession and to keep them there. But it believes also that these salaries should be based on training in subject-matter fields and on demonstrated teaching competence, rather than on the completion of more courses in "education," as is frequently the case at present.

3. Problems relating primarily to the student

The Problem of Proper Guidance

Assuming competent instruction, the primary problem in the case of the student is one of proper guidance. We believe that much advice to the effect that mathematics is not necessary or desirable is based on the misconception that the subject is inherently difficult and dull for most students. However, the phenomenal success of recent mathematical TV programs, and the widespread and perennial interest in mathematical puzzles and riddles is proof of a greater latent interest in mathematics than is ordinarily suspected of the general run of mankind. This interest can be kept alive and developed in students only by good teaching and by proper emphasis on the importance of mathematics in human affairs. It is therefore a duty of the mathematics teacher and the administrator properly to inform students, patrons, and those responsible for guidance, of the vital significance of the subject.

The Importance of Mathematics in Modern Life

The fact of the matter is that in every area of scientific investigation today, as well as in many types of skilled labor, a certain body of mathematical skills is required for effective participation. Moreover, these requirements are rapidly increasing and will continue to do so as our civilization becomes increasingly technical. Every student and parent should know this, and moreover, every high school student should continue his mathematical training to the full extent of his ability. Whether or not he goes on to college—and this is something he may not be sure of himself—this training will serve him in good stead if it is sound material and is well taught. What the colleges, science, industry and above all the vocation of citizenship need and cry for today is the help of thinkers, not dillettantes or robots.

Aids to Proper Guidance

There are various aids available for a proper guidance program. There is a guidance bulletin prepared by the National Council of Teachers of Mathematics, there is the widely circulated bulletin on the mathematical needs of prospective engineering students prepared by the University of Illinois, there are circulars prepared by industry, such as General Electric's *Why Study Mathematics?*, and there are helpful articles in such journals as the *Mathematics Teacher* and the

American Mathematical Monthly. A pamphlet listing these aids and suggesting proper methods in mathematics guidance would be of great value if it could be circulated extensively at little cost among educators, patrons, students and those responsible for guidance.

Guidance Through Informal Lectures

The committee is aware that not all guidance is planned as such, but may rather be a byproduct of other experiences. As an example of this, the committee has had the opportunity to observe how effectively good popular lectures on mathematics can represent its importance in the educational process as well as in modern life and thought. We therefore urge mathematics teachers and interested laymen to utilize every opportunity to speak in behalf of the subject, whether it be to groups of students or to groups of patrons.

The Responsibility of the Teacher

No amount of proper guidance can solve the problem, of course, in the absence of effective and inspirational teaching. Many teachers have found that striking exhibits, properly used, will create interest in and convey the importance of mathematics. There is unlimited room for ingenuity here. Frequently topics commonly presented only in a formal way can be made both meaningful and inspirational by a few simple indications of how one would use these ideas in the factory, the laboratory, the market place, or in the understanding of nature. Ideas for enrichment materials of this kind are to be found in the professional periodicals, with which every teacher should be thoroughly familiar. Every teacher ought also to feel an obligation to share with other teachers good ideas which may not be widely known, through the medium of such periodicals. These observations point up to the extent to which the problem reverts to the classroom teacher. Sound material, inspirationally taught, is one of the best means of advertising our subject and represents the greatest contribution to good education we can possibly make.

4. Problems relating to schools and curricula

Correspondence Courses for High School Students

The committee feels a great concern for the high school student who needs and desires adequate mathematical training but whose school cannot or will not offer the necessary courses. The committee believes that genuine intellectual ability is so precious a quality that it ought never be allowed to suffer for lack of the opportunities necessary for proper development. We believe therefore that the universities should follow the lead of such schools as the University of Wisconsin in developing and advertising energetically a program of correspondence courses in those secondary school subjects which are frequently neglected.

One effect of advertising such a program would of course be such as to make students more fully aware of the importance of mathematics in their prepara-

tion. It would also make school boards more fully aware of their responsibility for providing proper curricular opportunities. In the long run, one would therefore expect such a program to be self-liquidating, which is as it should be.

The Problem of a Good Curriculum

It is a matter of observation that guidance, instruction and administration in certain high schools is of such a high calibre that the graduates of these schools do outstandingly well in employment requiring technical skills as well as in college work and beyond. These are schools in which thorough teaching of fundamentals has been emphasized. This fact alone is evidence that the critics of modern education are dealing with genuine issues. Moreover, it emphasizes the need for making teachers, school officials, and patrons aware of the statistical flaws in the now notorious *Eight-Year Study*, flaws which render its conclusions of little value. The fact of the matter is that as yet, nobody has demonstrated that a properly balanced program of subject-matter courses, with the emphasis on methods of creative and critical thinking, is not the best preparation for college work or for life.

The Problem of Maintaining High Standards

In view of the above observations the committee wishes to go on record as favoring the subject-matter type of curriculum as opposed to the core type of program, specific subject-matter requirements for high-school graduation and for college entrance, and mathematics for all those able to comprehend it. This does not mean that the committee is opposed to changes in content of courses, in methods of instruction, or in methods of evaluating achievement. It does mean that the committee is opposed to such changes as imply lowered standards of scholarship and accomplishment. Certainly educational research ought to enable us to improve these standards rather than to urge us to lower them. When it does the latter, it runs counter to the realities of our times, and is based on false premises, faulty observation or errors of logic. It is indeed not the peculiar whim of the professors, nor of the college administrators, that standards be kept high; it is the demand of life itself. The real life-adjustment is no retreat into a core of mediocrity, but rather an honest facing of the fact that knowledge and the wisdom to use it constructively are the essence of survival.

5. The problem of cooperation

There are certain inherent dangers involved in an attempt to implement convictions such as have been expressed here. For example, the scholars have so long ignored their responsibilities to the elementary and secondary teachers that a rift has developed between the elementary and secondary schools on the one hand and the colleges on the other. Great care must therefore be exercised to make certain that what are intended to be constructive efforts may not be interpreted as unwarranted criticism or interference.

Another danger is that the perils and needs of the times may not merely permit the overdue strengthening of mathematics and science in the elementary and secondary schools to take place, but may in fact result in an overemphasis on these subjects, to the neglect of the humanities. We have as much or more reason to protest the creation of scientific robots as to protest the neglect of scientific subjects in the curriculum.

As a matter of fact, the problem is more than just how much arithmetic, or how much history, the pupil should have. The issue is rather this: Shall we or shall we not give our children both the fundamental knowledge and the training in creative and critical thinking that they need for intelligent and effective participation in a democratic society? This is a problem of the utmost seriousness! The time has come when we must join forces at all levels and in all subject matter fields, for the battle to guarantee our children the educational opportunities they need and deserve is destined to be a long, hard one, and it is only by means of the highest degree of cooperation that it can be won.

B. ACTIVITIES OF THE COMMITTEE

In addition to drawing up the preceding analysis and statement of principles, the committee has listened to reports from the following members of the University of Illinois staff:

(a) Professor B. E. Meserve, who outlined the plans and progress of a committee at the University of Illinois which is developing experimentally at the University High School a four-year, unified high school mathematics program which will be terminal after two years for students not preparing for a technical career, but which will continue through all four years for students whose interests or plans demand a more adequate mathematical training;

(b) Professor S. S. Cairns, who described the interests of the Research Education Division of the National Science Foundation which resulted in its setting up this summer's program at Boulder, Colorado, and which may result in its cooperating with efforts to provide similar opportunities for high school teachers; Professor Cairns also discussed the concern of the Scientific Manpower Commission for good college programs, indicated its possible interest in secondary programs and reported on the planning session for the Allerton House Conference on Education;

(c) Professor R. E. Pingry, who outlined plans of the joint MAA-NCTM committee for the study of teacher training programs in mathematics; and

(d) Dean Stanley H. Pierce, who outlined plans of the Secondary Schools Committee of the American Society for Engineering Education for the study of means of closer articulation of high school and college programs in mathematics.

Besides informing itself in this way on the activities of other committees having similar concerns, with the object of instituting cooperation and avoiding duplication of effort wherever possible, the committee has conveyed its concerns to other groups and individuals, including the following:

(i) to the Illinois Council of Teachers of Mathematics, information concerning our purposes and a request for an expression of support

(ii) to the National Council of Teachers of Mathematics, lists of proposed subjects for yearbooks and pamphlets, and support of a proposal for a national high school mathematics publication

(iii) to Dr. Everett Welker a proposal that he prepare for publication his statistical criticism of the *Eight-Year Study*, given before this group last year

(iv) to the editors of the *Mathematics Teacher* and the *American Mathematical Monthly* a proposal that they consider publishing critical articles on the *Eight-Year Study*

(v) to the Provost of the University of Illinois, an offer to provide all possible assistance to the mathematical sub-committee helping to prepare for the Allerton House Conference on Education

(vi) to the other committees with concerns similar to ours, offers of all possible cooperation

(vii) to Senator Charles Clabaugh of the School Problems Commission, an expression of our conviction that school boards should be required by law to provide, at their expense and by correspondence if necessary, advanced high school mathematics courses for all students whose future plans require them, and to allot study time for such courses from the student's school day.

Replies to these communications so far received have been uniformly favorable.

C. RECOMMENDATIONS OF THE COMMITTEE

In order to implement further its conclusions, the committee recommends to the Illinois Section of the Mathematical Association of America the following actions:

(a) that the Illinois Section recommend to the Mathematical Association of America that it appoint a Committee on Cooperation with Industry to capitalize on industry's known willingness to invest money in advertising the need for good mathematical and scientific training at the high school level and in workshops for high school teachers, designed to inform them of ways in which mathematics is applied to practical problems

(b) that the Illinois Section and the Illinois Council of Teachers of Mathematics explore the possibility of cooperating in the organization in Illinois of an annual summer workshop for high school teachers

(c) that the Illinois Section recommend to the MAA that it offer the NCTM aid, support and cooperation in the proposed launching of a national high school mathematics periodical

(d) that the members of the Illinois Section interested in teacher training be urged to offer to prepare for the NCTM, pamphlets on such topics as "How to Organize a Workshop," "How to Make a Good Assignment,"

and others of the many worthy topics for which the NCTM has been unable to find authors

(e) that the Illinois Section urge its members to join the Illinois Council and the National Council of Teachers of Mathematics as an expression of support of efforts of these organizations to improve the teaching of secondary mathematics

(f) that the Illinois Section recommend to the Mathematics Department of the University of Illinois that it prepare and arrange for the distribution of a companion to the bulletin on the mathematical needs of prospective engineers, the new bulletin being designed to list the mathematical needs of students planning to enter fields of study other than engineering

(g) that the Committee for the Strengthening of Mathematics Teaching be made, officially as well as in effect, a joint endeavor of the Illinois Section and the Illinois Council

(h) that the Illinois Section consider a moderate increase in its dues in order to be able to defray reasonable expenses of such committees as it may appoint, and finally

(i) that the Illinois Section approve this report and authorize its circulation among concerned members of
  the several mathematical organizations
  the public and the press
  the various agencies of the University of Illinois
  the legislature and committees thereof and
  the United States Office of Education.

Respectfully submitted,

MARY ENTSMINGER
Laboratory School
Southern Illinois University
MARTHA HILDEBRANDT
Proviso Township High School
FRANZ E. HOHN (Chairman)
University of Illinois
ANICE SEYBOLD
North Central College
HENRY SWAIN
New Trier Township High School

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-second annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, Navy Pier, Chicago, Illinois, on Friday afternoon and Saturday forenoon, May 8 and 9, 1953. Professor H. G. Ayre, Chairman of the Section, presided at all sessions.

There were seventy-nine in attendance, including the following fifty-eight members of the Association:

Beulah M. Armstrong, H. G. Ayre, Imogene C. Beckemeyer, Winifred V. Berglund, A. H. Black, R. L. Blair, Angeline J. Brandt, B. K. Brown, J. R. Brown, S. S. Cairns, Laura E. Christman, John Christopher, H. S. Clair, Flora Dinkines, F. W. Donaldson, I. K. Feinstein, Evelyn Frank, A. E. Gault, G. D. Gore, L. M. Graves, A. E. Hallerberg, M. C. Hartley, E. W. Hellmich, E. H. C. Hildebrandt, Martha Hildebrandt, F. E. Hohn, Rose L. Hornacek, C. A. Jacokes, E. C. Kiefer, Rose Lariviere, A. O. Lindstrum, Jr., W. C. McDaniel, A. W. McGaughey, B. E. Meserve, R. J. Mihalek, E. B. Miller, M. G. Moore, Elsie C. Muller, H. E. Nelson, Grace M. Nolan, F. S. Nowlan, Margaret Olmsted, C. E. Olsen, T. B. Ondrak, Gordon Pall, J. W. Peters, R. E. Pingry, Ruth B. Rasmusen, Haim Reingold, T. E. Rine, L. A. Ringenberg, M. Anice Seybold, A. T. Street, B. R. Ullsvick, L. R. Van Deventer, Arnold Wendt, Charles Craig Wilson, Alice K. Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Professor M. C. Hartley, University of Illinois, Navy Pier, Chicago; Vice-Chairman, Professor Rothwell Stephens, Knox College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. The Secretary-Treasurer's report was read and accepted. Then the retiring Sectional Governor reported on the work of the Board of Governors for this year.

The following program was presented:

1. *Content and function of pre-calculus mathematics for the preparation of secondary teachers*, by Professor B. R. Ullsvik, Illinois State Normal University.

The revision of content only will not improve pre-calculus mathematics for purposes of teacher education because of the importance of the teacher in inspiring students both for learning and for becoming good teachers. Accepting that "Any program of training teachers must be tied into what they teach," the undergraduate courses in mathematics should prepare for the teaching of high-school "general mathematics," as well as the traditional courses in algebra and geometry.

The present trends of student enrollments indicate that an increase in numbers can be expected and no appreciable change in the quality of students can be hoped for. The typical pre-calculus courses do not adequately prepare for the teaching of Euclidean geometry, and a review of high-school geometry would be more appropriate than merely more theorems in college geometry. There is much need for further investigation concerning the value of integration of pre-calculus courses rather than teaching compartmentalized college algebra, analytics and college geometry.

2. *Contribution of related areas to the preparation of teachers of secondary mathematics*, by Professor A. W. McGaughey, Bradley University.

The paper called attention to some studies which had been made in this area, pointing out that secondary school principals, teachers of mathematics on the secondary level and university professors all recommended that the candidate for a teaching certificate in secondary mathematics should obtain as broad a cultural background as possible, including economics, sociology and ethics as well as the physical sciences. The university professors in their report in the May, 1935 issue of this MONTHLY recommended that these courses be taken in preference to advanced work in mathematics or in theory of education beyond the legal requirements.

The author believes that the secondary teacher needs many illustrative examples during his beginning years and should be helped in gathering them. He suggested the possibility that this Association might sponsor such an endeavor.

3. *What the teacher training institutions can do to improve the teaching of secondary mathematics*, by Professor R. E. Pingry, University of Illinois.

Despite the high probability for a critical shortage of qualified mathematics teachers in secondary schools in the next ten years, colleges should continue to hold high standards for teacher education programs. Colleges should develop better selection procedures so poor prospects for teaching will not be certified. The mathematics education of teachers should be evaluated, and improved student teaching programs developed. More attention should be given to the problem of teaching junior high school mathematics and to developing in-service training programs for mathematics teachers who are already teaching.

4. *An adequate master's degree program for teachers of secondary mathematics*, by Professor E. H. C. Hildebrandt, Northwestern University.

The program is based on the questions most frequently raised by teachers with secondary school teaching experience. Requests include advanced work in mathematics related to the content of the secondary courses including a better understanding of the foundations of the calculus, an introduction to the fields of higher mathematics, and a systematic approach to problem-solving as required by the mathematics student, engineer, scientist and industrial mathematician. Problems of the mathematics curriculum and better teaching techniques in mathematics also require further study. Finally, are these courses alone adequate? Can the written thesis actually provide for additional training needed in this profession?

5. Address at dinner meeting: *Abstractions*, by Professor S. S. Cairns, University of Illinois.

The speaker planned an entertaining, non-technical talk that would interest all attending the dinner by comparing some general abstractions with some encountered in mathematics.

6. *What the high school principals and superintendents are doing to strengthen mathematics programs in the schools*, by Dr. N. E. Watson, Northfield Township High School, Northbrook, introduced by the Secretary.

Dr. Watson stressed the fact that mathematics is a language and that it must so be taught. He questioned the preparation of mathematics teachers and their ability to present the subject so that pupils could see its importance and functional aspects. He stressed the development of new and better instructional materials, different attitudes toward the place of mathematics among the other courses offered and a sense of humor in its presentation. He urged Master's degrees in mathematics for teachers of the subject.

He urged more mathematics, rather than less, for high school pupils but at their level of need and understanding. He said high schools are doing much counseling after testing programs in mathematics. He urged better courses, materials and points of view. He urged all mathematics teachers to consider themselves teachers of the language of mathematics.

7. *Report of the committee on the strengthening of mathematics in secondary schools*, by Professor F. E. Hohn and Committee.

This report is published in full in this issue, page 652.

E. C. KIEFER, *Secretary.*

6. *On the formulation of the definition of determinant for linear mappings*, by Professor H. B. Ribeiro, University of Nebraska.

7. *Remarks on teaching of geometry*, by Professor N. A. Court, University of Oklahoma.

EDWIN HALFAR, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The May meeting of the Wisconsin Section of the Mathematical Association of America was held at Mount Mary College, Milwaukee, Wisconsin, on May 2, 1953. Mr. A. C. Moeller presided in the absence of the chairman, Professor R. H. Bing.

Ninety-two persons attended the meeting, including the following thirty-eight members of the Association:

R. H. Bardell, E. H. Batho, Leon Battig, L. J. Berner, W. W. Bigelow, C. C. Braunschweiger, Leonard Bristow, R. C. Buck, E. G. H. Comfort, Rev. L. A. V. DeCleene, W. E. Deskins, H. P. Evans, F. R. Harding, R. C. Huffer, Rev. M. L. Jautz, J. F. Kenney, K. L. Leverance, A. P. Loomer, C. C. MacDuffee, Morris Marden, A. E. May, Genevieve L. Meyer, A. C. Moeller, Marianne S. Otto, O. E. Overn, G. A. Parkinson, H. P. Pettit, J. G. Renno, Jr., R. E. Schwartz, Sister M. Elizabeth, Sister Mary Felice, Sister Mary Petronia, Abraham Spitzbart, J. V. Talacko, W. J. Thomsen, R. D. Wagner, L. F. Wahlstrom, Louise A. Wolf.

The following officers were elected for the year 1953–54: Chairman, Dr. L. F. Wahlstrom, State Teachers College, Eau Claire; Secretary-Treasurer, Sister Mary Felice, Mount Mary College; Chairman of the Program Committee, Professor Abraham Spitzbart.

The following papers were presented:

1. *Geometry and modern algebra*, by Dr. C. W. Curtis, University of Wisconsin, introduced by Professor R. H. Bing.

The familiar concepts of algebraic curves and surfaces lead in a natural way to the definition of an algebraic manifold in n-dimensional space over a given field K. It was shown that geometrical questions concerning algebraic manifolds could be replaced by algebraic questions concerning ideals in the ring $K(x_1, \cdots, x_n)$ of polynomials in n variables with coefficients in K; in particular the purely geometric concept of an irreducible algebraic manifold can be replaced by the concept of a prime ideal in $K(x_1, \cdots, x_n)$. It was indicated how this transition can be accomplished by making use of two fundamental results first proved by Hilbert: the basis theorem for polynomial ideals, and the zero theorem.

2. *Variations on a theme*, by Professor H. P. Pettit, Marquette University.

Starting with a vector space, we assume the norm as not necessarily a positive definite quadratic form. Then the Schwarz inequality may not hold. The minimal cone, found by equating the norm to zero, may be real. A surface with constant radius turns out to be a quadric asymptotic to the minimal cone. Two vectors are called normal to each other if they are conjugate relative to the minimal cone. Using vector methods, one easily shows that the properties of perpendiculars in ordinary geometry have analogous theorems in this system, some of which were new to the writer. This variation from standard procedure gives the vector methods added power in the study of geometry.

6. *On the formulation of the definition of determinant for linear mappings*, by Professor H. B. Ribeiro, University of Nebraska.

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Mr. Richardson discussed the educational and experience requirements for the Certified Public Accountant's certificate. These include mathematics at the high school level and a commerce course at a university with an accounting major. The importance of arithmetic, algebra, geometry and trigometry in the training program should be stressed, while many other mathematical subjects provide important background and increase the analytical power of the candidate. An accountant must be accurate and quick and literally live with figures. A sure road to success in this profession is the use of mental facilities rather than frequent recourse to automatic calculators.

SISTER MARY FELICE, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The seventh annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Montana State University, Missoula, Montana, on June 19, 1953, in conjunction with the four hundred ninety-third meeting of the American Mathematical Society. Professor R. A. Rosenbaum, Chairman of the Section, invited Professors Harold Chatland, R. D. James, and S. G. Hacker to preside at the afternoon session.

Fifty-seven persons were in attendance, including the following forty-three members of the Association:

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The afternoon session consisted of the following invited hour address, three twenty-minute papers, two ten-minute papers, and a symposium by a panel of three speakers:

1. *The distribution of quadratic residues*, by Professor Leo Moser, University of Alberta. (By invitation.)

This paper was a historical outline of the theory of distribution of quadratic residues. After presenting the contributions of Gauss and Dirichlet, three main types of theorems were discussed. The first concerned existence and frequency of blocks of residues and of non-residues. Here the results of Jacobsthal, Bennet, Davenport, A. Brauer, and Perron were presented. Secondly, results were given showing that the least non-residue is small, but sometimes not very small. These results are due to Nagell, Vinogradoff, Brauer, Pillai, Ankeny and others. Finally, it was shown that in certain small ranges of integers, at least a certain fixed fraction of the numbers are residues and at least an equal fraction are non-residues. These results were obtained by Vandiver, Redei, and the author.

Mr. Richardson discussed the educational and experience requirements for the Certified Public Accountant's certificate. These include mathematics at the high school level and a commerce course at a university with an accounting major. The importance of arithmetic, algebra, geometry and trigometry in the training program should be stressed, while many other mathematical subjects provide important background and increase the analytical power of the candidate. An accountant must be accurate and quick and literally live with figures. A sure road to success in this profession is the use of mental facilities rather than frequent recourse to automatic calculators.

SISTER MARY FELICE, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

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2. *A risk function approach to the problem of tolerance limits*, by Professor Bernard Ostle, Montana State College, introduced by the Secretary.

Given a function $f(x)\geqq 0$ for $-\infty\leqq x\leqq\infty$ such that $\int_{-\infty}^{\infty}f(x)dx=1$, defining $F(x)=\int_{-\infty}^{x}f(x)dx$, and having a random sample of n values from the population specified by $f(x)$, the problem is to find two functions of the sample values, say $L_1(x_1, x_2, \cdots, x_n)$ and $L_2(x_1, \cdots, x_n)$ such that $P\{F(L_2)-F(L_1)\geqq\beta\}=\alpha$ where $0\leqq\alpha\leqq 1$ and $0\leqq\beta\leqq 1$. The question of most interest is to determine the minimal value of n which will satisfy the equation $P\{F(L_2)-F(L_1)\geqq\beta\}=\alpha$ once the forms of L_1 and L_2 have been decided upon. One approach is to consider a weight function $W(L_1, L_2)$ and a cost function $C(n)$, and to determine n so as to minimize the risk involved, that is to minimize

$$r(n)=\int\cdots\int W(L_1,\ L_2)\prod_{i=1}^{n}f(x_i)dx_i+C(n).$$

This has been done for certain choices of L_1, L_2 and W, and the results have been compared with solutions previously given.

3. *Potential theory*, by Professor M. G. Arsove, University of Washington.

From its inception at the end of the 18th century in the work of Legendre, Laplace, Gauss, Green, Dirichlet, and others, potential theory has been a powerful tool in the study of field problems of mathematical physics. Moreover, it has served as an important stimulus in the development of many branches of mathematical analysis. Classical potential theory had evolved so satisfactorily, in fact, that by the early 1900's its interest as a field of mathematical research had begun to wane. However, the advent of general measure theory and the pioneering work of F. Riesz in subharmonic functions were responsible for a renaissance of potential theory, this time as an instrument of pure mathematical research and with significant applications to complex function theory.

4. *The problem of difference sets*, by Professor T. G. Ostrom, Montana State University.

In this paper, the author traces the history of the difference set problem up to the most recent developments. A set of integers $\{a_0, a_1, \cdots, a_n\}$ is said to be a difference set mod N if the set of differences $\{a_i-a_j\}$ contains each non-zero residue mod N exactly once. The existence of a difference set mod N is equivalent to the existence of a cyclic projective plane with $n+1$ points on a line and a total of N points. In all known finite projective planes with $n+1$ points on a line, n is a prime power. The difference set problem is part of the more general problem for finite projective planes: "Must n be a prime power?"

5. *Mathematics in a women's college*, by Professor Andrewa R. Noble, Mills College.

In this paper the author discusses the courses, textbooks, requirements for the major, quality of students, and placement of graduates of the mathematics department at one college for women, Mills College, to show that the program differs little from that given in a coeducational institution of comparable size. The quality of students enrolled in mathematics classes, the size of classes, and light teaching loads are favorable factors at Mills College.

6. *The Math-Science Day at the University of Washington*, by Professor R. B. Leipnik, University of Washington.

A Spring Saturday in 1952 saw Math-Physics Day introduced on the University of Washington campus. About 1200 high-school students and 125 teachers attended from 70 schools. Though most "delegates" came from near Seattle, 20% came from over 100 miles. For students: job talks,

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 5, 1953.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, April, 1954.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1954.

NEBRASKA

NORTHERN CALIFORNIA

OHIO, April, 1954.

OKLAHOMA

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April, 1954.

SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.

SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.

SOUTHWESTERN, Arizona State College, Tempe.

TEXAS, Texas Technological College, Lubbock, April, 1954.

UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

WISCONSIN, State Teachers College, Eau Claire, May, 1954.

EMPLOYMENT OPPORTUNITIES

Beginning with the February 1954 issue, the MONTHLY will devote this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor, (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Washington) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, New York. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

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CLASSROOM NOTES

Edited by G. B. THOMAS, Massachusetts Institute of Technology

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The following persons presented papers at meetings of the Association and its Sections.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 10

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ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

NOTE ON THE SHAPE OF LEVEL CURVES OF GREEN'S FUNCTION

J. L. WALSH, Harvard University

The object of this note is to indicate some easily proved but geometrically striking properties of the level curves of Green's function. These properties, like convexity and star-shapedness, if possessed by the boundary of a given region, are possessed also by all the level curves; the properties serve to measure the relative nearness to circular shape of the boundary or a level curve. It turns out that in a precise manner the level curves become more nearly circular as they approach the pole of Green's function.

If O is the origin and Γ a smooth arc, the *vector-tangent angle* ψ of Γ at a point P of Γ is defined as the directed angle from the radius vector OP extended beyond P, to the tangent to Γ at P in the positive sense of the curve. The arc Γ is an arc of a circle with center O if and only if we have $\psi \equiv \pi/2$; the nearness of ψ to $\pi/2$ is an indication of the nearness of Γ to circular shape, and if Γ is closed may be measured either by $[\max \psi - \min \psi]$ or by $\max |\psi - \pi/2|$.

The angle $\psi - \pi/2$ has been used recently by P. Davis and Henry Pollak* as a measure of nearness to circularity of Γ *at the point* P, thus considered as a function of both Γ and P. These writers establish part (c) of Theorem 1, which is merely a rephrasing of an inequality due to Grunsky,† and establish a weaker form of Theorem 3, namely a form restricted to bounded simply connected regions. They also point out that the level curves become nearly circular as they approach the pole of Green's function, but do not mention the monotonic character of this variation.

THEOREM 1. *Let the simply connected region R of the $w(=u+iv)$-plane contain $w=0$ but not $w=\infty$, and let $G(u, v)$ be Green's function for R with pole in O. Let $C_r(0<r<1)$ denote the level locus $G(u, v) = -\log r$ in R.* (a) *If the boundary of R is a smooth Jordan curve C_1 for which the vector-tangent angle ψ satisfies the inequalities $\psi_0 \leqq \psi \leqq \psi_1$, the same inequalities hold for the vector-tangent angle of every C_r. Except in the trivial case that R is a circle with center O, we have with $r_1 < r_2 \leqq 1$,* $\max$ [*ψ on C_{r_1}*] $<\max$ [*ψ on C_{r_2}*], $\min$ [*ψ on C_{r_1}*] $>\min$ [*ψ on C_{r_2}*]. (b) *If R is star-shaped, then for the level locus C_r we have*

$$\left| \psi - \frac{\pi}{2} \right| \leqq \tan^{-1} \frac{2r}{1 - r^2} \cdot \tag{1}$$

(c) *If R is arbitrary, for C_r we have*

$$\left| \psi - \frac{\pi}{2} \right| \leqq \log \frac{1 + r}{1 - r} \cdot \tag{2}$$

* Trans. Amer. Math. Soc. vol. 72, 1952, pp. 82–103; Theorem 10 and Corollary 10.1.

† H. Grunsky, Jahresber, d. d. Math. Vereinigung, vol. 43, 1933, pp. 140–143; see also G. M. Golusin, Mat. Sbornik, vol. 1 (43), 1936, pp. 127–135.

Of course ψ is properly not single valued, but for each curve such as C_r we choose $0 \leqq \psi \leqq \pi$ at the first intersection of the prime vector with the curve, and determine ψ at other points of the curve by continuity.

If the function $w=f(z)\equiv z+a_2z^2+\cdots$ maps conformally $|z|<1$ onto R, C_r is the image (Kreisbild) of $|z|=r<1$. It is no loss of generality here to assume $f'(0)=1$. The inverse of $w=f(z)$ is $z=F(w)\equiv e^{-G(u,v)-iH(u,v)}$, where $H(u, v)$ is conjugate to $G(u, v)$ in R, so the circle $|z|=r$ corresponds to C_r: $|F(w)|\equiv e^{-G(u,v)}=r$. We have at any point of C_r, in the notation $z=re^{i\theta}$, $dz=izd\theta$, $dw=f'(z)dz$,

$$\psi = \arg dw - \arg w = \arg\left[\frac{izf'(z)}{f(z)}\right] = \frac{\pi}{2} + \operatorname{Im}\left[\log\frac{zf'(z)}{f(z)}\right]; \tag{3}$$

this last member is a function harmonic in $|z|<1$ even at $z=0$ (when suitably defined there), by virtue of $f'(z)\neq 0, f(z)/z\neq 0$, and under the hypothesis of (a) is continuous in $|z|\leqq 1$. By the principle of maximum for a harmonic function applied to the region $|z|<1$ we have

$$\max\ [\psi \text{ on } C_r] < \max\ [\psi \text{ on } C_1],$$
$$\min\ [\psi \text{ on } C_r] > \min\ [\psi \text{ on } C_1],$$

except that in the trivial case $zf'(z)/f(z)\equiv 1$ (which implies $f(z)\equiv z$), these inequalities are replaced by equalities. With $r_1<r_2$, the curves C_{r_1} and C_{r_2} are the images of the circles $|z|=r_1/r_2$ and $|z|=1$ under the transformation $w=f(r_2z)$, so (a) is established.

Let now R be star-shaped; if the boundary C_1 is a smooth Jordan curve we have on C_1 the inequality $0\leqq\psi\leqq\pi$, and hence by (a) the inequality $0<\psi<\pi$ holds on each C_r; if C_1 is not a smooth Jordan curve, it is nevertheless well known that each $C_{r'}(r'<1)$ is star-shaped, so each C_r lies in a star-shaped $C_{r'}(0<r<r'<1)$ and again by (a) we have $0<\psi<\pi$ on C_r. From (3) we may therefore write $(|z|<1)$

$$U(z) \equiv \operatorname{Im}\left[\log\frac{zf'(z)}{f(z)}\right], \qquad -\frac{\pi}{2} < U(z) < \frac{\pi}{2}, \qquad U(0) = 0. \tag{4}$$

We need to make use of the

LEMMA. *If the function $U(z)$ is harmonic and in modulus less than $\pi/2$ in the circle $|z|<1$, with $U(0)=0$, then in the circle $|z|\leqq r(<1)$ we have*

$$|U(z)| \leqq \tan^{-1}\frac{2r}{1-r^2}.$$

This lemma is proved with the aid of Schwarz's Lemma, by means of the function

THEOREM 2. *Let R be an infinite region whose boundary B is finite, and let $G(u, v)$ be Green's function for R with pole at infinity. If the circle C with center O and radius ρ_0 contains B, then on any arc of a level curve exterior to a concentric circle of radius $\rho(>\rho_0)$ we have*

$$\left|\psi - \frac{\pi}{2}\right| \leqq \sin^{-1}\frac{\rho_0}{\rho}\cdot$$

Theorem 2 follows at once from the fact* that the normal to a level curve must cut C. By inversion we have

THEOREM 3. *Let R be a region not necessarily finite with boundary B, and let $G(u, v)$ be Green's function for R with pole in the finite point O. If B lies exterior to a circle C with center O and radius ρ_0, then on any arc of a level curve interior to a concentric circle of radius $\rho(\rho<\rho_0)$ we have*

$$\left|\psi - \frac{\pi}{2}\right| \leqq \sin^{-1}\frac{\rho}{\rho_0}\cdot$$

The results of the present note indicate relative nearness to circular shape of the level curve C_r as measured globally but in terms of infinitesimal properties. A measure referring to global behavior without infinitesimal properties can be defined as

$$\frac{\min\ |w| \text{ on } C_r}{\max\ |w| \text{ on } C_r},$$

a number which is not greater than unity, and which can be shown to increase monotonically as r decreases and to approach unity as r approaches zero.†

THE GENERAL ELECTRIC MATHEMATICS FELLOWSHIP PROGRAM

E. B. ALLEN, Rensselaer Polytechnic Institute

A cooperative educational program involving industry and an educational institution is always an important event and deserves widespread publicity. When the program concerns the field of mathematics, it becomes notable as illustrating the great need for mathematics in the industries of today. Consequently, a brief description of the General Electric Mathematics Fellowship Program, as administered by Rensselaer Polytechnic Institute and the Educational Services Division of the General Electric Company, should be of interest to many engaged in mathematical work.

* Walsh, this MONTHLY, vol. 42, 1935, pp. 1–17.

† Walsh, this MONTHLY, vol. 44, 1937, pp. 202–213.

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For several years, the General Electric Company has sponsored two programs in science, each for fifty secondary school teachers, at Union College in Schenectady and at Case Institute of Technology in Cleveland. The success of these two programs led the General Electric Company to consider sponsoring a similar program for secondary school teachers of mathematics. The feasibility of attempting a mathematics program was discussed with officials at Rensselaer Polytechnic Institute and, as a result, the program was initiated with the Head of the Department of Mathematics at Rensselaer as Director.

The General Electric Mathematics Fellowship Program provides for six weeks of summer instruction for fifty secondary school teachers of mathematics from the New England States, New York, New Jersey, Pennsylvania, Delaware, Maryland, Virginia, and the District of Columbia. The Fellowships pay the tuition, board, room, supplies, and round trip fare to Troy from the teaching location of each Fellow.

The aims of the program in mathematics are: (1) to stimulate enthusiasm for the study of mathematics on the part of teachers and, through them, on the part of students and officers of public instruction, (2) to familiarize teachers with some of the important applications of mathematics, (3) to furnish teachers with information which may be of assistance in the guidance of their students, (4) to foster growth in the mathematical knowledge and background of the participants, and (5) to aid these teachers in making their secondary school mathematics courses of vital interest to students.

With these aims in view, the following three courses were specially designed for the General Electric Mathematics Fellows.

S11.01—*Applied Mathematical Analysis.* This course met six times a week. It included the following material: statistics (median, mode, standard deviation, probability, binomial and normal distribution, *etc.*), electricity and magnetism (fundamental definitions, electrostatics, electromagnetism, direct and alternating current circuits, machines), mechanics (fundamental definitions, Newton's Laws, applications).

S11.02—*Topics in Elementary Mathematics from an Advanced Viewpoint.* This course met six times a week. It contained a review of derivatives and the definite integral; trigonometric, exponential, and logarithmic functions (series treatment); elements of projective geometry; the number system.

S11.03—*Mathematics at Work.* This course consisted of two 2-hour laboratory periods a week and, in general, two trips a week to industrial plants and laboratories.

Nine laboratory exercises, to illustrate material from courses S11.01 and S11.02, were scheduled. They included such items as statistics, experiments in electricity, tabular and graphical methods for finding derivatives, mechanical methods for areas and integrals, calculating machines, special coordinate papers, rotating vectors, significant figures.

Eight trips to industrial plants and laboratories comprised the second part of this course. Visits were made to various laboratories of the General Electric

Company and to W. & L. E. Gurley, Troy, maker of surveying and mathematical instruments.

Further educational material was presented by a series of talks on insurance, employment opportunities in science and engineering, and the like; by visits to the Rensselaer computing laboratory and the observatory; and by showing educational films and the General Electric House of Magic.

It will be noted that the program had considerable content. This was a natural result of the decision to have the courses carry graduate credit towards the M.S. degree. This decision was based on the following facts: (1) secondary school teachers in many states need to take courses (or do other work) in order to retain their teaching certificates, (2) it was thought that courses of graduate caliber would be of considerable benefit to those who might have majored in education, (3) it was hoped that some of the Fellows might be encouraged to become candidates for a master's degree.

Applications for the fifty Fellowships were secured by sending announcements of the program to principals and heads of mathematics departments of secondary schools situated in the territory assigned to Rensselaer. These announcements described the program and contained a request that the Fellowships be brought to the attention of all teachers of mathematics. As a result of this coverage, nearly 300 completed applications were submitted.

The selection of individuals to whom Fellowships were to be awarded was made by a committee composed entirely of Rensselaer personnel. The geographical distribution of the Fellows for 1952 is shown herewith: Maine 2; New Hampshire, 2; Vermont, 2; Massachusetts, 8; Connecticut, 2; Rhode Island, 1; New York, 13; New Jersey, 5; Pennsylvania, 8; Delaware, 1; Maryland, 1; Virginia, 3; District of Columbia, 2; total, 50. Of these fifty Fellows, twenty-six were men and twenty-four were women.

The Fellows lived in units of the Institute's dormitories and dined together in one of the dining halls. These arrangements fostered discussions, not only of the program but also of methods and problems of secondary school teaching.

A number of social activities were provided for the Fellows by the Institute and by the General Electric Company. The Fellows also organized many special social events of their own.

In retrospect, it appears that the program was very much worth while. The courses were somewhat difficult, the trips were demanding, and the extracurricular activities were strenuous. In spite of this, there is general agreement that much was accomplished. Several Fellows used the summer experience to aid them in their regular teaching positions, several expressed a determination to do more advanced work in mathematics, several found ideas and applications useful in their regular teaching, and several found their standing enhanced by their holding a Fellowship. The results were sufficiently encouraging for the General Electric Company to decide to start a similar mathematics program at Purdue University in the summer of 1953. This program was available to to teachers in eleven states west of the territory served by Rensselaer.

ON THE UNIQUENESS OF THE DISTRIBUTION FUNCTION FOR THE BUFFON NEEDLE PROBLEM

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In the well known Buffon needle problem, a needle of length L is dropped on a board ruled with equidistant parallel lines of spacing D where $D \geqq L$; it is required to determine the probability that the needle will intersect one of the lines.

The needle position is determined by the distance, x, of its middle point from the nearest line, and by the acute angle, ϕ, between this perpendicular distance and the needle. In one solution of this problem [1], the variables x and ϕ are considered independent, and as a hypothesis, the distribution of probability for x and ϕ is assumed to be uniform. Since the needle will intersect one line if

$$x \leqq \frac{L}{2} \cos \phi,$$

the probability, P, that the needle intersects one of the lines is given by

$$P = \frac{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}L \cos \phi} dx d\phi}{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}D} dx d\phi}.$$

It follows, therefore, that

$$P = \frac{2L}{\pi D}.$$

Since experiments [2, 3] made agree well with this result, it is stated that this indirectly confirms the hypothesis of a uniform distribution in x and ϕ.

Using a very elegant method, and making no assumptions as to the distribution function, Barbier [1], also obtains $P = 2L/\pi D$.

It is the purpose of this paper to consider the distribution functions which lead to this probability.

We are therefore interested in solutions of the integral equation

$$\frac{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}L \cos \phi} F(x, \phi) dx d\phi}{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}D} F(x, \phi) dx d\phi} = \frac{2L}{\pi D}.$$

The solution of this integral equation is not unique, for there are infinitely

many functions of the form $F(x, \phi) = A + G(\phi)$ which satisfy the integral equation, where $G(\phi)$ satisfies

$$\int_0^{\frac{1}{2}\pi} G(\phi)d\phi = 0 \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} G(\phi) \cos \phi d\phi = 0.$$

One simple example is

$$G(\phi) = B[\cos 2\phi + 2 \cos 4\phi].$$

It is clear that the distribution function is not unique.

We will now consider the effects on the uniqueness of the distribution function if x and ϕ are assumed to be independent variables. Then, $F(x, \phi) = G(\phi)H(x)$, and

$$\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}L\cos\phi} G(\phi)H(x)dxd\phi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}D} G(\phi)H(x)dxd\phi} = \frac{2L}{\pi D}.$$

Thus

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}D} G(\phi)H(x)dxd\phi = KD.$$

Differentiating with respect to D yields

$$H(\tfrac{1}{2}D)\int_0^{\frac{1}{2}\pi} G(\phi)d\phi = K.$$

Hence $H(x) = \text{constant}$, and the distribution in x is uniform. Then for $G(\phi)$ we must have

$$\frac{\int_0^{\frac{1}{2}\pi} G(\phi) \cos \phi d\phi}{\int_0^{\frac{1}{2}\pi} G(\phi)d\phi} = \frac{2}{\pi}.$$

The solution of $G(\phi)$ is not unique, as was shown previously:

$$[i.e., G(\phi) = B + C[\cos 2\phi + 2 \cos 4\phi]].$$

Consequently, we cannot assume a uniform distribution on the basis that it gives the desired probability. However, if we take a slightly more involved version of the Buffon problem it will be shown that there is only one distribution function which leads to the probability one would obtain if a uniform distribution were assumed.

Let us now consider the needle problem with $L \geqq D$, instead of $L \leqq D$. Define

Here,

$$A = \int_0^{\frac{1}{2}\pi} G(\phi)d\phi.$$

Hence,

$$\frac{\int_0^{\phi_0} G(\phi)d\phi}{\int_0^{\frac{1}{2}\pi} G(\phi)d\phi} = \frac{2\phi_0}{\pi} + \frac{2(n-1)}{\pi}\,\frac{1-\sin\phi_0}{\cos\phi_0}.$$

Letting $\phi_0=0$, [*i.e.*, $D=L$], we get $n-1=0$. Thus $G(\phi)=$constant, and the distribution is uniform in X and ϕ.

References

1. J. V. Uspensky, Introduction to Mathematical Probability, McGraw-Hill, 1937, p. 251 and p. 254.
2. G. Castelnuovo, Calcolo delle Probabilità, vol. 1, p. 183.
3. A. DeMorgan. A Budget of Paradoxes, vol. 1, p. 282.

SOME PROPERTIES OF FIBONACCI NUMBERS

K. SUBBA RAO, Maharajah's College, Vizianagram, India

In this paper, I consider the series of Fibonacci 1, 2, 3, 5, 8, 13, $\cdots$, the law of formation of whose terms is that any term (from the third onwards) is equal to the sum of the two preceding terms. I denote the terms of this series by

$$U_1, U_2, U_3, U_4, \cdots$$

and I define $U_0=1$. I call the U's 'Fibonacci numbers.' Thus any three consecutive Fibonacci numbers are connected by the relation $U_n=U_{n-1}+U_{n-2}$.

Among the several known results concerning Fibonacci numbers, I quote below some interesting ones:

I $\qquad U_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

II $\qquad U_{2n} = U_n^2 + U_{n-1}^2$

III $\qquad U_{3n+2} = U_{n+1}^3 + U_n^3 - U_{n-1}^3$

Here,

$$A = \int_0^{\frac{1}{2}\pi} G(\phi)d\phi.$$

Hence,

$$\frac{\int_0^{\phi_0} G(\phi)d\phi}{\int_0^{\frac{1}{2}\pi} G(\phi)d\phi} = \frac{2\phi_0}{\pi} + \frac{2(n-1)}{\pi}\,\frac{1-\sin\phi_0}{\cos\phi_0}.$$

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II $$U_{2n} = U_n^2 + U_{n-1}^2$$

III $$U_{3n+2} = U_{n+1}^3 + U_n^3 - U_{n-1}^3$$

IV $\quad U_n^2 - U_{n-p}U_{n+p} = (-1)^{n-p+1}U_{p-1}^2$

V $\quad U_1 + U_2 + U_3 + \cdots + U_n = U_{n+2} - 2$

VI $\quad U_1 - U_2 + U_3 - \cdots + (-1)^{n-1}U_n = (-1)^{n-1}U_{n-1}.$

I now list below some results which can all be proved by the principle of mathematical induction:

$$U_1 + U_3 + U_5 + \cdots + U_{2n-1} = U_{2n} - 1$$
$$U_2 + U_4 + U_6 + \cdots + U_{2n} = U_{2n+1} - 1$$
$$U_1 + U_4 + U_7 + \cdots + U_{3n-2} = \tfrac{1}{2}(U_{3n} - 1)$$
$$U_2 + U_5 + U_8 + \cdots + U_{3n-1} = \tfrac{1}{2}(U_{3n+1} - 1)$$
$$U_3 + U_6 + U_9 + \cdots + U_{3n} = \tfrac{1}{2}U_{3n+2} - 1$$
$$U_{2n+1} = U_{n+1}^2 - U_{n-1}^2$$
$$U_1 + U_5 + U_9 + \cdots + U_{4n-3} = U_{2n-1}^2$$
$$U_3 + U_7 + U_{11} + \cdots + U_{4n-1} = U_{2n}^2 - 1$$
$$U_2 + U_6 + U_{10} + \cdots + U_{4n-2} = \tfrac{1}{2}(U_{4n-1} + U_{2n-1}^2)$$
$$U_4 + U_8 + U_{12} + \cdots + U_{4n} = \tfrac{1}{2}(U_{4n+3} - U_{2n+1}^2 - 2)$$
$$U_nU_{n+2} = U_{n+1}^2 + (-1)^n$$
$$U_1^2 + U_2^2 + U_3^2 + \cdots + U_n^2 = \tfrac{1}{2}(U_{2n+3} - U_{n+1}^2 - 2)$$
$$= U_nU_{n+1} - 1$$
$$U_2^2 + U_4^2 + U_6^2 + \cdots + U_{2n}^2 = \tfrac{1}{5}(3U_{2n+1}^2 + 2U_{2n}^2 - 4U_{2n-1}U_{2n+1} + 2n - 5)$$
$$U_1^2 + U_3^2 + U_5^2 + \cdots + U_{2n-1}^2 = \tfrac{1}{5}(3U_{2n}^2 + 2U_{2n+1}^2 - 6U_{2n-1}U_{2n+1} - 2n - 5)$$
$$U_1U_3 + U_3U_5 + \cdots + U_{2n-1}U_{2n+1} = \tfrac{1}{5}(3U_{2n+1}^2 - 2U_{2n}^2 - 3n - 1)$$
$$U_0U_2 + U_2U_4 + \cdots + U_{2n-2}U_{2n} = \tfrac{1}{5}(2U_{2n+1}^2 - 3U_{2n}^2 + 3n + 1)$$
$$U_0U_2 + U_1U_3 + \cdots + U_{2n-2}U_{2n} + U_{2n-1}U_{2n+1} = U_{2n}U_{2n+1} - 1$$
$$U_1U_2 + U_3U_4 + \cdots + U_{2n-1}U_{2n} = \tfrac{1}{5}(U_{2n+1}^2 + U_{2n}^2 - n - 2)$$
$$U_0U_1 + U_2U_3 + \cdots + U_{2n-2}U_{2n-1} = \tfrac{1}{5}(4U_{2n}^2 - U_{2n+1}^2 + n - 3)$$
$$U_0U_1 + U_1U_2 + \cdots + U_{2n-2}U_{2n-1} + U_{2n-1}U_{2n} = U_{2n}^2 - 1.$$

Finally the congruence relation

$$U_{(n+1)k+n} \equiv 0 \pmod{U_n}$$

may be proved by induction on k.

By using the identity

$$U_m^2 - U_{m-p}U_{m+p} = (-1)^{m-p+1}U_{p-1}^2,$$

I deduce the following results:

$$U_0U_4 + U_2U_6 + \cdots + U_{2n-2}U_{2n+2} = \tfrac{1}{5}(3U_{2n+1}^2 - 2U_{2n}^2 + 7n - 1)$$
$$U_1U_5 + U_3U_7 + \cdots + U_{2n-1}U_{2n+3} = \tfrac{1}{5}(2U_{2n+3}^2 - 3U_{2n+2}^2 - 7n - 6)$$
$$U_0U_4 + U_1U_5 + \cdots + U_{2n-2}U_{2n+2} + U_{2n-1}U_{2n+3} = \tfrac{1}{2}(U_{4n+5} - U_{2n+2}^2 - 4).$$

By starting with the identities

$$U_mU_{m+1}U_{m+2} = U_{m+1}^3 + (-1)^m U_{m+1}$$

and

$$U_{m+1}^3 = U_m^3 + U_{m-1}^3 + 3U_{m-1}U_mU_{m+1},$$

we can easily prove the following:

$$U_1U_2U_3 + U_3U_4U_5 + \cdots + U_{2n-1}U_{2n}U_{2n+1} = \tfrac{1}{4}(U_{2n+1}^3 - U_{2n+1})$$
$$U_0U_1U_2 + U_2U_3U_4 + \cdots + U_{2n-2}U_{2n-1}U_{2n} = \tfrac{1}{4}(U_{2n}^3 + U_{2n} - 2)$$
$$U_0^3 + U_2^3 + U_4^3 + \cdots + U_{2n}^3 = \tfrac{1}{4}(U_{2n+1}^3 + 3U_{2n+1})$$
$$U_1^3 + U_3^3 + U_5^3 + \cdots + U_{2n-1}^3 = \tfrac{1}{4}(U_{2n}^3 - 3U_{2n} + 2)$$
$$U_1^3 + U_2^3 + U_3^3 + \cdots + U_n^3 = \tfrac{1}{4}(U_{n+1}^3 + U_n^3 + (-1)^n 3U_{n-1} - 2).$$

By making use of the identity

$$U_{m+1}^3 = U_{m-1}U_{m+1}U_{m+3} + (-1)^m U_{m+1}$$

and the preceding results, it may be proved that

$$U_0U_2U_4 + U_2U_4U_6 + \cdots + U_{2n-2}U_{2n}U_{2n+2} = \tfrac{1}{4}(U_{2n+1}^3 + 7U_{2n+1} - 8)$$
$$U_1U_3U_5 + U_3U_5U_7 + \cdots + U_{2n-1}U_{2n+1}U_{2n+3} = \tfrac{1}{4}(U_{2n+2}^3 - 7U_{2n+2} + 6)$$
$$U_0U_2U_4 + U_1U_3U_5 + U_2U_4U_6 + \cdots + U_{n-1}U_{n+1}U_{n+3}$$
$$= \frac{1}{10}[U_{3n+7} + (-1)^{n+1}16U_n - 5].$$

I now prove the following main

THEOREM. *If k is an integer >1, then* i) *a positive integer l, less than k and depending on k, can be found such that*

$$U_{n_1+n_2+\cdots+n_k-l} < U_{n_1}U_{n_2}\cdots U_{n_k} < U_{n_1+n_2+\cdots+n_k-l+1}$$

whenever each of the $n_1, n_2, \cdots, n_k$ is $>n_0$, n_0 depending on k. ii) *there lie exactly mk Fibonacci numbers between U_n^k and U_{n+m}^k, n being $>n_0$.*

Proof of (i). Let l be the greatest integer in

$$\left(\frac{k-1}{2}\right)\cdot\frac{\log 5}{\log\left(\frac{1+\sqrt{5}}{2}\right)}-k+2,$$

so that

$$\left(\frac{k-1}{2}\right)\cdot\frac{\log 5}{\log\left(\frac{1+\sqrt{5}}{2}\right)}-k+1<l<\left(\frac{k-1}{2}\right)\cdot\frac{\log 5}{\log\left(\frac{1+\sqrt{5}}{2}\right)}-k+2.$$

We have

$$\begin{aligned} U_{n_1}U_{n_2}\cdots U_{n_k} &= \left(\frac{1}{\sqrt{5}}\right)^k\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n_1+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n_1+1}\right\} \\ &\quad\cdots\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n_k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n_k+1}\right\} \\ &= \frac{1}{5^{k/2}}\left(\frac{1+\sqrt{5}}{2}\right)^{n_1+n_2+\cdots+n_k+k}\left\{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n_1+1}\right\} \\ &\quad\cdots\left\{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n_k+1}\right\} \end{aligned}$$

and

$$\begin{aligned} &U_{n_1+n_2+\cdots+n_k-l+1} \\ &\qquad = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n_1+n_2+\cdots+n_k-l+2}\left\{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n_1+n_2+\cdots+n_k-l+2}\right\}. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n_1,\cdots,n_k\to\infty}\frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l+1}} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+l-2}}{5^{\frac{1}{2}(k-1)}} \\ &< \frac{1}{5^{\frac{1}{2}(k-1)}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}(k-1)\log 5/[\log((1+\sqrt{5})/2)]} \\ &= \frac{5^{\frac{1}{2}(k-1)}}{5^{\frac{1}{2}(k-1)}} = 1. \end{aligned}$$

Therefore

$$\frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l+1}} < 1$$

if each of $n_1, n_2, \cdots, n_k$ is $>m_0$.
Similarly

$$\lim_{n_1,\cdots,n_k\to\infty} \frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+l-1}}{5^{\frac{1}{2}(k-1)}}$$

$$> \frac{1}{5^{\frac{1}{2}(k-1)}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}(k-1)\log 5/[\log((1+\sqrt{5})/2)]} = 1.$$

Therefore

$$\frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l}} > 1$$

if each of $n_1, n_2, \cdots, n_k$ is $>N_0$. Hence, if each of $n_1, n_2, \cdots, n_k$ is $>n_0=\max(m_0, N_0)$ we have (i).

COROLLARY. *Putting $n_1=n_2=\cdots=n_k=n$ in the above inequalities, we have for $n>n_0$,*

$$U_{kn-l} < U_n^k < U_{kn-l+1},$$

where l has the value specified in (i) *above.*

Proof of (ii). From the above corollary, we have

$$U_{kn-l} < U_n^k < U_{kn-l+1},$$

and replacing n by $(n+1)$, we have

$$U_{kn+k-l} < U_{n+1}^k < U_{kn+k-l+1}.$$

Hence between U_n^k and U_{n+1}^k lie the k Fibonacci numbers

$$U_{kn-l+1},\ U_{kn-l+2},\ \cdots,\ U_{kn-l+k}.$$

By extending the argument, it can be deduced that between U_n^k and U_{n+m}^k there lie exactly mk Fibonacci numbers.

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TRACE FUNCTIONS ON ALGEBRAS WITH PRIME CHARACTERISTIC

HANS ZASSENHAUS, McGill University

The natural trace function on the matrix algebra F_d of degree d over a field F is defined as

$$f((\alpha_{ik})) = \operatorname{tr}((\alpha_{ik})) = \sum_{i=1}^{d} \alpha_{ii}.$$

It satisfies the rules

$$f(a+b) = f(a) + f(b) \tag{1}$$

$$f(\lambda a) = \lambda f(a), \tag{2}$$

$$f(ab) = f(ba), \tag{3}$$

for a, b in F_d, λ in F.

Introducing the Lie-multiplication in F_d by

$$a \bigcirc b = ab - ba, \tag{4}$$

the rules (1)–(3) become equivalent to (1), (2) and

$$f(a \bigcirc b) = 0 \tag{3a}$$

for $a, b \in F_d$.

In an associative ring A of prime characteristic p the identity

$$(a_1 + a_2)^p = a_1^p + a_2^p + \Lambda(a_1, a_2) \tag{5}$$

holds, with $\Lambda(a_1, a_2)$ a certain sum of Lie-products of the form

$$a_{i_1} \bigcirc (a_{i_2} \bigcirc (\cdots (a_{i_{p-1}} \bigcirc a_{i_p}) \cdots)).*$$

If A is commutative then (5) assumes the simpler form

$$(a_1 + a_2)^p = a_1^p + a_2^p. \tag{5a}$$

Let f be a function defined on an associative algebra A over a field F of prime characteristic p and assume that the values of f are in F satisfying (1)–(3). It follows from (1), (3a) and (5) that

$$f((a+b)^p) = f(a^p) + f(b^p) \tag{6}$$

and more generally

$$f\left(\left(\sum_{i=1}^{r} \lambda_i a_i\right)^p\right) = \sum_{i=1}^{r} \lambda_i^p f(a_i^p). \tag{7}$$

* Hans Zassenhaus, Über Lie'sche Ringe mit Primzahl Charakteristik, Hamburger Abhandlungen, Bd. 13, p. 89, 1939.

E.g. for $A = F_d$, $f = \text{tr}$ the matrices $e_{rs} = (\delta_{ir}\delta_{ks})$ form a basis of F_d over F with the multiplication rule $e_{rs}e_{uv} = \delta_{su}e_{rv}$ such that for $a = (\alpha_{ik})$,

$$a = \sum \alpha_{ik}e_{ik}^{*}$$

and, according to (7), (5a),

$$\text{tr}\,(a^p) = \sum \alpha_{ik}^p(\text{tr}\,e_{ik})^p = \sum \alpha_{ii}^p = (\sum \alpha_{ii})^p = (\text{tr}\,a)^p,$$

a rule which may be obtained also by using ordinary tools of matrix theory.

DEFINITION. *A function f defined on an associative algebra A over a field F of prime characteristic p with values in an algebraically closed extension Ω of F is called a trace function A over F if it satisfies the rules* (1)–(3) *and the rule*

$$f(a^p) = f(a)^p \qquad \textit{for a contained in } A. \tag{8}$$

An example is given by the natural trace function on a matrix algebra of characteristic p. More generally, every representation Δ of an associative algebra A over a field F of characteristic $p>0$ by matrices of degree d with coefficients in Ω leads to the trace function $\text{tr}\,\Delta(a)$ on A over F. This follows from the conditions

$$\begin{aligned} \Delta(a + b) &= \Delta a + \Delta b, \\ \Delta(\lambda a) &= \lambda\Delta a, \\ \Delta(ab) &= \Delta a \cdot \Delta b, \end{aligned} \tag{9}$$

for $a, b \in A$, $\lambda \in F$ which every representation must satisfy. It is the purpose of this paper to show that the only trace functions on A over F are the traces defined by representations, and to give an application to the theory of representations of finite groups for characteristic $p>0$.

From now on A will always be an associative algebra over a field F of characteristic $p>0$.

PROPOSITION 1. *The trace functions on A form a module $T(A/F)$ of characteristic p.*

Proof obvious.

Iterating (8) we obtain the rule

$$f(a^{p^j}) = f(a)^p \qquad (j = 0, 1, 2, \cdots) \tag{8a}$$

for trace functions. In particular,

(8b) if an equation $a^{p^j} = 0$ holds then $f(a) = 0$.

This leads to

* The mere summation symbol indicates summation over all indices occurring twice.

PROPOSITION 2. *Each trace function on A over F vanishes on the radical $R(A)$ of A.*

For any element a of A there must be a non-trivial linear relation between the infinitely many elements $a, a^p, a^{p^2}, \cdots$ say

$$\sum_{j=0}^{m} \lambda_j a^{p^j} = 0, \qquad \text{with } \lambda_j \in F, \lambda_m = 0.$$

For a trace function on A over F it follows that

$$0 = f\left(\sum_{j=0}^{m} \lambda_j a^{p^j}\right) = \sum_{j=0}^{m} \lambda_j f(a)^{p^j}.$$

Hence

PROPOSITION 3. *The values of a trace function on A over F are algebraic over F.*

If A is the ring sum of the algebras A_1 and A_2 over F then every trace function f on A induces a trace function f_i on $A_i (i=1, 2)$. The trace function f on A over F is uniquely determined by the pair f_1, f_2 according to the rule

$$f(a_1 + a_2) = f_1(a_1) + f_2(a_2) \qquad \text{for } a_1 \in A_1,\ a_2 \in A_2. \tag{9}$$

Conversely any pair of trace functions f_i on A_i $(i=1, 2)$ determines a trace function f on A according to (9). Hence

PROPOSITION 4. *The module formed by the trace functions on a ring sum of two algebras over F is direct sum of the modules formed by the trace functions on the summands.*

For the matrix algebra of degree d over F we obtain for each trace function f the equations

$$f(e_{rs}) = f(e_{rs} \bigcirc e_{ss}) = 0, \qquad f(e_{rr} - e_{ss}) = f(e_{rs} \bigcirc e_{sr}) = 0$$

in case of $r \neq s$. Hence for $a = \sum \alpha_{ik} e_{ik}$ it follows that

$$\begin{aligned} f(a) &= \sum \alpha_{ik} f(e_{ik}) = \sum \alpha_{ii} f(e_{ii}) = \sum \alpha_{ii}(f(e_{ii} - e_{11}) + f(e_{11})) \\ &= \sum \alpha_{ii} f(e_{11}) = \operatorname{tr} a \cdot f(e_{11}). \end{aligned}$$

Furthermore

$$f(e_{11})^p = f(e_{11}^p) = f(e_{11});$$

hence $f(e_{11})$ belongs to the prime field. We have obtained

PROPOSITION 5. *The module of the trace functions on a matrix algebra A of characteristic d is generated by the natural trace function so that there are exactly p trace functions on a matrix algebra.*

Let $b_1, b_2, \cdots, b_n$ be a basis of the associative algebra A over F. There are linear relations

$$b_i \bigcirc b_k = b_ib_k - b_kb_i = \sum \gamma_{ikl}b_l, \tag{10}$$

$$b_i^p = \sum \gamma_{il}b_l \tag{11}$$

with $\gamma_{ikl}, \gamma_{il}$ contained in F. In view of the rules (1)–(3), (8) each trace function f on A over F satisfies the rules

$$f(\sum \lambda_l b_l) = \sum \lambda_l f(b_l), \tag{12}$$

$$\sum \gamma_{ikl}f(b_l) = 0, \tag{13}$$

$$f(b_i)^p = \sum \gamma_{il}f(b_l). \tag{14}$$

Conversely we have

PROPOSITION 6. *If a set of n constants $f(b_1), f(b_2), \cdots, f(b_n)$ belonging to the algebraically closed extension Ω of F satisfies* (13), (14), *then it defines a trace function on A over F according to* (12).

Proof: The rules (1)–(3) follow from (12), (13) by obvious computations. Using (1)–(3), (14) and (7) we obtain

$$\begin{aligned} f((\sum \lambda_i b_i)^p) &= \sum \lambda_i^p f(b_i^p) = \sum \lambda_i^p f(\sum \gamma_{il}b_l) \\ &= \sum \lambda_i^p \sum \gamma_{il} f(b_l) = \sum \lambda_i^p f(b_i)^p \\ &= (\sum \lambda_i f(b_i))^p = (f(\sum \lambda_i b_i))^p \end{aligned}$$

and hence (8).

From propositions 1–6 we have the following

THEOREM ON TRACE FUNCTIONS. *The trace functions on an associative algebra A over a field F of prime characteristic p with values in the algebraically closed extension Ω of F coincide with the trace functions belonging to the representations of A over F by matrices of finite degree with coefficients in Ω. The number of the trace functions on A over F is equal to p^ρ where ρ denotes the number of classes of equivalent absolutely irreducible representations of A over F.*

Proof: We recapitulate that a representation Δ of A over F by matrices of degree d with coefficients in Ω is called absolutely irreducible if there are d^2 matrices linearly independent over Ω among the matrices $\Delta(a)$ with a contained in A. Denoting by A_Ω the result of extending the associative algebra A with basis $b_1, b_2, \cdots, b_n$ over F to an associative algebra with basis $b_1, b_2, \cdots, b_n$ over Ω with the same rules of multiplication, we conclude from proposition 6 that for every trace function f on A over F by the formula

$$f_\Omega(\sum \Lambda_l b_l) = \sum \Lambda_l f(b_l) \qquad \text{with } \Lambda_l \in \Omega, \tag{15}$$

there is defined a trace function f_Ω on A_Ω over Ω. Conversely every trace function on A_Ω over the algebraically closed field Ω has its values, according to proposition 2, entirely in Ω. Hence it induces a trace function on A over F. And this correspondence between the trace functions on A_Ω over Ω and the trace functions on A over F provides an isomorphism between the modules $T(A_\Omega/\Omega)$ and $T(A/F)$. According to McLagan-Wedderburn the difference algebra of A_Ω over its radical decomposes into the ring sum of ρ matrix algebras over Ω where by known results ρ coincides with the number of classes of equivalent absolutely irreducible representations of A over F. Due to proposition 2 the number of trace functions of A_Ω over Ω coincides with the corresponding number for $A_\Omega - R(A_\Omega)$ over Ω. From propositions 4, 5 it follows that this number is p^ρ.

Since each trace function on a matrix algebra over F is obtained by multiplying the natural trace function with an element of the prime field, we conclude that it may be interpreted as the trace of a multiple of the natural representation by matrices. In view of the construction leading to proposition 4 each trace function on a ring sum of finitely many matrix algebras over F is the trace of a certain fully reducible representation. Hence any trace function on A_Ω over Ω is the trace of a sum of irreducible representations of A_Ω over Ω by matrices of finite degree with coefficients in Ω, or what amounts to the same, any trace function on A over F is the trace of the sum of absolutely irreducible representations of A over F, *q.e.d.*

By application of proposition 6 and the main theorem to the group algebra of a finite group over the prime field of characteristic $p>0$ we obtain

PROPOSITION 7. *Let ρ_p denote the number of classes of equivalent absolutely irreducible representations of a finite group G for characteristic $p>0$. There are p^{ρ_p} trace functions on G. They are characterized as functions f on G with values in the algebraic algebraically closed field Ω of characteristic p satisfying*

$$f(xy) = f(yx) \tag{16}$$

$$f(x^p) = f(x^p) \tag{17}$$

for $x, y \in G$.

Equivalent to (16), (17) are the rules

$$f(txt^{-1}) = f(x) \tag{16a}$$

$$f(x^{p^j}) = f(x)^{p^j} \tag{17a}$$

for $x, y \in G$, j a non-negative integer.

DEFINITION. *Two elements a, b of G are p-conjugate if there is an equation*

$$xa^{p^j}x^{-1} = yb^{p^k}y^{-1}$$

with x, $y \in G$ and j, k non-negative integers. We write

$$a \underset{p}{\sim} b$$

to indicate the p-conjugacy of a and b, whereas

$$a \sim b$$

indicates that a and b are conjugate under G.

By definition

$$b \underset{p}{\sim} a \qquad \text{if } a \underset{p}{\sim} b,$$

$$a \underset{p}{\sim} b \qquad \text{if } a \sim b,$$

$$a^{p^j} \underset{p}{\sim} a \qquad (j = 0, 1, 2, \cdots).$$

The relation $a \sim b$ has the 3 properties of an equivalence relation. Due to the identity

$$(xax^{-1})^m = xa^m x^{-1},$$

we have

$$a^m \sim b^m \qquad \text{if } a \sim b$$

and

$$a^m \underset{p}{\sim} b^m \qquad \text{if } a \underset{p}{\sim} b.$$

Furthermore $a \underset{p}{\sim} b$ if and only if $a^{p^j} \sim b^{p^k}$ holds for some non-negative integers j, k. Suppose that we also have $b^{p^l} \sim c^{p^m}$. Then

$$a^{p^{j+l}} = (a^{p^j})^{p^l} \sim (b^{p^k})^{p^l} = (b^{p^l})^{p^k} \sim (c^{p^m})^{p^k} = c^{p^{m+k}},$$

$$a \underset{p}{\sim} c.$$

Hence p-conjugacy satisfies the 3 requirements for an equivalence relation. The elements of G are distributed among classes of p-conjugate elements each of which consists of some classes of conjugate elements. Let the order of G be $n = p^\nu \cdot n'$ with n' prime to p. Then for $a \in G$ we have $a \underset{p}{\sim} a^{p^\nu}$ so that the order of a^{p^ν} divides n'. Hence each element of G is p-conjugate to an element with order prime to p, *i.e.*, to a p-regular element. Let a be a p-regular element, then from $a^n = 1$ follows $a^{n'} = 1$.

Among the classes of conjugate elements represented by a, a^p, a^{p^2}, $\cdots$ there must be repetitions. Let $a^{p^{d(a)}}$ be the first element conjugate to a previous element say

$$a^{p^{d(a)}} \sim a^{p^i} \qquad \text{with } 0 \leqq i < d(a)$$

and denote by p' a solution of the congruence $pp' \equiv 1(n')$; then in case $i > 0$ we have

$$a^{p^{d(a)-1}} = (a^{pp'})^{p^{d(a)-1}} = (a^{p^{d(a)}})^{p'} \sim (a^{p^i})^{p'} = a^{p^{i-1}}$$

contrary to the minimum property of d. Hence $i = 0$, $a^{p^{d(a)}} \geqq a$, $a^{p^{2d(a)}} \geqq a^{p^{d(a)}}, \cdots$

$$a^{p^{jd(a)}} \sim a \qquad (j = 0, 1, 2, \cdots)$$

$$a^{p^i} \sim a^{p^{i+jd(a)}} \qquad (i, j = 0, 1, 2, \cdots).$$

It follows that there are precisely $d(a)$ classes of conjugate elements containing an element a^{p^i} and these $d(a)$ classes are represented by the elements $a, a^p, \cdots, a^{p^{d(a)-1}}$. Every p-regular element which is p-conjugate to a must be in one of the $d(a)$ classes of conjugate elements represented by $a, a^p, \cdots, a^{p^{d(a)-1}}$.

Now we answer the following question: Let a be p-regular, b an element of order $p^\mu q$ where p does not divide q, and let

$$a^{p^j} \sim b^{p^k} \tag{18}$$

for some non-negative integers j, k. Under which conditions is

$$a^{p^l} \sim b^{p^m}, \qquad \text{with some non-negative integers } l, m? \tag{19}$$

Answer: The necessary and sufficient conditions are

$$m \geqq \mu \text{ and the congruence } l + k \equiv j + m(d(a)).$$

Proof: Assume (19). Since a is p-regular, the same is true for a^{p^l} and b^{p^m}, hence $m \geqq \mu$. Furthermore

$$a^{p^{l+k}} \sim b^{p^{m+k}} \sim a^{p^{j+m}}.$$

But due to the previous considerations two powers a^{p^r} and a^{p^s} are conjugate under G if and only if $r \equiv s(d(a))$. Hence $l+k \equiv j+m(d(a))$. Conversely let $m \geqq \mu$ and $l+k \equiv j+m(d(a))$; then

$$a^{p^{l+k}} \sim a^{p^{j+m}}$$

and b^{p^m} is p-regular,

$$b^{p^{m+k}} = (b^{p^k})^{p^m} \sim (a^{p^j})^{p^m} = a^{p^{j+m}} \sim a^{p^{l+k}},$$

$$b^{p^m} = (b^{p^m})^{(pp')^k} = (b^{p^{m+k}})^{p'^k} \sim (a^{p^{l+k}})^{p'^k} = (a^{(pp')^k})^{p^k} = a^{p^k},$$

which completes the proof.

For the construction of the trace functions on G we choose a representative system $a_1, a_2, \cdots, a_r$ of the classes of p-conjugate elements such that each representative is p-regular. Due to (16a) and (17a) and the relation

$$a_i^{p^{d(a_i)}} \sim a_i,$$

we obtain for each trace function f on G the equations

$$f(a_i)^{p^{d(a_i)}} = f(a_i)$$

which are equivalent to the statement that $f(a_i)$ belongs to the Galois field $GF(p^{d(a_i)})$ of $p^{d(a_i)}$ elements. Furthermore for x contained in G we have the relation

$$x^{p^j} \sim a_i^{p^k} \qquad \text{with some } i,\ j,\ k, \tag{20}$$

from which it follows that

$$f(x)^{p^j} = f(a_i)^{p^k} \tag{21}$$

which determines $f(x)$ uniquely once the values $f(a_1), f(a_2), \cdots, f(a_r)$ are known.

Conversely let us assign to each representative a_i one of the $p^{d(a_i)}$ elements of $GF(p^{d(a_i)})$ as the value $f(a_i)$ and let us define $f(x)$ according to (20) and (21) for every element x of G. Due to the answer which we gave to the question above it follows that the value of $f(x)$ is independent of the relation (20) connecting x with the representative a_i. Since $(x^{p^l})^{p^j} \sim a^{p^{k+1}}$ we find that $f(x^{p^l})^{p^j} = f(a_i)^{p^{k+1}}$ and hence $f(x^{p^l}) = f(x)^{p^l}$. If $y \geqq x$ then

$$y^{p^j} \sim x^{p^j} \sim a_i^{p^k},$$
$$f(y)^{p^j} = f(a_i)^{p^k},$$
$$f(y) = f(x).$$

Hence f is a trace function.

According to the previous construction, the number of trace functions on G is p^σ where

$$\sigma = \sum_{i=1}^{r} d(a_i).$$

The exponent of p coincides with the number of classes of conjugate p-regular elements.

Hence we have obtained another proof of the theorem of Richard Brauer* that the number ρ_p of classes of absolutely irreducible representations of a finite group G for characteristic $p>0$ coincides with the number of classes of conjugate p-regular elements.† In addition we have found an explicit construction of the p^{ρ_p} trace functions on G for characteristic p.

* Über die Darstellung von Gruppen in Galois'schen Feldern. Actualités scientifiques et industrielles 195, 1935.

† This application has been suggested to me by a lecture of Professor R. Brauer, in which he makes use of (1), (2), (3) in proving his theorem.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

ON INEQUALITIES FOR ANALYTIC FUNCTIONS

C. T. RAJAGOPAL, Madras, India

Let $f(z)$ be regular for $|z| < R$. Let

$$A(r) = \max \mathfrak{R}f(z) \quad \text{for} \quad |z| \leqq r, \quad 0 \leqq r < R,$$
$$M(r) = \max |f(z)| \quad \text{for} \quad |z| \leqq r, \quad 0 \leqq r < R,$$
$$A(R) = \lim_{r \to R-0} A(r), \qquad M(R) = \lim_{r \to R-0} M(r).$$

Then the following two inequalities, of which the second includes Cauchy's, may be proved by the use of Schwarz's lemma, as explained by Jensen in a comprehensive paper [**1**] and by the author in an independent note [**2**].

$$(1) \qquad \frac{|f^{(n)}(0)|}{n!} \leqq \frac{2}{R^n} [A(R) - \mathfrak{R}f(0)], \qquad n \geqq 1.$$

$$(2) \qquad \frac{|f^{(n)}(0)|}{n!} \leqq \frac{1}{R^n} \frac{[M(R)]^2 - |f(0)|^2}{M(R)}, \qquad n \geqq 1.$$

The case $n = 1$ of each of the inequalities (1), (2) has been generalized as below by Lindelöf [**1**, pp. 13, 24].

$$(1') \qquad |f'(z)| \leqq \frac{2R}{R^2 - r^2} [A(R) - \mathfrak{R}f(z)], \qquad 0 \leqq |z| = r < R.$$

$$(2') \qquad |f'(z)| \leqq \frac{R}{R^2 - r^2} \frac{[M(R)]^2 - |f(z)|^2}{M(R)}, \qquad 0 \leqq |z| = r < R.$$

An advanced problem recently proposed by H. S. Shapiro [**3**] suggests the following results deducible from (1), (1′), and supplementary to (2), (2′).

THEOREM. *If $f(z)$ is regular and non-vanishing for $|z| < R$ and if $k \geqq 1$ is the least positive integer for which $f^{(k)}(0) \neq 0$, then*

$$(3) \qquad \frac{|f^{(n)}(0)|}{n!} \leqq \frac{2}{R^n} \frac{M(R)}{e}, \qquad k \leqq n \leqq 2k - 1;$$

and further

$$(3') \qquad |f'(z)| \leqq \frac{2R}{R^2 - r^2} \frac{M(R)}{e}, \qquad 0 \leqq |z| = r < R.$$

Proofs of (3), (3′). To prove (3), consider

$$F(z) = \log f(z)$$

where the logarithm has its principal value. The condition $f(z) \neq 0$ for $|z| < R$ ensures that $F(z)$ is regular for $|z| < R$. Further, for $|z| \leqq r < R$,

$$\mathcal{R}F(z) = \log |f(z)| \leqq \log M(r).$$

The hypothesis that either $f'(0) \neq 0$, or $f^{(k)}(0) \neq 0$ with $f^{(n)}(0) = 0$ for $n \leqq k-1$, $k \geqq 2$, gives

$$f(z) = f(0) + \sum_{n=k}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \qquad k \geqq 1,$$

or

$$F(z) = \log f(z) = \log f(0) + \sum_{n=k}^{2k-1} \frac{1}{n!} \frac{f^{(n)}(0)}{f(0)} z^n + \sum_{n=2k}^{\infty} a_n z^n, \qquad k \geqq 1,$$

for $|z| < R$, $\log f(z)$ being analytic in this region. Hence, equating the coefficients of z^n, $k \leqq n \leqq 2k-1$, in the power series for $F(z)$ in $|z| < R$ and in the above representation of $F(z)$, we get

$$F^{(n)}(0) = \frac{f^{(n)}(0)}{f(0)}, \qquad 1 \leqq k \leqq n \leqq 2k-1.$$

Consequently we can replace $f(z)$ by $F(z)$ in (1) and obtain, for $k \leqq n \leqq 2k-1$,

$$\frac{1}{n!}\left|\frac{f^{(n)}(0)}{f(0)}\right| = \frac{|F^{(n)}(0)|}{n!} \leqq \frac{2}{R^n}\left[\log M(R) - \log |f(0)|\right],$$

i.e.,

$$\frac{|f^{(n)}(0)|}{n!} \leqq \frac{2}{R^n}\left[|f(0)| \log \frac{M(R)}{|f(0)|}\right].$$

When the expression within square brackets in the right-hand member of the last inequality is replaced by its absolute maximum which corresponds to $|f(0)| = M(R)/e$, the inequality becomes identical with (3).

(3′) may be derived from (1′) in the same way as (3) from (1). Alternatively, (3′) may be deduced from (3) by arguing as follows. The function

$$g(z) = f\left[\frac{R^2(z + z_0)}{R^2 + \bar{z}_0 z}\right], \qquad |z_0| < R,$$

satisfies the same conditions as $f(z)$. Hence (3) holds for $g(z)$ with $n = 1$, *i.e.*,

$$|g'(0)| \leqq \frac{2}{R}\frac{M(R)}{e}, \quad \text{or} \quad |f'(z_0)| \frac{R^2 - |z_0|^2}{R^2} \leqq \frac{2}{R}\frac{M(R)}{e}.$$

The last inequality is the same as (3′) with z_0 in place of z.

REMARKS ON (3), (3′). *The inequality* (3) *cannot be improved by replacing its right-hand member by a smaller number.* For, in the example

$$(4) \qquad f(z) = \exp\left(\frac{z^k+1}{z^k-1}\right), \qquad R = 1,$$

where k is a positive integer, we have $f^{(n)}(0)=0$ for $n \leqq k-1$ when $k>1$, and

$$\frac{|f^{(k)}(0)|}{k!} = \frac{2}{e} = \frac{2}{R^k}\frac{M(R)}{e}$$

whether $k>1$ or $k=1$.

(3′) *again cannot be improved to*

$$|f'(z)| \leqq \frac{2R}{R^2-r^2}\left(\frac{M}{e}-\delta\right), \qquad \delta > 0.$$

For, when $f(z)$ is the function in (4), with $k=1$, the left-hand member of the above inequality tends to $2M/eR$ as $r \to 0$, while the right-hand member tends to $2(M/e-\delta)/R$, thus leading to a contradiction for all z sufficiently close to 0.

It may also be pointed out in regard to (3′) that its right-hand member must contain a factor such as $1/(R-r)$ which tends to infinity as $r \to R$. A consideration of the function $f(z)$ in (4), with $k=1$, is sufficient to show this.

An example to show that (3) *is not necessarily true when* $f(z)$ *vanishes in* $|z|<R$ is furnished by the function

$$(5) \qquad f(z) = \frac{z^k-\alpha}{1-\alpha z^k}, \qquad 0 < \alpha < \sqrt{1-\frac{2}{e}}, \qquad R = 1,$$

where k is a positive integer. This function is such that $f^{(n)}(0)=0$ for $n \leqq k-1$ when $k>1$; and, whether $k>1$ or $k=1$,

$$\frac{|f^{(k)}(0)|}{k!} = 1-\alpha^2 > \frac{2}{e} = \frac{2}{e}\frac{M(R)}{R^k}.$$

The function $f(z)$ in (5) shows also that (3′) *is not always true when* $f(z)$ *vanishes in* $|z|<R$.

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2. C. T. Rajagopal, Mathematics Student, vol. 15, 1947, pp. 5–7.
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for α_{ij} in F. Let θ be a generator of the multiplicative cyclic group F^*. Then any $\alpha_{ij}\neq 0$ may be written as $\alpha_{ij}=\theta^{k_{ij}}$ so that

$$\alpha_{ij}e_{11} = (\theta e_{11})^{k_{ij}}, \qquad \alpha_{ij} \neq 0 \text{ in } F. \tag{3}$$

Combining (2) and (3), we have either $a=0$ or (by suppressing all zero terms)

$$a = \sum y^{m-i+1}(\theta e_{11})^{k_{ij}}y^{j-1}.$$

That is, y and θe_{11} generate A.

The referee points out that if B is any ring with unity element, then formula (1) holds for the matrix units e_{ij} of the ring B_m of all $m\times m$ matrices with elements in B. This implies that if B is generated by a subset S, then B_m is generated by y and the elements of $e_{11}S$ (since e_{11} itself is generated by the elements of $e_{11}S$). Our theorem is a corollary of this result.

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1. A. A. Albert, Two element generation of a separable algebra, Bull. Amer. Math. Soc., vol. 50, 1944, pp. 786–788.
2. A. A. Albert, Modern Higher Algebra, Chicago, 1937.

A THEOREM ON A SPECIAL CLASS OF NEAR-VANDERMONDE DETERMINANTS

V. L. Shapiro, Rutgers University

Let us call a determinant of the form

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ x_1^4 & x_2^4 & \cdots & x_n^4 \\ x_1^6 & x_2^6 & \cdots & x_n^6 \\ \vdots & \vdots & & \vdots \\ x_1^{2(n-1)} & x_2^{2(n-1)} & \cdots & x_n^{2(n-1)} \end{vmatrix} \tag{1}$$

a near-Vandermonde determinant. The motivation for this definition arises from the fact that when $x_1x_2 \cdots x_n$ is factored out of the above determinant, what is left is nearly a Vandermonde determinant.

Spécial classes of Vandermonde and near-Vandermonde determinants appear in a natural manner in the study of trigonometrical series. (See for example Hardy [1, p. 31], where it is shown how Fourier in his work was led to the study of systems of equations whose determinants are essentially Vandermonde.) In the author's study of localization phenomenon in double trigonometric series, Shapiro [2], a special class of near-Vandermonde determinants arose quite naturally, namely the class of determinants

for α_{ij} in F. Let θ be a generator of the multiplicative cyclic group F^*. Then any $\alpha_{ij}\neq 0$ may be written as $\alpha_{ij}=\theta^{k_{ij}}$ so that

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$$\Delta_n = \begin{vmatrix} 1 & 2 & \cdots (n-1) & n \\ 1 & 2^2 & \cdots (n-1)^2 & n^2 \\ 1 & 2^4 & \cdots (n-1)^4 & n^4 \\ 1 & 2^6 & \cdots (n-1)^6 & n^6 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 2^{2(n-1)} & \cdots (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix}$$

which occurs when $x_i = i$ in (1). It became necessary in the above-mentioned work to show that there is an infinite number of n for which $\Delta_n \neq 0$, which is the theorem to be proved in this note.

THEOREM. $\Delta_n \neq 0$ *for an infinite number of* n.

Let p be any odd prime number. Set $n = \frac{1}{2}(p+1)$. It will be shown that for n so chosen, $\Delta_n \neq 0$. The first row of Δ_n can be rewritten as

$$n-(n-1) \qquad n-(n-2) \cdots n-1 \qquad n-0$$

and consequently we see that $\Delta_n = A - B$, where

$$A = n \begin{vmatrix} 1 & 1 & \cdots 1 & 1 \\ 1 & 2^2 & \cdots (n-1)^2 & n^2 \\ 1 & 2^4 & \cdots (n-1)^4 & n^4 \\ 1 & 2^6 & \cdots (n-1)^6 & n^6 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 2^{2(n-1)} & \cdots (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix} \tag{2}$$

and

$$B = \begin{vmatrix} n-1 & n-2 & \cdots 1 & 0 \\ 1 & 2^2 & \cdots (n-1)^2 & n^2 \\ 1 & 2^4 & \cdots (n-1)^4 & n^4 \\ 1 & 2^6 & \cdots (n-1)^6 & n^6 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 2^{2(n-1)} & \cdots (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix}. \tag{3}$$

Now the determinant in (2) is a special case of the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots 1 & 1 \\ x_1 & x_2 & \cdots x_{n-1} & x_n \\ x_1^2 & x_2^2 & \cdots x_{n-1}^2 & x_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots x_{n-1}^{n-1} & x_n^{n-1} \end{vmatrix} = \prod_{i<j}(x_j - x_i) \qquad \begin{array}{l} i = 1, \cdots, n-1 \\ j = 2, \cdots, n \end{array}$$

where $x_i = i^2$ for $i = 1, \cdots, n$. Consequently A contains as a factor $n^2-(n-1)^2 = 2n-1$. But $2n-1 = p$. Therefore p divides A.

Expanding B by means of the first row, we see that the cofactor of each element in this row is a Vandermonde determinant of order $n-1$, multiplied by the square of constants which are less than or equal to n^2. Thus, for example the cofactor of $n-1$ is

$$\begin{vmatrix} 2^2 & \cdots (n-1)^2 & n^2 \\ 2^4 & \cdots (n-1)^4 & n^4 \\ 2^6 & \cdots (n-1)^6 & n^6 \\ \vdots & \vdots & \vdots \\ 2^{2(n-1)} & \cdots (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix} = 2^2 \cdots (n-1)^2 n^2 \begin{vmatrix} 1 & \cdots 1 & 1 \\ 2^2 & \cdots (n-1)^2 & n^2 \\ 2^4 & \cdots (n-1)^4 & n^4 \\ \vdots & \vdots & \vdots \\ 2^{2(n-2)} & \cdots (n-1)^{2(n-2)} & n^{2(n-2)} \end{vmatrix}. \tag{4}$$

The determinant on the right side of (4) contains as a factor $n^2-(n-1)^2 = 2n-1 = p$. Therefore p divides the cofactor of $n-1$. Likewise p divides the cofactor of $n-2, n-3, \cdots, 2$. 0 multiplied by its cofactor gives a number divisible by p. So in order to show that p does not divide B, it only remains to show that p does not divide the cofactor of 1. But the cofactor of 1 is

$$(-1)^n 1 \cdot 2^2 \cdots (n-2)^2 n^2 \begin{vmatrix} 1 & 1 & \cdots 1 & 1 \\ 1 & 2^2 & \cdots (n-2)^2 & n^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{2(n-2)} & \cdots (n-2)^{2(n-2)} & n^{2(n-2)} \end{vmatrix} = (-1)^n 1 \cdot 2^2 \cdots (n-2)^2 n^2 \prod_{i<j} (j^2 - i^2) \tag{5}$$

$$i = 1, \cdots, n-2; \; j = 2, \cdots, n-2, n.$$

Noticing that $\prod_{i<j}(j^2-i^2) = \prod_{i<j}(j-i)(j+i)$ where $j-i \leqq n-1$ and $j+i \leqq 2n-2$, we conclude that p does not divide the right side of (5) and consequently that p does not divide B. On the other hand we have already shown that p does divide A. Therefore Δ_n cannot be equal to zero; which concludes the proof of the theorem.

References

1. G. H. Hardy, Divergent Series, Oxford, 1949.

2. V. L. Shapiro, Square summation and localization of double trigonometric series, Part I (Ph.D. Dissertation, University of Chicago, 1952).

NOTE ON ONE TYPE OF INDETERMINATE FORM

HARRY FURSTENBERG, Yeshiva University

In what follows, we shall try to obtain an expression for $\lim_{n\to\infty}\{[\phi(n)]^{n^{-r}}\}$, where $\phi(n)\geqq 0$ and $\phi(\infty)=\infty$. In case $r=1$, use is made of the series $\sum_{n=0}^{\infty}\phi(n)z^n$ and its radius of convergence ρ. For on the one hand $1/\rho=\overline{\lim}_{n\to\infty}\{\phi(n)^{1/n}\}$ and on the other $1/\rho=\lim_{n\to\infty}\{\phi(n)/\phi(n-1)\}$ when this limit exists. When both limits exist, they are equal and $\lim_{n\to\infty}\{\phi(n)^{1/n}\}=\lim_{n\to\infty}\{\phi(n)/\phi(n-1)\}$. For $r>1$, we proceed as follows:

Let $\lim_{n\to\infty}A_n^{(0)}$ exist and equal A. Define $A_n^{(r)}$ by

$$A_n^{(r)} = \sum_{p=1}^{n} A_p^{(r-1)}.$$

We shall now prove:

THEOREM I.

$$\lim_{n\to\infty} \frac{A_n^{(r)}}{n^r} = \frac{A}{r!}\cdot$$

Define $E_1(s)=s$, $E_r(s)=\sum_{p=1}^{s}E_{r-1}(p)$. A simple induction proves that

$$E_r(s) = \binom{s+r-1}{r}$$

and therefore for a fixed r, $\lim_{s\to\infty}(E_r(s)/s^r)=1/r!$. $A_n^{(r)}$ can be written as $\sum_{p=1}^{n}E_{r-1}(n-p+1)A_p^{(0)}$. Choose n_0 such that for all positive k, $|A-A_{n_0+k}^{(0)}|<\epsilon$, where ϵ is an arbitrary positive number. Our theorem states that

$$\left|\frac{A}{r!}-\frac{A_n^{(r)}}{n^r}\right| \quad\text{or}\quad \left|\frac{A}{r!}-\frac{\sum_{p=1}^{n}E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right|$$

can be made arbitrarily small by choosing n large enough. Let

$$\Gamma_{n,n_0} = \left|\frac{\sum_{p=1}^{n_0}E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right| \quad\text{and}$$

$$\Delta_{n,n_0} = \left|\frac{A}{r!}-\frac{\sum_{p=n_0+1}^{n}E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right|.$$

It will be sufficient to prove that $\Gamma_{n,n_0}+\Delta_{n,n_0}$ can be made arbitrarily small. Now

$$\Delta_{n,n_0} = \left| \left(\frac{A}{r!} - \frac{A}{r!}\left(\frac{n-n_0}{n}\right)^r \right) + \left(\frac{A}{r!}\left(\frac{n-n_0}{n}\right)^r - \frac{AE_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r \right) \right.$$

$$\left. + \left(\frac{AE_r(n-n_0)}{n^r} - \frac{\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1)A_p^{(0)}}{n^r} \right) \right|.$$

Therefore Δ_{n,n_0} is less than or equal to the sum of the absolute values of the three terms. This may be written as

$$\begin{aligned} \Delta_{n,n_0} \leqq A \Bigg\{ &\left| \frac{1}{r!} - \frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r \right| \\ &+ \left| \frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r - \frac{E_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r \right| \Bigg\} \\ &+ \left| \frac{\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1)A}{n^r} - \frac{\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1)A_p^{(0)}}{n^r} \right| \\ \leqq A \Bigg\{ &\left| \frac{1}{r!} - \frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r \right| \\ &+ \left| \frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r - \frac{E_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r \right| \Bigg\} \\ &+ \left| \frac{E_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r \epsilon \right| \end{aligned}$$

since $|A-A_p^{(0)}|<\epsilon$ for $p>n_0$ and $\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1)=E_r(n-n_0)$. Using the fact that $\lim_{n\to\infty}((n-n_0)/n)^r=1$ and $\lim_{s\to\infty}(E_r(s)/s^r)=1/r!$ we find that for a fixed n_0, n may be chosen so large that Δ_{n,n_0} differs from $\epsilon/r!$ by less than an arbitrary η. At the same time we can choose

$$n > \left[n_0 \sum_{p=1}^{n_0} E_{r-1}(n-p+1)A_p^{(0)} \right]^{1/r}.$$

Then $\Gamma_{n,n_0}<1/n_0$ and $\Gamma_{n,n_0}+\Delta_{n,n_0}<1/n_0+\epsilon/r!+\eta$. By first choosing n_0 sufficiently large $1/n_0+\epsilon/r!$ can be made arbitrarily small, and then by choosing n large enough with respect to n_0, η can be made arbitrarily small. Theorem I results immediately.

We may now proceed to evaluate $\lim_{n\to\infty}\{[\phi(n)]^{n^{-r}}\}$. Let this limit exist and equal x. Then

$$\log x = \lim_{n\to\infty} \left\{ \frac{\log \phi(n)}{n^r} \right\}.$$

$$\phi(n) = \phi(n-1)\left\{\prod_{\nu=1}^{n}\left(\frac{\nu+2}{\nu}\right)^{\nu}\right\}.$$

To solve this we write

$$\frac{\phi(n)}{\phi(n-1)} = \prod_{\nu=1}^{n}\left(\frac{\nu+2}{\nu}\right)^{\nu}; \qquad \frac{\phi(n-1)}{\phi(n-2)} = \prod_{\nu=1}^{n-1}\left(\frac{\nu+2}{\nu}\right)^{\nu}.$$

From (1),

$$\lim_{n\to\infty}\left[\phi(n)\right]^{n^{-2}} = \lim_{n\to\infty}\left\{\frac{\left[\dfrac{\phi(n)}{\phi(n-1)}\right]}{\left[\dfrac{\phi(n-1)}{\phi(n-2)}\right]}\right\} = \lim_{n\to\infty}\left(\frac{n+2}{n}\right)^{n/2}$$

$$= \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n/2} = e.$$

NOTE ON FINDING THE INTEREST RATE

H. E. Stelson, Michigan State College

In this note some formulas are presented for very accurate determination of the rate of interest in an annuity or installment payment plan. Consider the ordinary annuity formula

$$B = Ra_{\overline{n|}} = R\,\frac{1-(1+r)^{-n}}{r} \tag{1}$$

where

R = periodic payment*

n = number of periodic payments

B = unpaid balance at the beginning of the credit period

r = periodic rate (unknown).

Now formula (1) can be written in the form

$$B = \frac{I}{\dfrac{nr}{1-(1+r)^{-n}} - 1} \tag{2}$$

where

$I = Rn - B$, the cost of the loan.

The right member of (2) may be expanded in a series as follows:

* The rate of interest in installment payment plans, by the author, this Monthly, vol. 56, 1949, pp. 257–261.

$$\phi(n) = \phi(n-1)\left\{\prod_{\nu=1}^{n}\left(\frac{\nu+2}{\nu}\right)^{\nu}\right\}.$$

To solve this we write

$$\frac{\phi(n)}{\phi(n-1)} = \prod_{\nu=1}^{n}\left(\frac{\nu+2}{\nu}\right)^{\nu}; \qquad \frac{\phi(n-1)}{\phi(n-2)} = \prod_{\nu=1}^{n-1}\left(\frac{\nu+2}{\nu}\right)^{\nu}.$$

From (1),

$$\lim_{n\to\infty}\,[\phi(n)]^{n^{-2}} = \lim_{n\to\infty}\left\{\frac{\left[\dfrac{\phi(n)}{\phi(n-1)}\right]}{\left[\dfrac{\phi(n-1)}{\phi(n-2)}\right]}\right\} = \lim_{n\to\infty}\left(\frac{n+2}{n}\right)^{n/2}$$

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The right member of (2) may be expanded in a series as follows:

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$$r_2 = \frac{2M}{n+1}\left[\frac{1+\dfrac{n+2}{3(n+1)}M}{1+\dfrac{(2n+1)}{3(n+1)}M}\right] \tag{6}$$

with an error

$$\epsilon_2 < \frac{2(n-1)(2n+1)(n+2)M^4}{135(n+1)^4}$$

$$r_3 = \frac{2M}{n+1}\left[\frac{1+\dfrac{7n+11}{15(n+1)}M}{1+\dfrac{4n+2}{5(n+1)}M+\dfrac{(n-1)(2n+1)}{45(n+1)^2}M^2}\right]. \tag{7}$$

This value of r_3 expanded in a series is the same as the series for r for the first four terms. Hence the error ϵ_3 must be the order of M^5/n.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

SIMPLE PROBLEMS ON ARC LENGTH

H. S. THURSTON, University of Alabama

The author of a popular text-book* in elementary calculus uses the determination of the circumference of a circle as an illustration of the use of the formula $s=\int\sqrt{1+y'^2}$. In a footnote he points out that a vicious circle is involved (although it develops that the girth of a vicious circle is the same as that of its gentle and saddle-broken cousin, *viz.* $c=2\pi r$) since it was necessary to know in advance that the circumference is $2\pi a$ in order to evaluate arcsin 1 which arises in the solution. He adds that the problem was chosen as an illustration "in order not to diminish the limited supply of simple problems."

A question arises as to the significance of the word "simple." One can easily infer that there is only a finite number of curves the determination of whose arc length is within the capabilities of the college student. On the other hand, the author may mean that there is a limited supply of problems which the average student can work in, let us say, fifteen minutes per problem. While the

* Love, Differential and Integral Calculus, 4th edition, Macmillan, 1943.

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A question arises as to the significance of the word "simple." One can easily infer that there is only a finite number of curves the determination of whose arc length is within the capabilities of the college student. On the other hand, the author may mean that there is a limited supply of problems which the average student can work in, let us say, fifteen minutes per problem. While the

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writer is inclined to agree with the latter interpretation, he wishes to point out that there is an unlimited supply of problems on this topic which involve only the usual techniques of integration.

It will be conceded, I am sure, that a simplification is achieved if $1+y'^2$ is a perfect square. Such functions as $y=\log\cos x$ and $y=\cosh x$ lend themselves to this type of simplification because of the identities $1+\tan^2 x=\sec^2 x$ and $1+\sinh^2 x=\cosh^2 x$. Problems involving the arc lengths of these curves are found in nearly every text-book in elementary calculus. The functions may be regarded as special cases of an infinite set of functions now to be considered.

Let $y=F(x)+G(x)$. Then $1+y'^2=1+F'^2+2F'G'+G'^2$. If we choose as F' and G' functions whose indefinite integrals can be found and such that $4F'G'+1=0$, then $1+y'^2=(F'-G')^2$ and $s=\int_a^b\sqrt{1+y'^2}=\int_a^b|F'(x)-G'(x)|\,dx$. The infinite variety of choices for F' and G' may well furnish both authors and instructors with a refreshingly new set of problems on the topic under consideration.

As noted above $y=\log\cos x$ and $y=\cosh x$ may be regarded as functions of the type just indicated. The definition of $\cosh x$ represents this function in the form $F(x)+G(x)$ with $4F'G'+1=0$, whereas $y=\log\cos x$ may be written as $y=\log(\cos\frac{1}{2}x-\sin\frac{1}{2}x)+\log(\cos\frac{1}{2}x+\sin\frac{1}{2}x)=F(x)+G(x)$, the condition involving F' and G' again being satisfied.

Of special interest, if simplicity is desired, are those functions for which $F'=ax^m$ and $G'=bx^n$. On imposing the condition $4F'G'+1=0$ we are led to the equations $m+n=0$, $4ab+1=0$, whence, if m is neither 1 nor -1,

$$y=\frac{ax^{m+1}}{m+1}+\frac{x^{1-m}}{4a(m-1)} \tag{1}$$

and

$$y=\frac{ax^2}{2}-\frac{1}{4a}\log x,\qquad y=a\log x-\frac{1}{8a}x^2 \tag{2}$$

for $m=1$ and $m=-1$, respectively.

In preparing a set of problems for classroom use the complexity of the functions may vary from one as simple as $y=3x/4$ (by putting $a=1$ and $m=0$ in (1)) to something of the type

$$y=1/4\left[\left(\log\frac{x-2}{\sqrt{x^2+4}}+\arctan\tfrac{1}{2}x\right)+(8\log x-4x+x^2-x^3/3)\right]$$

constructed from

$$F'=\frac{x}{(x-2)(x^2+4)},\qquad G'=\frac{-(x-2)(x^2+4)}{4x},\qquad x>2.$$

It is readily seen that there is no limit to the complexity of the horrible monstrosities which may be constructed after this pattern, and no end to the supply.

AN EVERYWHERE CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTION

JOHN MCCARTHY, Princeton University

The following is an especially simple example. It is

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g(2^{2^n} x)$$

where $g(x)=1+x$ for $-2 \leqq x \leqq 0$, $g(x)=1-x$ for $0 \leqq x \leqq 2$ and $g(x)$ has period 4.

The function $f(x)$ is continuous because it is the uniform limit of continuous functions. To show that it is not differentiable, take $\Delta x = \pm 2^{-2^k}$, choosing whichever sign makes x and $x+\Delta x$ be on the same linear segment of $g(2^{2^k}x)$. We have

1. $\Delta g(2^{2^n}x)=0$ for $n>k$, since $g(2^{2^n}x)$ has period $4 \cdot 2^{-2^n}$.
2. $|\Delta g(2^{2^k}x)|=1$.
3. $|\Delta \sum_{n=1}^{k-1} 2^{-n} g(2^{2^n}x)| \leqq (k-1) \max |\Delta g(2^{2^n}x)| \leqq (k-1) 2^{2^{k-1}} 2^{-2^k} < 2^k 2^{-2^{k-1}}$.

Hence $|\Delta f/\Delta x| \geqq 2^{-k} 2^{2^k} - 2^k 2^{2^{k-1}}$ which goes to infinity with k.

The proof that the present example has the required property is simpler than that for any other example the author has seen.

Weierstrass gave the example $F(x)=\sum_{n=0}^{\infty} b^n \cos (a^n \pi x)$ for $b<1$ and $ab>1+3\pi/2$ which is discussed in Goursat-Hedrick, *Mathematical Analysis*.

A complete discussion of the construction of functions with various singular properties is given in Hobson, *Functions of a Real Variable*, volume II, Cambridge, 1926.

PREMIUM AND DISCOUNT ON BONDS

C. B. READ and J. R. HANNA, University of Wichita

In teaching the mathematics of investment it is discovered that the terms "premium on a bond" and "discount on a bond" are not defined in the same manner by various authors. Since the discrepancies involve the same concept, the discussion is restricted to "premium." Definitions found are:

I. The premium is the excess of the purchase price over the redemption value.

II. The premium is the excess of the purchase price over the face value.

An analysis of some recent books shows five using definition I, thirteen using definition II, two using the term without definition, and three using more than one definition.

[The word premium as applied to bonds is used in two senses: premium with respect to the purchase price and premium with respect to redemption price, a fact not always made clear in texts. An author may be so well aware of this that he fails to realize how lack of explicit definition may confuse the student.]

Without entering into a discussion of which definition is in use by investment firms, it seems unfortunate that authors do not mention alternative definitions. The student is justifiably confused when the author has used definition II

and then proceeds to discuss "amortization of the premium" using definition I for the "premium" without, in many cases, giving any reason for the change, or even a hint that a change has been made.

A GRAPHICAL PROCEDURE

J. P. RUSSELL, Polytechnic Institute of Brooklyn

The purpose of this note is to point out a simple graphical procedure for solving a differential equation of the following type:

$$\frac{dy}{dx} = \frac{\phi(y) + x}{\psi(x) + y}, \tag{1}$$

where $\phi(y)$ and $\psi(x)$ are continuous functions. The construction is an extension of the method of Liénard [1] used in plotting phase trajectories in non-linear problems.

One first plots the curves Γ_1: $[x = -\phi(y)]$, and Γ_2: $[y = -\psi(x)]$ as in Figure 1. Through an arbitrary point, $P(x, y)$, a line is drawn parallel to the x axis intersecting Γ_1 at A, and a line through P parallel to the y axis is drawn intersecting Γ_2 at B. One then draws a line, PD, perpendicular to AB and a small section of the integral curve of (1) passing through P is obtained by taking a small segment of PD through P. At a point close to P on PD repeat the entire procedure, and so on. The proximity of the successive points taken will determine the accuracy of the procedure.

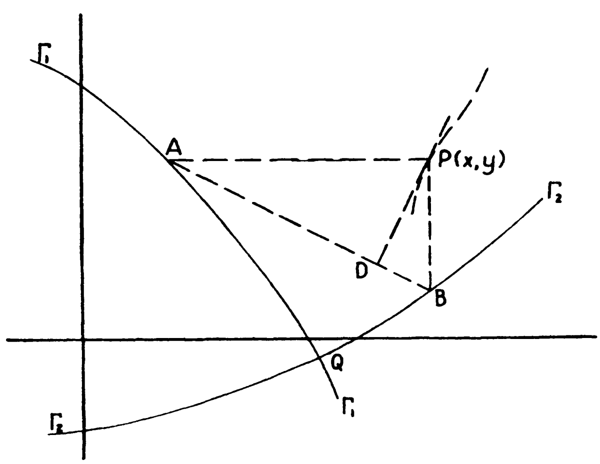


FIG. 1

The justification of the construction proceeds as follows:

The coordinates of P are $[x, y]$. Hence, the coordinates of A become $[-\phi(y), y]$; and the length of $\overline{AP}$ is therefore $\phi(y) + x$. In similar manner, the coordinates

of B are $[x, -\psi(x)]$, and hence the length of $\overline{BP}$ is $y+\psi(x)$. The slope of the line AB is $-(\psi(x)+y)/(\phi(y)+x)$, and the slope of PD is $(\phi(y)+x)/(\psi(x)+y)$, required by equation (1).

It may be remarked that the integral curves will cut Γ_1 with zero slope, and Γ_2 with vertical slope; except at Q, which we see is a singular point of the differential equation.

References

1. J. J. Stoker, Non-linear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950, pp. 31–36.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1091. *Proposed by J. Lambek, McGill University, and Leo Moser, University of Alberta*

Given a sequence of integers $n_1 < n_2 < \cdots < n_i < \cdots$ such that for $j > i$ the decimal representation of n_j does not begin (on the left) with the decimal representation of n_i. Prove that

$$\sum_i 1/n_i \leqq 1 + 1/2 + 1/3 + \cdots + 1/9.$$

E 1092. *Proposed by N. A. Court, University of Oklahoma*

The homothetic center of the orthic and tangential triangles of a given triangle (T) (see the proposer's *College Geometry*, 2nd ed., p. 98, art. 191) is the pole of the orthic axis of (T) with respect to the circumcircle of (T).

E 1093. *Proposed by H. S. Wilf, Nuclear Development Associates, White Plains, N. Y.*

Define $S_0=1$, $S_1=3$, $S_{n+1}=2S_n^2-1$ for $n\geqq 1$. Find

$$\lim_{n\to\infty} \frac{S_n}{2^n S_0 S_1 \cdots S_{n-1}}.$$

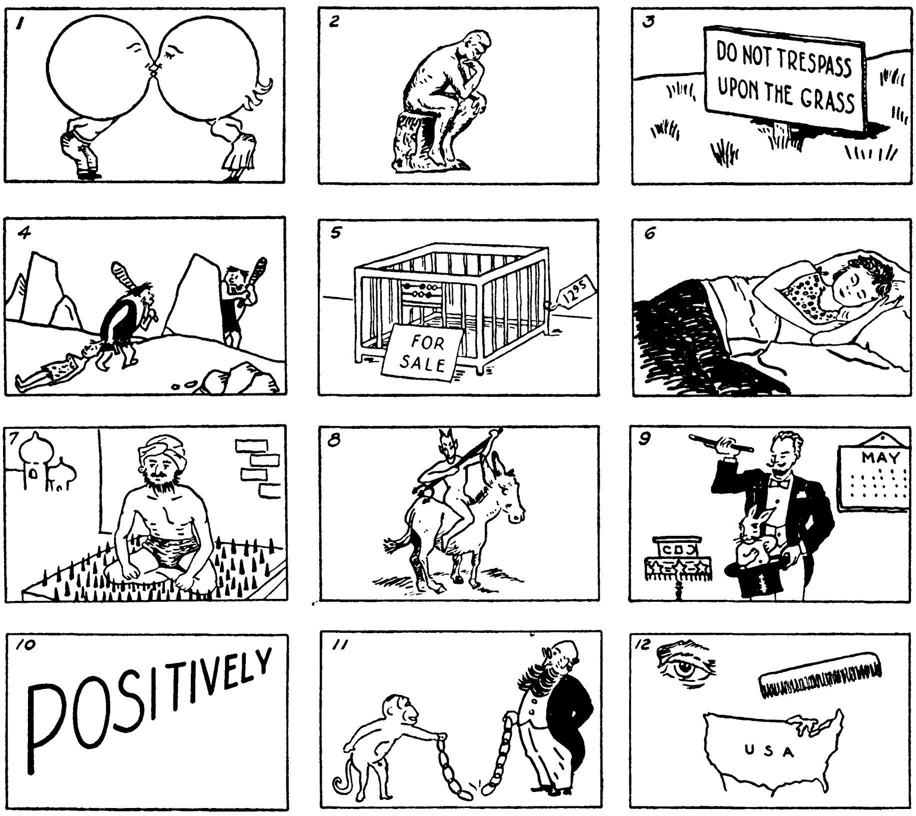
E 1094. *Proposed by Azriel Rosenfeld, Columbia University*

(1) Construct a function defined everywhere on a closed (or open) interval, which takes on each of its values exactly twice on this interval.

(2) Prove that no such function can be continuous.

E 1095. *Proposed by Leon Bankoff and C. W. Trigg, Los Angeles, Calif.*

Translate each of the following sketches into a mathematical term.



SOLUTIONS

A Quadratic Having Roots Differing by Unity

E 1061 [1953, 262]. *Proposed by Walter Penney, Washington, D. C.*

Solve for n, given that the equation

$$\sum_{i=1}^{n} (x+i-1)(x+i) = 10n$$

has roots r and $r+1$.

Solution by Julian Braun, Washington, D. C. Expansion of the left member and further reduction yields

$$x^2 + nx + (n^2 - 31)/3 = 0, \qquad n > 0. \tag{1}$$

The sum of the roots must be $2r+1 = -n$, and the product of the roots must be $r(r+1) = (n^2-31)/3$, a simultaneous solution of which yields $n = 11$ and $r = -6$.

Also solved by P. M. Anselone, J. L. Baker, W. E. Briggs, Bernard Chovitz, P. L. Chessin, J. E. Darraugh, Fred Discepoli, S. H. Eisman, Herbert Emich, D. G. Frood and D. A. Trumpler (jointly), Lloyd Fulk, Harry Furstenberg, H. M. Gehman, A. Gregory, Douglas Holdridge, P. F. Hultquist, A. R. Hyde, Herbert James, John Jones, Jr., A. E. Livingston, David Mandelbaum, R. G. McDermot, George Millman, C. S. Ogilvy, M. W. Oliphant, S. Parameswaran, W. O. Pennell, W. J. Pervin, C. F. Pinzka, Azriel Rosenfeld, C. M. Sandwick, Sr., Milton Scharf, Nathan Schwid, Michael Skalskyj, O. E. Stanaitis, A. V. Sylwester, F. Underwood, Roscoe Woods, and the proposer. Late solution by H. J. Hauer.

Frood and Trumpler found $n = 11$ by using the fact that the roots of (1) will differ by unity if and only if the discriminant is equal to 1.

Fulk and Woods considered the more general problem of finding n given that

$$\sum_{i=1}^{n} (x + i - 1)(x + i) = pn$$

has roots r and $r+q$, where p and q are considered as given.

A Set of Six Positive Integers

E 1062 [1953, 262]. *Proposed by Leo Moser, University of Alberta*

(1) Find six positive integers, not exceeding 24, such that the sums of the numbers in the possible subsets of those numbers will all be different.

(2) Prove that no seven positive integers, not exceeding 24, can have sums of all subsets different.

Solution by C. F. Pinzka, Princeton, N. J. (1) Let the six integers be $n_1 = a$, $n_2 = a+b$, $n_3 = a+b+c$, $\cdots$, $n_r = a+b+c+d+e+f$, so that a, b, c, d, e, f are all positive. If we arbitrarily assume that the "obvious" order of the subsets (*i.e.*, a sum of k numbers is greater than a sum of j numbers if $k > j$, and in two ordered sums of k numbers each that sum is greater in which one first finds a term exceeding the corresponding term in the other sum) results in an increasing sequence of sums, a set of inequalities results. The crucial inequalities (inequalities including the remaining ones) are

$$c > e + f \geqq 2,$$

$$d > f \geqq 1,$$

$$a > c + 2d + 2e + f \geqq 10,$$
$$b > d + 2e + f \geqq 5,$$

obtained from

$$n_3 + n_4 > n_2 + n_6,$$
$$n_4 + n_5 > n_3 + n_6,$$
$$n_1 + n_2 + n_3 > n_5 + n_6,$$
$$n_2 + n_3 + n_4 > n_1 + n_5 + n_6.$$

The "minimum" set $(a, b, c, d, e, f) = (11, 6, 3, 2, 1, 1)$ gives 11, 17, 20, 22, 23, 24 as a set of positive integers having the desired property. Since an arbitrary assumption was made above, there is no guarantee of uniqueness.

(2) The number of subsets of 4 or fewer numbers out of a set of 7 is (excluding the null set)

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 98.$$

If no integer in these sets exceeds 24, then 1 and 90 will be obvious lower and upper bounds for the sums of these subsets of 4 or fewer numbers. Since $90 < 98$, the sums cannot all be distinct.

Also solved by W. E. Briggs, Michael Skalskyj, and the proposer.

An Inequality

E 1063 [1953, 263]. *Proposed by J. V. Whittaker, U.C.L.A.*

Show that if $a \geqq 2$ and $x > 0$, then $a^x + a^{1/x} \leqq a^{x+1/x}$, equality holding if, and only if, $a = 2$ and $x = 1$.

Solution by A. E. Livingston, University of Washington. Let $f(x, a) = (a^x + a^{1/x})/a^{x+1/x}$ for $a \geqq 2$ and $x > 0$. We first observe that $f(x, a) = f(1/x, a)$ and, hence, that the problem will be solved if we show that $f(x, a) \leqq 1$ for $x \geqq 1$ with equality if and only if $a = 2$ and $x = 1$. We next observe that $f(x, a)$ is a strictly decreasing function of a for each x and, hence, that it is sufficient to prove that $f(x, 2) < 1$ for $x > 1$, it being clear that $f(1, 2) = 1$. We will obtain this latter inequality by showing that $F(x) = 2^x f(x, 2) - 2^x < 0$ for $x > 1$. It is easily verified that

$$F'(x) = 2^{x-1/x}(\ln 2)(1 + 1/x^2 - 2^{1/x}) = 2^{x-1/x}(\ln 2)g(1/x).$$

The proof will be complete if we can show that $g(1/x) < 0$ for $x > 1$. Now

$$g''(t) = 2 - 2^t(\ln 2)^2 > 2 - 2(\ln 2)^2 > 0$$

for $0 < t < 1$, so that $g(t)$ is strictly convex for these t. Since $g(t)$ is continuous and $g(0) = g(1) = 0$, it follows that $g(t) < 0$ for $0 < t < 1$ or that $g(1/x) < 0$ for $x > 1$.

Also solved by F. J. Duarte, A. R. Hyde, M. S. Klamkin, R. Larivière, M. J. Pascual, W. O. Pennell, O. E. Stanaitis, and the proposer.

A Converse of the Mean Value Theorem

E 1064 [1953, 263]. *Proposed by Jacob Samoloff and Albert Wilansky, Lehigh University*

Let $f(x)$ be continuous and $f'(x)$ exist in a neighborhood of $x=c$. Suppose that there exists a continuous function $\theta(h)$, with $0<\theta(h)<1$, satisfying the equation

$$f(c+h)-f(c)=hf'[c+h\theta(h)].$$

Does it follow that $f'(x)$ is continuous at $x=c$?

Solution by J. V. Whittaker, U.C.L.A. We have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0}f'(c+h\theta(h))=\lim_{q\to 0}f'(c+q),$$

for $\theta(h)$, being positive and continuous in a closed interval containing 0, must assume a positive minimum value in this interval, so that $q=h\theta(h)\to 0$ is equivalent to $h\to 0$. Therefore, $f'(x)$ is continuous at $x=c$.

Also solved by Sidney Glusman, D. S. Greenstein, Douglas Holdridge, A. E. Livingston, M. S. Klamkin, and Nathan Schwid.

Maximum Section of a Solid Right Circular Cylinder

E 1065 [1953, 263]. *Proposed by C. S. Ogilvy, Hamilton College*

Find the largest plane section of a given solid right circular cylinder.

Editorial Note. It is a mistake to assume, as did all the attempted solutions, that the maximum section is the maximum *elliptical* section. Consider, for example, a cylinder whose altitude h is equal to four times the radius r. Then the maximum elliptical section has an area $E=\sqrt{5}\pi r^2\approx 7r^2$. But the area of a rectangular section through the axis of the cylinder is $R=8r^2$.

The correct solution depends upon the fact that the maximum section of a closed convex surface having a center of symmetry must pass through the center of symmetry. V. L. Klee, Jr., has observed that this is a direct application of the Brunn-Minkowski Theorem. Therefore, the maximum section of a solid right circular cylinder must pass through the center of the cylinder, and therefore will be an ellipse, a part of an ellipse, or a rectangle. Let us consider the second type of central section. Denoting the area of the section by A and the angle the section makes with a base of the cylinder by θ, one may easily show that

$$A/2r^2=\sqrt{1+(k/u)^2}(u\sqrt{1-u^2}+\sin^{-1}u),\qquad 0<u\leqq 1,$$

where $k=h/2r$ and $u=k\cot\theta$. One may now apply standard maximum-minimum methods, but the equation $d(A/2r^2)/du=0$ probably can be solved only by approximation methods.

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where $k=h/2r$ and $u=k\cot\theta$. One may now apply standard maximum-minimum methods, but the equation $d(A/2r^2)/du=0$ probably can be solved only by approximation methods.

x, y, z, u have one and only one real solution in terms of the parameters a, b, where $0<a<5$, $0<b<5$:

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{5-a}+\frac{u^2}{5-b}=1$$

$$(x-z-4)^2+(y-u-2)^2=5.$$

SOLUTIONS

Divisors and Multiples of a Set of Integers

4352 [1949, 479]. *Proposed by Paul Erdös, University of Notre Dame*

Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(1<a_i \leqq n)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

where $2, 3, \cdots, p_k$ are the first k primes.

Solution by George Szekeres, University of Adelaide, Australia. Given n and $1 \leqq a_i \leqq n$, $i=1, \cdots, k$, let $M(n; a_1, \cdots, a_k)$ denote the set of all m, $1 \leqq m \leqq n$ for which either $a_i | m$ or $m | a_i$ for at least one a_i. Let $|M|$ denote the number of integers belonging to the set M and write

$$\lambda(a) = \alpha_1 + \cdots + \alpha_r \qquad \text{if } a = p_1^{\alpha_1} \cdots p_r^{\alpha}.$$

Given k and n let a_i, $i=1, \cdots, k$, be a system of numbers for which $|M(n; a_1, \cdots, a_k)|$ is a maximum and (if there are several such systems) one for which $\lambda(a_1a_2 \cdots a_k)$ is a minimum. We shall show that under these conditions each a_i is a prime. Clearly we may assume that $a_i \neq a_j$ if $i \neq j$.

(1) $a_i = p^\alpha$, p prime, implies $\alpha=1$.

For, if $\alpha>1$, take a system a_j' defined by $a_i'=p$, $a_j'=a_j$ for $j\neq i$. Obviously $M(n; a_1', \cdots, a_k') \supseteq M(n; a_1, \cdots, a_k)$ and $\lambda(a_1' \cdots a_k') < \lambda(a_1 \cdots a_k)$.

(2) $a_i=p$, p prime, implies $(a_j, p)=1$ for $j \neq i$.

Suppose $a_j = p^\alpha a_j'$, $\alpha>0$, $(a_j', p)=1$, for some $j\neq i$. By (1) $a_j'>1$. Write $a_\nu'=a_\nu$ for $\nu \neq j$. Then as before,

$$\text{(I)} \qquad M' = M(n; a_1', \cdots, a_k') \supseteq M(n; a_1, \cdots, a_k) = M.$$

For the only elements of M which are possibly not in M' are divisible by p hence divisible by $a_i = a_i'$, hence are in M'.

(3) Let $a_i = p^\alpha d$, $(p, d)=1$, $\alpha \geqq 1$, $d>1$, then $d \nmid a_j$ for $j \neq i$.

Suppose $d | a_j$, $j \neq i$, and write $a_i'=p$, $a_\nu'=a_\nu$ for $\nu \neq i$. Again (I) is true since the only members of M which are possibly not in M' are divisors of d, hence of $a_j = a_j'$, hence are in M'.

(4) Suppose now that $a_1, \cdots, a_r$ are composite and $a_{r+1}, \cdots, a_k$ are primes. By (2), $(a_\mu, a_\nu)=1$, if $1 \leqq \mu \leqq r$, $r<\nu\leqq k$. Let $q_1, \cdots, q_k$ be distinct primes such that (i) q_1 is the smallest prime factor of $a_1 \cdots a_r$, and (ii) every a_i is divisible

by at least one q_ν. Clearly such a system exists and we may assume that $a_\nu = q_\nu$ for $r < \nu \leqq k$. We shall show that

$$\text{(II)} \qquad |M'| = |M(n; q_1, \cdots, q_k)| \geqq |M(n; a_1, \cdots, a_k)| = |M|.$$

For $\nu = 1, \cdots, k$, let $m_\nu \geqq 1$ be the greatest divisor of a_ν which is relatively prime to $q_1 \cdots q_k$, *e.g.*, $m_\nu = 1$ for $\nu > r$. Obviously the members of M which are not in M' must be divisors of some m_ν and every $d | m_\nu$, $d > 1$ has this property. All we have to show is that there are at least as many members of M' not in M as the number of $d > 1$, $d | m_\nu$.

Suppose first that $d | m_\nu$, $1 < d < m_\nu$, and let α be the largest integer such that $A(d) = q_1^\alpha d \leqq n$. Clearly $\alpha \geqq 1$ and $A(d)$ is a member of M'. On the other hand $A(d)$ is not in M, *i.e.*, it is not a divisor of a_i and not divisible by a_i for any i. For $i = \nu$ this is clear, for $i \neq \nu$ it follows from (3). In fact if $A(d)$ is a divisor of a_i (*i.e.*, $i \leqq r$) then it must be equal to a_i, since $pq_1^\alpha d > n$ for any prime factor p of a_i, hence $a_i | A(d)$. But if a_i divides $A(d)$ then it must have the form $q_1^\beta d_1$, d_1 a divisor of d, hence of a_ν, against (3). It is also clear that for $d_1 \neq d_2$, $A(d_1) \neq A(d_2)$.

Suppose next that $d = m_\nu > 1$, *i.e.*, $\nu \leqq r$, and let their number be s. We must find s distinct numbers not exceeding n and different from all the $A(d)$ such that they are members of M' but not of M. For $i = 1, \cdots, r$, let β_i be the largest integer such that $q_i^{\beta_i} \leqq n$, and γ_i the largest integer such that $A_i = q_i^{\beta_i} q_1^{\gamma_i} \leqq n$. The numbers A_i are evidently members of M' and different from the $A(d)$. We have to show that at least s of them are not in M. Now if A_i is a divisor of a_j then it must be identical with a_j by the same argument as before, hence is divisible by a_j. Of course, it may happen that A_i is divisible by an a_j (or even several a_j) with $m_j = 1$. But two different A_i's cannot be divisible by the same a_j since this would imply that a_j is a power of q_1. It follows that there are at least as many A_i not in M as there are a_ν's with $m_\nu > 1$, *i.e.*, s. The numbers $A(d)$ and A_i together give therefore at least as many members of M', not in M, as the number of $d > 1$, $d | m_\nu$, *q.e.d.*

But $\lambda(q_1 \cdots q_k) < \lambda(a_1 \cdots a_k)$ if $r > 0$, hence $r = 0$ and every a_i is a prime. The remark that

$$|M(n; q_1, \cdots, q_k)| \leqq |M(n; 2, 3, \cdots, p_k)|$$

concludes the proof.

Eratosthenian Averages

4445 [1951, 422]. *Proposed by Paul Erdös, University of Notre Dame*

Split the set of primes $p_1 < p_2 < \cdots$ into two classes q_i and r_i so that $\sum 1/q_i = \sum 1/r_i = \infty$. Define $\mu'(k) = 0$ if k is a multiple of one of the r's, otherwise $\mu'(k) = \mu(k)$, where $\mu(k)$ is the Möbius symbol (0 if k has a square factor, $+1$ if k has an even number of distinct prime factors, -1 if it has an odd number). Prove that

$$\sum_{k=1}^{\infty} \frac{\mu'(k)}{k} = 0.$$

Editorial Note. After having provided an independent proof, the proposer finds the statement and proof of the problem in A. Wintner, *Eratosthenian Averages*, 1943, pp. 70–71.

Squarefull Integers

4459 [1951, 636]. *Proposed by D. J. Newman, Harvard University*

Find an asymptotic expression for the number of integers, not exceeding x, each of which has the property that each of its prime divisors divides it to the second power at least.

Editorial Note. The following solution, with a somewhat more accurate estimate, is quoted by an anonymous contributor from discussions among his colleagues.

Following Sklar [1953, 55], let us call such integers "squarefull" and denote the number of squarefull integers not exceeding x by $A(x)$. Let $c_n = 1$ if n is squarefull, $c_n = 0$ otherwise. Then $A(x) = \sum_{n \leqq x} c_n$. If we set

$$f(s) = \sum_{n=1}^{\infty} c_n/n^s$$

then (see, *e.g.*, Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 2, p. 828)

$$\sum_{n \leqq x} c_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)x^s}{s}\, ds. \tag{1}$$

From the values of c_n it follows that

$$f(s) = \prod_p (1 + p^{-2s} + p^{-3s} + \cdots + p^{-ks} + \cdots),$$

where the product is extended over all primes p. $f(s)$ can be written as

$$\begin{aligned} f(s) &= \prod_p (1 + p^{-s}(p^s - 1)^{-1}) = \prod_p (1 - p^{-6s})/(1 - p^{-2s})(1 - p^{-3s}) \\ &= \zeta(2s)\zeta(3s)/\zeta(6s), \end{aligned}$$

where $\zeta(s)$ stands for the Riemann ζ-function. The integral in (1) can be evaluated by Cauchy's theorem on residues and, if R_j are the residues of the integrand, by (1) and the definition of $A(x)$,

$$A(x) = \sum_j R_j. \tag{2}$$

The only singularities of the integrand are poles and the corresponding residues

can be evaluated as follows:

(a) the poles $s=1/2$ and $s=1/3$ of the numerator have the residues $Ax^{1/2}$ and $Bx^{1/3}$ respectively, with $A=\zeta(3/2)/\zeta(3)$ and $B=\zeta(2/3)/\zeta(2)$;

(b) the zero $s=0$ of the denominator leads to the residue $x^0\zeta(0)=-1/2$;

(c) $s=-n/3$, the real zeros of $\zeta(6s)$ lead to residues $a_nx^{-n/3}$; for $n=1$ in particular we have the residue $a_1x^{-1/3}$, where $a_1\neq 0$ (its value can be computed but is immaterial for our purpose);

(d) $\rho_\nu/6$ are the complex zeros of $\zeta(6s)$, with $\rho_\nu=\sigma_\nu+it_\nu$, the complex zeros of $\zeta(s)$. Correspondingly we have the residues $x^{\rho_\nu/6}R_\nu=x^{\sigma_\nu/6}\cdot x^{it_\nu/6}R_\nu$ (R_ν can be computed; if e.g., ρ_ν is a simple root, we easily obtain $R_\nu=\zeta(\rho_\nu/3)\zeta(\rho_\nu/2)/\rho_\nu\zeta'(\rho_\nu)$, etc.) If $\sigma=\overline{\lim}\,\sigma_\nu$ then the sum of these residues can be written as

$$x^{\sigma/6}\sum_\nu R_\nu x^{-((\sigma-\sigma_\nu)+it_\nu)/6} = x^{\sigma/6}Q(x), \qquad |Q(x)| < \infty.$$

Adding together all residues we obtain by (2)

$$A(x) = Ax^{1/2} + Bx^{1/3} + Q(x)\cdot x^{\sigma/6} - 1/2 + O(x^{-1/3}), \tag{3}$$

where A and B have the indicated numerical values.

Assuming the Riemann hypothesis, $\sigma=\sigma_\nu=1/2$. Setting

$$C = \lim_{x\to\infty} \sum_\nu R_\nu x^{it_\nu/6},$$

(3) becomes $A(x)=Ax^{1/2}+Bx^{1/3}+Cx^{1/12}-1/2+O(x^{-1/3})$ if the limit defining C exists; otherwise $A(x)=Ax^{1/2}+Bx^{1/3}+O(x^{1/12})$. If we do not assume the Riemann hypothesis we still know that $\sigma<1$ and (3) becomes

$$A(x) = \{\zeta(3/2)/\zeta(3)\}x^{1/2} + \{\zeta(2/3)/\zeta(2)\}x^{1/3} + O(x^{1/6}).$$

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

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Trigonometry for Today. By M. Brooks, A. C. Schock, and A. I. Oliver. New York, Harper & Brothers, 1951. ix+200+Index and Tables, \$2.96.

Plane Trigonometry. By L. M. Kells, W. F. Kern, and J. R. Bland. New York, McGraw-Hill, 1951. xi+180+Appendices, Answers, Index, Tables. \$3.50.

In the preface to this, the third edition of Kells, Kern and Bland, the authors aver that they have made improvements "by eliminating every item that did not have a definite purpose. . . ." They have added a short chapter on vectors, and numerous new exercises. These occur especially in the chapter on graphs and the chapter on the addition formulas. The major features of the predecessors of this edition have been left intact.

In view of its elementary style, and relatively simple exercises, it would appear that the Brooks, Schock and Oliver is intended primarily for high school purposes. However, the emphasis which it places on the notion of "function" renders it rather unique among books at this level. The book begins with a chapter on coordinates, functions, and graphs. The trigonometric functions are then defined for arbitrary angles, and the Pythagorean identities are established. The graphs of the trigonometric functions are then taken up, and the student is directed to think of the functions in terms of their graphs. This point of view carries over into the chapter on reduction formulas which follows. With the exception of a chapter on the addition formulas, and a short chapter of "supplementary topics," the remainder of the book is concerned with the solution of triangles and logarithms.

H. T. MUHLY
State University of Iowa

Calculus. By C. R. Wylie, Jr. New York, McGraw-Hill Book Company, 1953. x+565 pages. $6.00.

The organization of this text differs from that of the traditional text mainly in the first chapter. Here the area in the first quadrant under the curve $y=x^2$ from $x=0$ to $x=1$ is found. After the customary use of rectangles for upper and lower approximations, the author makes use of the formula for the sum of the squares of the first n consecutive integers, which the student should already have encountered in problem lists on mathematical induction. Then by a frankly intuitive limiting process, the author finds both approximations to have the limit 1/3 and takes the area to be 1/3. After more facility is developed in finding areas by arithmetical means, the student is shown the symbol for definite integral and given a slight inkling of its further use. Also in the first chapter, the concept of slope is strongly associated with that of derivative.

Chapter two does careful work with limits and continuity, including $\lim_{\theta\to 0} (\sin \theta/\theta)$ and the number e. Chapters three and four complete the study of differentiation, covering all the traditional topics. In chapter five on integration, the definite integral is used to introduce the indefinite. Improper integrals follow immediately after a small amount of practice on proper integrals. Chapters six and seven complete the practice on formal integration. The formulas are derived with considerable rigor, but the amount of practice given the student is somewhat less than in many texts. Chapters eight, nine, and ten take care of rates, motion, and geometrical and physical applications of integration. Chapter eleven is an excellent explanation of hyperbolic functions. In chapter

twelve, analytic geometry of three dimensions follows, rather than precedes, the careful work on partial derivatives. This reviewer is glad to see the integral test of convergence included in chapter thirteen on series and to see that the comparison tests, so difficult for students to apply, are not the first tests given. Chapter fourteen includes Maclaurin's and Taylor's expansions, operations with series, and excellent material on Fourier's series. Chapters fifteen on multiple integration and sixteen on differential equations are followed by an excellent appendix containing a glossary and by tables.

The book abounds in examples, many from physics, and all completely solved. The author wisely suggests that the teacher adapt the text to his purposes by a suitable selection of examples. The examples are largely responsible for making the book so long that rather small type must be used. Further objection to the typography is the absence of sufficiently outstanding type for important statements. The individual formulas for differentiation are scattered and buried, and are not even numbered in a single sequence. The same is true for integration formulas. The reviewer has noticed several minor misprints and omissions.

The style is refreshingly direct. On page one the author admonishes the student that it is important for him to want to succeed in the study of the calculus. The excellent chapter introductions and summaries help the student to evaluate the work of each chapter and to relate it to the work as a whole. In these discussions the author takes the student into his confidence and supplies the answers to many of the student queries which most textbooks leave for the good teacher to answer.

This is a carefully written book. To use it as effectively as its own author no doubt does, a teacher would need to be in sympathy with the author's departure from the traditional course in a few places, notably in the beginning chapter and in the first chapter on integration.

ANICE SEYBOLD
North Central College

Complex Analysis. By Lars V. Ahlfors. New York, McGraw-Hill Book Company, 1953. xi+247 pp. $5.00.

Professor Ahlfors has fulfilled expectations that he would produce a scholarly and novel treatment of analytic function theory. The only disappointment which the reviewer experiences is that the author stopped too soon. One could wish for a continued discussion of more material in the same vein.

It has been generally recognized that expositions of analytic functions in English have failed to make sufficient use of relevant topological tools. This is the more surprising in view of the great mutual influence that complex variable theory and topology have exerted upon each other. In the present book Ahlfors remedies this deficiency by making systematic use of topological techniques, and he has also provided brief but reasonably adequate introductions to the topological ideas which he uses. At the same time he has retained the flavor of

classical function theory; and he has avoided the assembly-line format wherein every paragraph is labeled Theorem, Definition, Corollary, Remark, and the like. Since the book is self-contained and presupposes only a moderate mathematical maturity, it is suitable for a text in a first course in complex variables for beginning graduate students; or, since the point of view and treatment are quite different from that in standard texts, it can be used as the basis for a second course for students who already have considerable background. The reviewer tried the latter experiment, with encouraging results. The reasoning is close and condensed and requires alertness on the part of the student, but the writing and proofreading have been done with considerable care, so that a close scrutiny of details, such as places where, in the author's words, "a voluntary gap serves the purpose of saving half a page of unconstructive and dull reasoning" will be rewarding.

The most novel feature of the book is the treatment of Cauchy's theorem. The proof, which is spread over some 40 pages, interspersed with applications, introduces topology in stages. The key to the procedure is the lemma that $\int_C f(z)dz$ vanishes for every closed curve C in a region S, *regardless of the connectivity of* S, if $f(z)$ is continuous and is the derivative of an analytic function $F(z)$ in S. Since this lemma is independent of the topology of S, it is relatively easy to prove by standard methods of calculus from the Cauchy-Riemann equations for $F(z)$.

Among the numerous topics treated in the book, the following may be given special mention: Geometric properties of linear transformations, homology theory and Cauchy's formula based on the winding number of a curve, the openness of analytic mappings, normal families and the Riemann mapping theorem, the Dirichlet problem and harmonic measures, the theory of Riemann surfaces, algebraic functions, linear differential equations, and homotopy theory and the monodromy theorem.

P. W. KETCHUM
University of Illinois

NEW BOOKS RECEIVED

A First Course in Functions of a Complex Variable. By Wilfred Kaplan, Cambridge, Mass., Addison-Wesley Press, 1953. 7+134 pages. $3.50.

Foundations of Combinatorial Topology. By L. S. Pontryagin. Rochester, New York, Graylock Press, 1952. 12+99 pages. $3.00.

Three Pearls of Number Theory. By A. Y. Khinchin. Rochester, New York, Graylock Press, 1952. 64 pages. $2.00.

Foundations of the Nonlinear Theory of Elasticity. By V. V. Novozhilov. Rochester, New York. Graylock Press, 1953. 6+233 pages. $4.00.

50–100 Binomial Tables. By H. G. Romig. New York, John Wiley and Sons, Inc., 1953. 27+172 pages. $4.00.

Analytical Geometry of Three Dimensions. By W. H. McCrea. New York, Interscience Publishers, Inc., 1953. 7+144 pages. $1.25.

Squaring the Circle. By E. W. Hobson. New York, Chelsea Publishing Company, 1953. 381 pages approx. (4 books published in one). $3.25.

Mathematical Aspects of the Quantum Theory of Fields. By K. O. Friedrichs. New York, Interscience Publishers, Inc. 8+272 pages. $5.00.

Engineering Statistics and Quality Control. By I. W. Burr. New York, McGraw-Hill Book Co., Inc., 11+442 pages. $7.00.

A Refresher Course in Mathematics. By F. J. Camm. New York, Emerson Books, Inc. (251 West 19th Street, New York 11, N. Y.), 1953. 240 pages. $2.95.

Relativity and Reality. By E. G. Barter. New York, The Philosophical Library, Inc., 1953. 11+131 pages. $4.75.

Mathematics and Statistics for Economists. By Gerhard Tintner. New York, Rinehart and Company, Inc., 1953. 14+363 pages. $6.50.

Analytic and Projective Geometry. By D. J. Struik. Cambridge, Mass., Addison-Wesley Press, Inc., 1953. 9+291 pages. $6.50.

Analytische Geometrie. By Gunter Pickert. Leipzig, Germany, Akademische Verlagsgesellschaft, Geest & Portig K.-G., 1953. x+397 pages. DM 26.

Probability Tables for the Analysis of Extreme-Value Data. National Bureau of Standards Applied Mathematics Series 22, ii, 32 pages. 25 cents.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

PERSONAL ITEMS

At Brooklyn College: Associate Professors Samuel Borofsky and C. B. Boyer have been promoted to professorships; Mrs. Margaret Y. Woodbridge has been promoted to an assistant professorship; Professor Edward Fleisher has retired with the title of Professor Emeritus.

Brown University announces the following: Professor R. E. Gilman's leave of absence has been extended through 1953–54 in order that he may continue work in Washington; Mrs. Mildred Carlen Brunschwig has resigned from her position as Registrar of the Graduate School.

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Cornell University reports the following: Professor J. B. Rosser has received a Fulbright Fellowship for the year 1953–54 and is studying at the University of Paris; Assistant Professor G. A. Hunt has been promoted to an associate professorship; Mr. Walter Feit, previously a teaching assistant at the University of Michigan, Dr. I. S. Gál, formerly at the Institute for Advanced Study, Assistant Professor J. K. Goldhaber of the University of Connecticut, Dr. C. S. Herz, formerly National Science Foundation Fellow at Princeton University, and Dr. Steven Orey, previously a teaching assistant at the University, have been appointed to instructorships; Mr. R. C. Lesser of the Office of Statistical Services of Massachusetts Institute of Technology has been appointed Director of the Cornell Computing Center; Professor Pierre Samuel has returned to his position at Clermont-Ferrand, France.

At Georgia Institute of Technology: Assistant Professor J. R. Garrett has been promoted to an associate professorship; Dr. B. M. Drucker, Dr. J. A. Nohel, and Dr. Henry Sharp, Jr., have been appointed to assistant professorships; Assistant Professor W. B. Evans and Instructor W. R. Carnes have returned to resume their teaching duties after two years with the Air Force.

Kansas State College announces: Dr. J. M. Marr, previously a teaching assistant at the University of Tennessee, and Assistant Professor W. L. Stamey of the University of Georgia, Atlanta Division, have been appointed to assistant professorships; Mr. Albert Derin, formerly a graduate student at the University of Chicago and Mr. J. V. Guida, who has been a graduate student at the University of Missouri, have been appointed to instructorships; Assistant Professor Albert Furman is on sabbatical leave for the year 1953–54 and is studying at the University of Kansas; Professor P. M. Young has been appointed acting Dean of Students for the current academic year.

Mills College reports the following: Assistant Professor Grayson Schmidt, formerly head of the Department of Mathematics and Physics has resigned; Dr. Andrewa R. Noble has been appointed Assistant Professor of Mathematics and Physics and Head of the Department.

Oberlin College announces the following: Chairman E. P. Vance is on sabbatical leave for the year 1953–54; Professor R. R. Stoll will serve as Acting Chairman of the Department of Mathematics during the year; in addition to appointments announced previously, Dr. J. D. Baum of Yale University and Dr. A. D. Martin of Washington University have been appointed to instructorships; during the past summer Professor Stoll assisted Professor Artin at the Summer Conference on Collegiate Mathematics at Boulder, Colorado.

,University of Buffalo announces that Mr. D. O. McKay, formerly a teaching fellow at the University, and Mr. P. J. Schillo, who has been a graduate student at the University, have been appointed to instructorships.

At the University of Michigan: Assistant Professor R. C. Lyndon of Princeton University has been appointed to an assistant professorship; Instructor Edwin Weiss of Massachusetts Institute of Technology has been appointed to an instructorship; Associate Professor E. D. Rainville has been promoted to a

professorship; Assistant Professors P. S. Jones and George Piranian have been promoted to associate professorships; Instructor Frank Harary has been promoted to an assistant professorship; Assistant Professor N. H. Anning has retired with the title of Assistant Professor Emeritus; Associate Professor Wilfred Kaplan is on sabbatical leave for the year 1953–54; Associate Professor Hans Samelson is on leave for the year 1953–54 and is at the Institute for Advanced Study; Professor A. H. Copeland will be on sabbatical leave for the second semester of 1953–54.

Mr. Silvio Aurora, previously a graduate student at Columbia University, has a position as a tutor at Queens College.

Dr. Aaron Bakst has been appointed to an assistant professorship in the School of Commerce, New York University.

Acting Assistant Professor B. J. Ball of the University of Virginia has been appointed to an assistant professorship.

Dr. W. E. Barnes is now Head of the Ballistics and Statistical Theory Branch, Computation and Ballistics Department, United States Naval Proving Ground, Dahlgren, Virginia.

Dr. Max Beberman of the University of Illinois has been promoted to the position of Assistant Professor of Education.

Mr. V. W. Beck, formerly an electrical engineer with American Television and Radio, St. Paul, Minnesota, is now a research engineer with Minneapolis Honeywell Company.

Mr. J. S. Becker is now an engineering assistant in the Aviation Division, Studebaker Corporation, South Bend, Indiana.

Mr. P. R. Beesack has been appointed to a research assistantship at Washington University.

Mr. Jonas Beraru, previously a design engineer with Ford Instrument Company, Long Island City, New York, has accepted a position as a mathematician with the Reeves Instrument Corporation, New York City.

Dr. Gerald Berman of Illinois Institute of Technology has been promoted to an assistant professorship.

Mr. H. A. Bernhard, recently a graduate student at Columbia University, is now employed as a research engineer by the Boeing Airplane Company, Melbourne, Florida.

Miss Ida M. Bernhard, who has been Supervisor of Mathematics at Southwest Texas State Teachers College, is now Consultant in Secondary Education, Texás Education Agency, Austin, Texas.

Mr. H. H. Berry has accepted a position as a numerical analyst with the General Electric Company, Cincinnati, Ohio.

Associate Professor M. T. Bird of San Jose State College has been promoted to a professorship.

Mr. S. R. Bodner of Polytechnic Institute of Brooklyn has been appointed to a research assistantship.

Mr. J. R. Boyd, who has been teaching at San Marcos High School, Texas, has accepted a position at Chance Vought Aircraft, Dallas, Texas.

Dr. J. W. Brace, recently a graduate student at Cornell University, has been appointed to an instructorship at the University of Maryland.

Dr. W. E. Briggs of the University of Colorado has been appointed Research Assistant.

Dr. Paul Brock is now Head of Technical Services, Computer Division, Consolidated Engineering Corporation, Pasadena, California.

Professor D. M. Brown has been promoted to the position of Head of the Department of Data Reduction and Computation, Willow Run Research Center, University of Michigan.

Associate Professor H. K. Brown of Northeastern University has been promoted to the position of Professor of Mechanical Engineering and Director of Engineering Graduate Study.

Mr. A. L. Buchman, formerly of Hutchinson Central High School, Buffalo, New York, is teaching now at Technical High School, Buffalo.

Assistant Professor Emily E. Calkins of the College of William and Mary has been promoted to an associate professorship.

Miss Dorothy I. Carpenter of Denison University has been appointed to an associate professorship at Ashland College.

Mr. R. J. Cary, previously a student at Harpur College, has accepted a position with the Ansco Company, Binghamton, New York.

Mr. P. L. Chessin, formerly of Cooper Union School of Engineering, has a position as Associate Engineer in the Air Arm Division, Westinghouse Electric Corporation, Baltimore, Maryland.

Associate Professor D. E. Christie has been awarded a Faculty Fellowship from the Fund for the Advancement of Education; he is on leave of absence from Bowdoin College for the year 1953–54 and is Visiting Fellow in the Department of Mathematics, Princeton University.

Associate Professor B. G. Clark of Vanderbilt University has been promoted to a professorship.

Mr. M. J. Cleveland has a position as a mathematician in the Underwater Explosions Research Division, Norfolk Naval Shipyard, Portsmouth, Virginia.

Assistant Professor D. E. Coffey of Lawrence Institute of Technology has been promoted to an associate professorship.

Associate Professor L. W. Cohen is on leave of absence from Queens College and has been appointed Program Director for Mathematical Sciences for the National Science Foundation.

Assistant Professor C. H. Cook of Western State College, Austin, Texas, has accepted a position as Aerophysics Engineer with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. K. L. Cooke of State College of Washington has been promoted to an assistant professorship.

Professor N. A. Court of the University of Oklahoma has retired with the

title of Professor Emeritus.

Mr. D. E. Deal of Ball State Teachers College, Indiana, has been promoted to an assistant professorship.

Dr. W. E. Deskins of the University of Wisconsin has been appointed to an instructorship at Ohio State University.

Mr. David DeVol of the University of Colorado has accepted a position with the firm of George B. Buck, Consulting Actuary, New York City.

Mr. A. R. DiDonato, formerly with the du Pont Experimental Station, Wilmington, Delaware, is employed as a mathematical physicist at the Melpar Corporation, Alexandria, Virginia.

Dr. H. P. Edmundson, previously a graduate student at the University of California at Los Angeles, has accepted a position with the Department of Defense, Washington, D. C.

Dr. J. E. Flanagan of the University of Illinois has been appointed to an assistant professorship at the Carnegie Institute of Technology.

Mr. A. J. Flynn, formerly an assistant at Illinois State Normal University, has a position as an analyst in the Engineering Division, Eureka Williams Corporation, Bloomington, Illinois.

Dr. J. R. Foote, previously at the Wright-Patterson Air Force Base, Dayton, Ohio, has been appointed to an assistant professorship at the University of Oklahoma.

Mr. D. E. Freeland has been appointed Research Assistant in the Statistical Laboratory at Purdue University.

Mr. W. H. From, who has been with Aircraft Armaments, Inc., Baltimore, Maryland, has accepted a position at the American Machine and Foundry Company, Boston, Massachusetts.

Mr. G. A. Galloway, formerly head of the Department of Mathematics of Flat River Junior College, Missouri, is now Superintendent of Schools, Kane Unit Schools, Fairfield, Illinois.

Mr. R. H. Gillespie has accepted a position as Senior Research Assistant at the Electro-Metallurgical Company, Niagara Falls, New York.

Mr. W. H. Glenn, Jr., who has been Assistant Curriculum Coordinator for the Pasadena City Schools, has been appointed Chairman of the Department of Mathematics of Pasadena City College.

Dr. Herbert Goertzel, previously of the Oak Ridge National Laboratory, is now with the AEC Computing Facility, New York University.

Dr. Lillian Gough of the University of Buffalo has been appointed to an instructorship at Oswego State Teachers College, New York.

Mr. F. D. Grogan, who has been Resident Inspector for the Dallas Chemical Procurement District, Texas, is now Resident Inspector, Army Chemical Corps, Rocky Mountain Arsenal, Denver, Colorado.

Mr. H. M. Gurk, formerly a graduate student at the University of Pennsylvania, has been appointed to an instructorship in the Moore School of Electrical Engineering of the University.

Mr. F. S. Hawthorne of Hofstra College has been promoted to an assistant professorship.

Dr. C. M. Hebbert, who has retired from his position at the Bell Telephone Laboratories, has been appointed to a professorship at the Polytechnic Institute of Brooklyn.

Dr. Melvin Henriksen of Purdue University has been promoted to an assistant professorship.

Mr. D. M. Hester, who has been teaching at Liberty High School, Texas, has been appointed Head of the Department of Mathematics of Highland Junior College, Kansas.

Assistant Professor W. N. Huff of the University of Oklahoma has been promoted to an associate professorship.

Mr. O. C. Juelich, previously an assistant at Ohio State University, is engaged as an aerodynamicist with North American Aviation, Columbus, Ohio.

Mr. R. A. Kennedy, formerly a student at Illinois Institute of Technology, has accepted a position as Quality Control Supervisor with Johnson and Johnson, Chicago, Illinois.

Dr. R. J. Koch of Tulane University has been appointed to an assistant professorship at Louisiana State University.

Mr. Sidney Kravitz, previously at Picatinny Arsenal, Dover, New Jersey, has a position with Eastern Engineering Company, Hackensack, New Jersey.

Dr. R. J. Lambert of the National Security Agency, Washington, D. C., has accepted a position as Assistant Professor of Mathematics at Iowa State College; he also retains a position as Consultant with the Agency.

Mr. W. D. Lambert has a position as a consultant in the Mapping and Charting Research Laboratory, Columbus, Ohio.

Mr. David Loev, formerly a student at the University of Pennsylvania, has accepted a position as an engineer with the Burroughs Corporation, Philadelphia, Pennsylvania.

Associate Professor Lee Lorch of Fisk University has been promoted to a professorship.

Mr. R. A. Magda, previously a student at the University of Detroit, is teaching at Assumption High School, Windsor, Ontario, Canada.

Mr. J. N. Mangnall, formerly with Bell Aircraft Corporation, Niagara Falls, New York, is engaged as Associate Mathematician at Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. J. A. Mansour of the University of Detroit has been promoted to an instructorship.

Mr. W. A. Mary is teaching at Webster Junior High School, Collinsville, Illinois.

Brother Thomas Matthews is now President of Christian Brothers College, Memphis, Tennessee.

Mr. H. T. McAdams of the Aluminum Research Laboratories, East St. Louis, Illinois, has accepted a position as a research physicist with the Cornell

Aeronautical Laboratory, Buffalo, New York.

Mr. P. J. McCarthy has been appointed to a research assistantship at the University of Notre Dame.

Associate Professor A. W. McGaughey of Bradley University has been promoted to a professorship.

Mr. R. T. McLean of the College of Steubenville has been promoted to an assistant professorship.

Mr. F. A. McMahon, previously with Sperry Gyroscope Company, Great Neck, New York, is now a training supervisor at General Precision Laboratory, Inc., Pleasantville, New York.

Research Associate Paul Meier of Johns Hopkins University has been promoted to the position of Assistant Professor of Biostatistics.

Mr. H. E. Menke of Heidelberg College has accepted a position as a mathematician with the National Machinery Company, Tiffin, Ohio.

Mr. R. B. Merkel, who has been teaching in the Sacramento Unified School District, has been appointed to an assistant instructorship at Sacramento State College.

Mr. A. C. Moeller has been promoted to an assistant professorship in the Department of Electrical Engineering, Marquette University.

Assistant Instructor R. E. Montgomery of Rutgers University has been appointed to an instructorship at Trinity College, Connecticut.

Mr. A. J. Mortola of St. Peter's College has been appointed to an assistant professorship at Manhattan College.

Mr. T. D. Nagle, formerly a student at the University of Bridgeport, is employed now by Pratt and Whitney Aircraft Corporation, East Hartford, Connecticut.

Mr. R. J. Oravec, who was an actuarial student with Equitable Life Assurance Society of United States, New York City, is engaged now as an operations research scientist with Republic Aviation, Farmingdale, New York.

Dr. L. L. Pennisi of the University of Illinois, Navy Pier, Chicago, has been promoted to an assistant professorship.

Associate Professor H. H. Pixley of Wayne University has been appointed Associate Dean of Administration.

Mr. D. W. Pounder, previously a senior aerodynamicist at A. V. Roe Canada, Ltd., Toronto, has a position as Aerodynamic Engineer with deHavilland Propellers, Ltd., Hatfield, Hertfordshire, England.

Associate Professor L. L. Rauch of the University of Michigan has been promoted to the position of Professor of Instrumentation.

Dr. D. B. Ray, previously a graduate student at Cornell University, has been awarded a Jewett Fellowship and will study at Princeton University during 1953–54.

Mr. J. D. Rice, assistant at Rice Institute, has been appointed to an assistant professorship at Lamar State College of Technology.

Mr. M. B. Ritterman of Long Island University has a position as Senior

position at the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Miss Evelyn L. Trennt of Springfield Junior College, Illinois, has been appointed to an instructorship at Milwaukee Downer Seminary.

Mr. Carl Tross, previously at the Wright Air Development Center, Dayton, Ohio, is now with Lockheed Aircraft Corporation, Burbank, California.

Mr. S. V. S. Walker, formerly a student at The Citadel, has a position as Teacher and Assistant Principal at North Charleston High School, South Carolina.

Professor Emeritus F. B. Wiley of Denison University has resumed his teaching duties at the University.

Mr. L. R. Chase, a teacher at Rogers High School, Newport, Rhode Island, died on December 6, 1952. He had been a member of the Association for twenty-seven years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FOURTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The fourteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 6, 1954. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to undergraduate students in universities and colleges of the United States and Canada who have not received a college degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, Box 30, Kent State University, Kent, Ohio, by a postcard request. All applications must be filed with the Director not later than February 10th, 1954. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

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The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to the contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the thirteen previous competitions and examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August–September 1947, December 1948, August–September 1949, 1950, 1951, October 1952 and 1953.

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CALENDAR OF FUTURE MEETINGS

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

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The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

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ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY

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MISSOURI, University of Missouri, Columbia, Spring, 1954.

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OKLAHOMA

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EMPLOYMENT OPPORTUNITIES

Beginning with the February 1954, issue, the MONTHLY will devote this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

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To be published
Spring, 1954

Houghton Mifflin Company

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN